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JUSTIFICATION OF POINT ELECTRODE MODELS IN ELECTRICAL IMPEDANCE TOMOGRAPHY

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Abstract. The most accurate model for real-life electrical impedance tomography is the complete electrode model, which takes into account electrode shapes and (usually unknown) contact impedances at electrode-object interfaces. When the electrodes are small, however, it is tempting to formally replace them by point sources. This simplifies the model considerably and completely eliminates the effect of contact impedance.

In this work we rigorously justify such a point electrode model for the important case of having difference measurements ("relative data") as data for the reconstruction problem. We do this by deriving the asymptotic limit of the complete model for vanishing electrode size. This is supplemented by an analogous result for the case that the distance between two adjacent electrodes also tends to zero, thus providing a physical interpretation and justification of the so-called backscattering data introduced by two of the authors.

1. Introduction

The aim of *electrical impedance tomography* is to produce images of the admittance within an electrically conducting object (such as the human body) from boundary measurements of current and voltage, cf. the overview articles of Barber and Brown [1], Cheney, Isaacson and Newell [5], Borcea [3, 4], Lionheart [15], Bayford [2], and the book edited by Holder [11]. To alleviate modelling errors and measurement noise, many practical applications of impedance tomography utilize difference measurements (sometimes called "relative data"): For a given set of boundary current patterns, the measured voltages are compared with a set of reference potentials to generate an image of the corresponding *admittance change* inside the object. Common examples are time-difference and frequency-difference measurements.

The most accurate mathematical (forward) model for impedance tomography is known as the *complete electrode model*. This model takes into account both the shunting effect on the conducting electrodes and the contact impedance between the electrodes and the imaged object. It has been experimentally verified to be capable of predicting real-life measurements up to measurement precision, cf. Cheng, Isaacson, Newell and Gisser [6], and Somersalo, Cheney and Isaacson [20].

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In many practical applications, the size of the electrodes seems negligibly small compared to the total boundary area and to the inevitable modelling errors, such as inaccurate positioning of the electrodes; of the many possible examples, consider, e.g., the geophysical applications of impedance tomography in [17, 18, 19]. It is therefore tempting to formally replace small electrodes by *point electrodes* modelled by delta distributions. For difference data this has the additional effect of eliminating the (usually unknown) contact impedances.

In this work we will give a mathematically rigorous justification for using this kind of *point electrode model* by deriving it as an asymptotic limit of the complete electrode model when the electrodes' diameter h tends to zero. More precisely, we will show that the relative approximation error decays like h^2, if the electrodes are replaced by point sources located at their centers. The precise formulation of this main contribution of our paper is given in Section 2.4 below. It is supplemented by an analogous result for the case that the diameter of the electrodes, i.e., h, is as small as the distance between two adjacent electrodes: We prove that in the limit $h \to 0$ the corresponding real-world measurements converge to the so-called *backscattering data* introduced by two of the authors ([8, 9]) before. For backscatter data, however, we can only prove a convergence rate $O(h)$ as $h \to 0$.

For completeness, it should be mentioned that the connection between the complete electrode model and the so-called *continuum model* of impedance tomography (see, e.g., [3]) has previously been studied in [12, 13, 14]. However, the philosophy of these articles differs from the approach of this work: In [12, 13, 14] it has been investigated in what sense the current-to-voltage map of the complete electrode model approximates the Neumann-to-Dirichlet boundary operator of the continuum model as the electrodes get smaller, their number increases, and their coverage of the object boundary is getting better and better. In the present work, the locations and the number of the electrodes are fixed and the only thing that is altered is the electrode size.

The outline of this work is as follows. In Section 2 we give the precise mathematical specifications of the two relevant electrode models, and comment on their well-posedness for relative data; in a separate subsection we summarize all the geometrical assumptions on the finite size electrodes that we are going to impose when we let their diameter h go to zero. Afterwards, in Section 2.4, we present the main result of this work, the proof of which is postponed to Section 3. Finally, in Section 4 we provide our asymptotic result for backscatter data.

2. The Setting and Main Result

In what follows, we assume that $\Omega \subset \mathbb{R}^n$, $n = 2$ or $n = 3$, is a bounded domain (i.e., an open and connected set) with C^∞-boundary and connected complement. The outer unit normal of $\partial\Omega$ is denoted by ν. Throughout, let $\sigma_0 \in C^\infty(\overline{\Omega}; \mathbb{C}^{n\times n})$ be a smooth and (real) symmetric background admittance. We assume that the true admittance inside Ω is a compactly supported perturbation of σ_0, i.e.,

$$\sigma = \sigma_0 + \kappa, \tag{1}$$

where $\kappa \in L^\infty(\Omega; \mathbb{C}^{n\times n})$ is (real) symmetric and supported away from the boundary $\partial\Omega$. Furthermore, both σ and σ_0 are assumed to satisfy (see, e.g., [3])

$$\mathrm{Re}(\sigma\xi \cdot \overline{\xi}) \geq c\|\xi\|^2_{\mathbb{C}^n}, \quad |\sigma\xi \cdot \overline{\xi}| \leq C\|\xi\|^2_{\mathbb{C}^n}, \quad c, C > 0, \tag{2}$$

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for all $\xi \in \mathbb{C}^n$. Here, and throughout this work, $0 < c < C$ denote generic constants (c a small one, C a large one) that are independent of h and that may change from one occasion to the next.

2.1. **Complete electrode model.** To begin with, let us recapitulate the complete electrode model, where we employ superscripts h to serve as a measure for the size of the diameter of the electrodes. Later we will drive h to zero in the asymptotic analysis.

Within the complete electrode model the boundary of Ω is assumed to be partly covered by M electrodes, which are taken to be ideal conductors, and which are identified with the open, simply connected, and mutually disjoint parts $e_m^h \subset \partial\Omega$, $m = 1, \dots, M$, of the surface that they cover. The union of these electrodes is denoted by E^h. All electrodes may be used both for current injection and voltage measurement, and the corresponding electrode net currents and voltages are denoted by $\{I_m\}, \{U_m^h\} \subset \mathbb{C}$, respectively. Due to the principle of charge conservation, the total current vector $I = [I_m]_{m=1}^M$ belongs to the space

$$\mathbb{C}_\diamond^M := \Big\{ Z = [Z_m]_{m=1}^M \in \mathbb{C}^M \;\Big|\; \sum_{j=1}^M Z_j = 0 \Big\}.$$

During electrode measurements, a thin and highly resistive layer is formed at the electrode-object interface [6]. It is characterized by the contact impedances $\{z_m\}$ that in our analysis are assumed to be complex numbers with positive real parts.

The corresponding forward problem is as follows: Given a current pattern $I \in \mathbb{C}_\diamond^M$, find $(u^h, U^h) \in (H^1(\Omega) \oplus \mathbb{C}^M)/\mathbb{C} =: \mathcal{H}$ that satisfies

$$\begin{aligned}
&\nabla \cdot \sigma \nabla u^h = 0 && \text{in } \Omega, \\
&\nu \cdot \sigma \nabla u^h = 0 && \text{on } \partial\Omega \setminus \overline{E^h}, \\
&u^h + z_m\, \nu \cdot \sigma \nabla u^h = U_m^h && \text{on } e_m^h, \quad m = 1, \dots, M, \\
&\int_{e_m^h} \nu \cdot \sigma \nabla u^h \, \mathrm{d}S = I_m, && m = 1, \dots, M,
\end{aligned} \tag{3}$$

in an appropriate weak sense (cf. [20]). Note that in the factor space $\mathcal{H}$ the quotient is taken with respect to constant shifts of both, u^h and U^h, simultaneously. This reflects the freedom in the choice of the ground level of potential. By slight abuse of notation, we will subsequently identify complex numbers with the corresponding constant functions (over an appropriate domain), and refer to equivalence classes of factor spaces with respect to $\mathbb{C}$ to identify elements (be it functions, numbers, or tuples of both of them) that only differ by additive shifts. Unless there is a possibility of confusion, we also do not distinguish between equivalence classes and representative elements of them.

With this understanding the equations in (3) uniquely determine the electromagnetic potential u^h *within* Ω, and the potentials $\{U_m^h\}$ *on* the electrodes, and there holds

$$\begin{aligned}
\|(u^h, U^h)\|_{\mathcal{H}}^2 &= \inf_{c \in \mathbb{C}} \Big(\|u^h - c\|_{H^1(\Omega)}^2 + \sum_{m=1}^M \|U_m^h - c\|_{L^2(e_m^h)}^2 \Big) \\
&\leq C \sum_{m=1}^M |I_m|^2 / |e_m^h|.
\end{aligned} \tag{4}$$

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Furthermore, we have a similar inequality for the flux of the component u^h of the solution across the boundary, i.e.,

$$\|\nu \cdot \sigma \nabla u^h\|^2_{L^2(\partial\Omega)} \le C \sum_{m=1}^{M} |I_m|^2/|e^h_m|. \tag{5}$$

Both in (4) and (5), the constant $C = C(\Omega, \sigma, \{z_m\}) > 0$ is independent of the electrode configuration. We refer to the material in [20], [12, Theorem 2.3], and [13, Lemma 2.1] for a proof of these results.

Real-life electrode measurements of impedance tomography provide a noisy version of the current-to-voltage map

$$R^h : I \mapsto U^h, \quad \mathbb{C}^M_\diamond \to \mathbb{C}^M/\mathbb{C}. \tag{6}$$

Accordingly, we denote by $(u^h_0, U^h_0) \in \mathcal{H}$ the reference potential for the background admittance σ_0, i.e., the solution of (3) with σ replaced by σ_0. $R^h_0 : I \mapsto U^h_0$ is the corresponding reference measurement operator.

2.2. **Point electrode model.** Alternatively, we consider a very simplistic electrode model with electrodes of infinitesimal size at the points $x_m \in \partial\Omega$, $m = 1, \dots, M$, where boundary currents are treated as isolated delta distributions. In other words, the corresponding forward problem reads

$$\nabla \cdot \sigma \nabla u = 0 \quad \text{in } \Omega, \qquad \nu \cdot \sigma \nabla u = f \quad \text{on } \partial\Omega, \tag{7}$$

where

$$f = \sum_{m=1}^{M} I_m \, \delta_{x_m} \in H^{(1-n)/2-\varepsilon}(\partial\Omega), \qquad \text{for any } \varepsilon > 0, \tag{8}$$

with $I = [I_m]_{m=1}^M \in \mathbb{C}^M_\diamond$ being the same as in Subsection 2.1, and δ_{x_m} being the Dirac delta distribution on $\partial\Omega$ supported in x_m. It follows from the standard theory of elliptic boundary value problems that (7)–(8) has a unique solution $u \in H^{(4-n)/2-\varepsilon}(\Omega)/\mathbb{C}$ satisfying

$$\|u\|_{H^{(4-n)/2-\varepsilon}(\Omega)/\mathbb{C}} \le C\|f\|_{H^{(1-n)/2-\varepsilon}(\partial\Omega)} \le C\|I\|_{\mathbb{C}^M}, \tag{9}$$

for any $\varepsilon > 0$ and some $C = C_\varepsilon > 0$; see, e.g., [16] and [9, (A.5)].

Since the Dirichlet boundary value of u is (only) in $H^{(3-n)/2-\varepsilon}(\partial\Omega)/\mathbb{C}$ (cf. the trace theorems in [16, Chapter 2]), the boundary potential is not well defined at the discrete point x_m — unless I_m equals zero —, and thus there is no natural way of defining counterparts of the voltages U^h and the measurement operator R^h of the complete electrode model within this point electrode setting. However, there does exist a counterpart for difference measurements, i.e., for the relative voltages $U^h - U^h_0$, and the relative current-to-voltage map $R^h - R^h_0$. To this end, consider the reference potential $u_0 \in H^{(4-n)/2-\varepsilon}(\Omega)/\mathbb{C}$ that solves (7)–(8) for the background admittance σ_0 and set $w := u - u_0$. Then the vector of point evaluations $W := \big[w(x_m)\big]_{m=1}^M \in \mathbb{C}^M/\mathbb{C}$ is well-defined; see Lemma 2.1 below. In our main result we prove that W provides an approximation of the corresponding relative voltages $U^h - U^h_0$ of the complete electrode model, if the diameter of the finite size electrodes is small. An immediate corollary is that the measurement operator

$$A : I \mapsto W, \quad \mathbb{C}^M_\diamond \to \mathbb{C}^M/\mathbb{C}, \tag{10}$$

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approximates the corresponding relative measurement map $R^h - R_0^h$ of the complete electrode model.

Lemma 2.1. *The relative potential $w = u - u_0$ satisfies the estimate*

$$\|w\|_{H^r(\partial\Omega)/\mathbb{C}} \leq C\|I\|_{\mathbb{C}^M}$$

for any $r \in \mathbb{R}$ and $C = C(r) > 0$. In particular, $w|_{\partial\Omega}$ belongs to $C^\infty(\partial\Omega)/\mathbb{C}$.

Proof. Let us fix $\varepsilon > 0$ and $r \geq 3/2$; obviously, the latter choice can be made without loss of generality. Let Ω_0 and Ω_1 be auxiliary C^∞-domains with connected complements, such that $\operatorname{supp}\kappa \subset \Omega_0$, $\overline{\Omega}_0 \subset \Omega_1$ and $\overline{\Omega}_1 \subset \Omega$. In addition, let D be a smooth neighborhood of $\partial\Omega_1$ with the property $\overline{D} \subset \Omega \setminus \overline{\Omega}_0$. Since $\nabla \cdot \sigma_0 \nabla w = 0$ in $\Omega \setminus \overline{\Omega}_0$, it follows from (9) and a slight modification of [9, Lemma A.1], i.e., from interior regularity for elliptic equations, that

$$\|w\|_{H^{r+1/2}(D)/\mathbb{C}} \leq C\|w\|_{H^{(4-n)/2-\varepsilon}(\Omega\setminus\overline{\Omega}_0)/\mathbb{C}} \leq C\|I\|_{\mathbb{C}^M}.$$

In particular, using the trace theorem, we see that w satisfies

$$\nabla \cdot \sigma_0 \nabla w = 0 \quad \text{in } \Omega \setminus \overline{\Omega}_1, \qquad \nu \cdot \sigma_0 \nabla w = 0 \quad \text{on } \partial\Omega, \qquad \nu \cdot \sigma_0 \nabla w = g \quad \text{on } \partial\Omega_1$$

for some mean-free g with $\|g\|_{H^{r-1}(\partial\Omega_1)} \leq C\|I\|_{\mathbb{C}^M}$. Hence, the claim follows from the combination of [16, Chapter 2, Remark 7.2] and the trace theorem. □

2.3. **Geometrical assumptions.** We now list our assumptions on the interplay between the two electrode models introduced in Subsections 2.1 and 2.2 above. As before, we denote by e_m^h, $m = 1, \ldots, M$, the finite size electrodes, and by x_m the positions of the corresponding point electrodes. As already mentioned, we associate with h the size of the electrodes from the complete electrode model, and we let h float within some interval $0 < h < h_0$, where $h_0 > 0$ is kept fixed.

To be precise, we assume throughout that there is a fixed convex reference domain $Q \subset \mathbb{R}^{n-1}$ with $|Q| = 1$ and $0 \in Q$, such that, for each positive parameter $h < h_0$, the electrode $e_m^h \subset \partial\Omega$, with m fixed, is given by a one-to-one parameterization

$$e_m^h = X_m^h(Q^h) \qquad \text{with} \qquad Q^h = hQ\,,$$

which stands for

$$e_m^h \;=\; \bigl\{x = X_m^h(s) \bigm| s \in Q^h\bigr\}\,.$$

We assume that X_m^h is a diffeomorphism between Q^h and e_m^h, i.e., both X_m^h and its inverse are infinitely times continuously differentiable, and that there are universal constants $0 < c < C < \infty$, independent of h and m, such that the surface element $\mathrm{d}S$ on e_m^h satisfies

$$\mathrm{d}S \;=\; \sigma_m^h(s)\,\mathrm{d}s \qquad \text{with} \qquad c \leq \sigma_m^h(s) \leq C\,, \tag{11}$$

where $\mathrm{d}s$ is the Lebesgue's volume element of Q^h. To be precise, by the first part of (11) we mean that

$$\int_{e_m^h} g\,\mathrm{d}S \;=\; \int_{Q^h} (g \circ X_m^h)\,\sigma_m^h\,\mathrm{d}s$$

for any integrable function g on e_m^h, i.e., σ_m^h is the local stretching factor corresponding to the parameterization X_m^h. In addition, we need some extra control over the first and second order derivatives of X_m^h, namely we require that

$$\|X_m^h\|_{C^2(\overline{Q}_h)} \leq C\,. \tag{12}$$

It is easy to see that under these assumptions there holds

$$\|\psi \circ X_m^h\|_{H^1(Q^h)} \;\leq\; C\|\psi\|_{H^1(e_m^h)} \tag{13}$$

for every $\psi \in H^1(e_m^h)$, and

$$|\Gamma| \;\leq\; C|(X_m^h)^{-1}(\Gamma)| \tag{14}$$

for any smooth curve $\Gamma \subset e_m^h$. (In fact, for (13) and (14) to hold the assumption on the second order derivatives of X_m^h is redundant; however, such an assumption is needed in the proof of Lemma 3.2 below.) We point out that because of the above stipulations the area covered by e_m^h is given by

$$|e_m^h| \;=\; \int_{e_m^h} \mathrm{d}S \;=\; \int_{Q^h} \sigma_m^h(s)\,\mathrm{d}s \;=\; h^{n-1}\int_Q \sigma_m^h(hs)\,\mathrm{d}s \begin{cases} \leq Ch^{n-1} \\ \geq ch^{n-1}\,. \end{cases} \tag{15}$$

Finally, concerning the interplay between the two electrode models, we assume that $x_m \in e_m^h$, more precisely, that

$$x_m \;=\; X_m^h(0)\,,$$

and, to enable an $O(h^2)$ approximation property to be established below we require that

$$\int_{Q^h} s\,\sigma_m^h(s)\,\mathrm{d}s \;=\; 0\,, \tag{16}$$

i.e., that the origin (the preimage of x_m under X_m^h) is a (weighted) center of mass of Q^h.

Some interpretation of the above assumptions may be useful:

- For $n = 2$, i.e., in two space dimensions, the boundary of Ω is a closed curve, and it is most natural to assume that e_m^h are electrodes of length $|e_m^h| = h$, say. In this case one can choose $Q = [-1/2, 1/2]$, $Q^h = [-h/2, h/2]$, and let X_m^h be, for all $h > 0$, an arc length parameterization of the boundary. In this case $\sigma_m^h \equiv 1$, and the condition (16) is equivalent to saying that the position $x_m = X_m^h(0)$ of the point electrode is half way (along the boundary of Ω) between the two end points of the electrode e_m^h.
- In three space dimensions, one can think of Q^h being a planar reference shape for each of the electrodes, that is shrinking with decreasing size parameter h. Think of these electrodes as being elastic, so that they can be attached to the surface $\partial\Omega$ around x_m. The corresponding deformation is determined by X_m^h, with

$$\sigma_m^h(s) \;=\; \left|\frac{\partial X_m^h}{\partial s_1}(s) \times \frac{\partial X_m^h}{\partial s_2}(s)\right|, \qquad s = (s_1, s_2)\,,$$

being the local stretching factor. In order to satisfy condition (16), one has to make sure that the position of the point electrode is some kind of center of e_m^h (which is not the center of mass, though, as the latter does not usually sit on $\partial\Omega$).

2.4. **The main result.** We are now ready to formulate the main result of this paper. Recall that, given a current pattern I, we denote by U^h and U_0^h the voltages on the finite size electrodes corresponding to the admittance σ of (1) and to the background admittance σ_0, respectively. Similarly, in the framework of point electrodes, u and u_0 stand for the respective potentials of Subsection 2.2, and $W := \big[(u-u_0)(x_m)\big]_{m=1}^M \in \mathbb{C}^M/\mathbb{C}$ contains the relative voltages on the point electrodes.

Theorem 2.2. *Under the assumptions from Subsection 2.3,*

$$\big\|(U^h - U_0^h) - W\big\|_{\mathbb{C}^M/\mathbb{C}} \le Ch^2 \|I\|_{\mathbb{C}^M},$$

where $C>0$ is independent of $h \in (0,h_0)$ and $I \in \mathbb{C}_\diamond^M$.

For the corresponding measurement maps of the complete electrode model and the point electrode model,

$$R^h : I \mapsto U^h, \quad R_0^h : I \mapsto U_0^h, \quad \text{and} \quad A : I \mapsto W,$$

we deduce that A is an accurate approximation of $R^h - R_0^h$, provided that the parameter h, which measures the diameter of the electrodes, is relatively small.

Corollary 2.3. *Under the assumptions from Subsection 2.3,*

$$\big\|(R^h - R_0^h) - A\big\|_{\mathcal{L}(\mathbb{C}_\diamond^M, \mathbb{C}^M/\mathbb{C})} \le Ch^2,$$

where $C>0$ is independent of $h \in (0,h_0)$.

Remark 2.4. Suppose that the point electrode locations $\{x_m\}$ are not the centers of the corresponding finite size electrodes $\{e_m^h\}$ in the sense of (16). In such a case Theorem 2.2 and Corollary 2.3 are no longer valid. However, as long as only $x_m \in e_m^h$, $m=1,\ldots,M$, one can still obtain the weaker convergence rate

$$\big\|(R^h - R_0^h) - A\big\|_{\mathcal{L}(\mathbb{C}_\diamond^M, \mathbb{C}^M/\mathbb{C})} \le Ch. \tag{17}$$

In fact, the proof of Lemma 3.2 below could be shortened considerably, and also the geometrical assumptions about the finite size electrodes could be weakened, if the aim was only to prove an $O(h)$-estimate.

Furthermore, it is easy to see from the proof of Theorem 2.2 that its assertion remains valid, if one or all x_m deviate from the center of the respective electrode(s) by $O(h^2)$.

3. Proof of the Main Result

This section is devoted to proving Theorem 2.2. For a current pattern $I \in \mathbb{C}_\diamond^M$, let (u^h, U^h) and (u_0^h, U_0^h) be the solution pairs of (3) corresponding to the admittance σ from (1) and the background admittance σ_0, respectively, and set $(w^h, W^h) := (u^h - u_0^h, U^h - U_0^h)$.

We begin with a refinement of the inequality (5) for the complete electrode model, which – in contrast to (5) – is only valid on the electrodes.

Lemma 3.1. *The component u^h of the solution to (3) satisfies*

$$\|\nu \cdot \sigma \nabla u^h\|_{H^1(E^h)}^2 \le C \sum_{m=1}^M |I_m|^2/|e_m^h|,$$

where $C>0$ is independent of the electrode configuration, and of h, in particular.

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Proof. Because σ is smooth in a neighborhood of $\partial\Omega$, the Neumann-to-Dirichlet map corresponding to the first equation of (3) is bounded from the subspace of $L^2(\partial\Omega)$-functions with zero integral mean to $H^1(\partial\Omega)/\mathbb{C}$ (cf., e.g., [9, Theorem A.3]), and thus (5) gives

$$\|u^h\|^2_{H^1(\partial\Omega)/\mathbb{C}} \le C \sum_{m=1}^M |I_m|^2/|e_m^h|.$$

Hence, it follows from the third equation of (3) that (cf. [7, p. 146, Remark 6])

$$\begin{aligned}
\|\nu\cdot\sigma\nabla u^h\|^2_{H^1(E^h)} &\le C\sum_{m=1}^M \|U_m^h - u^h\|^2_{H^1(e_m^h)} \\
&\le C\sum_{m=1}^M \left(\|u^h\|^2_{H^1(e_m^h)/\mathbb{C}} + \|U_m^h - u^h\|^2_{L^2(e_m^h)}\right) \\
&\le C\left(\sum_{m=1}^M |I_m|^2/|e_m^h| + \sum_{m=1}^M \|U_m^h - u^h\|^2_{L^2(e_m^h)}\right).
\end{aligned}$$

For the second term on the right hand side we have

$$\begin{aligned}
\sum_{m=1}^M \|U_m^h - u^h\|^2_{L^2(e_m^h)} &\le C\sum_{m=1}^M \left(\|U_m^h - c\|^2_{L^2(e_m^h)} + \|c - u^h\|^2_{L^2(e_m^h)}\right) \\
&\le C\left(\|u^h - c\|^2_{H^1(\Omega)} + \sum_{m=1}^M \|U_m^h - c\|^2_{L^2(e_m^h)}\right),
\end{aligned}$$

where we have applied the trace theorem. By taking the infimum over $c\in\mathbb{C}$, and using (4) it thus follows that

$$\sum_{m=1}^M \|U_m^h - u^h\|^2_{L^2(e_m^h)} \le C\sum_{m=1}^M |I_m|^2/|e_m^h|,$$

which completes the proof. □

Next we provide a first result (in a comparatively weak norm) on how well u^h approximates the potential u of Subsection 2.2 for point electrodes on $\partial\Omega$.

Lemma 3.2. *Under the assumptions from Subsection 2.3 we can find for every $\epsilon > 0$ some $C_\varepsilon > 0$ such that*

$$\|\nu\cdot\sigma\nabla(u^h - u)\|_{H^{-(n+3)/2-\varepsilon}(\partial\Omega)} \le C_\varepsilon h^2 \|I\|_{\mathbb{C}^M}$$

for every $0 < h < h_0$ and all $I \in \mathbb{C}^M_\diamond$.

Proof. 1. Let $\varphi \in C^\infty(\partial\Omega)$ be fixed, and denote $\nu\cdot\sigma\nabla u^h|_{\partial\Omega}$ by f^h. Note that it follows from Lemma 3.1 and (15) that

$$\|f^h\|^2_{H^1(e_m^h)} \le Ch^{1-n}\|I\|^2_{\mathbb{C}^M}. \tag{18}$$

According to the boundary conditions of (3) and (7), we can therefore rewrite

$$\begin{aligned}
\left|\langle \nu\cdot\sigma\nabla(u^h-u),\varphi\rangle\right| &= \left|\int_{\partial\Omega} f^h\varphi\,\mathrm{d}S - \sum_{m=1}^{M} I_m\varphi(x_m)\right| \\
&= \left|\sum_{m=1}^{M}\int_{e_m^h} f^h\left(\varphi-\varphi(x_m)\right)\mathrm{d}S\right| \\
&\le \sum_{m=1}^{M}\left|\int_{e_m^h}(f^h-I_m/|e_m^h|)\big(\varphi-\varphi(x_m)\big)\,\mathrm{d}S + \frac{I_m}{|e_m^h|}\int_{e_m^h}\big(\varphi-\varphi(x_m)\big)\,\mathrm{d}S\right|,
\end{aligned}$$

and hence,

$$\begin{aligned}
\left|\langle \nu\cdot\sigma\nabla(u^h-u),\varphi\rangle\right| \le\; &\sum_{m=1}^{M}\|f^h-I_m/|e_m^h|\|_{L^2(e_m^h)}\|\varphi-\varphi(x_m)\|_{L^2(e_m^h)} \\
&+\sum_{m=1}^{M}\frac{|I_m|}{|e_m^h|}\left|\int_{e_m^h}\varphi\,\mathrm{d}S-\varphi(x_m)|e_m^h|\right|.
\end{aligned} \tag{19}$$

The terms that enter on the right-hand side of (19) will now be treated separately for any fixed $m\in\{1,\dots,M\}$.

2. To begin with, we remark that, according to (3), $\psi = f^h - I_m/|e_m^h|$ has vanishing integral mean over e_m^h, i.e.,

$$0 = \int_{e_m^h}\psi(x)\,\mathrm{d}S = \int_{Q^h}\psi(X_m^h(s))\sigma_m^h(s)\,\mathrm{d}s = h^{n-1}\int_Q \psi(X_m^h(hs))\sigma_m^h(hs)\,\mathrm{d}s\,.$$

Therefore the Poincaré-Friedrichs inequality for the domain Q yields

$$\begin{aligned}
\int_Q\left|\psi(X_m^h(hs))\sigma_m^h(hs)\right|^2\mathrm{d}s &\le C\int_Q\left|\nabla_s\Big(\psi(X_m^h(hs))\sigma_m^h(hs)\Big)\right|^2\mathrm{d}s \\
&\le Ch^2h^{1-n}\int_{Q^h}\left|\nabla_s\Big(\psi(X_m^h(s))\sigma_m^h(s)\Big)\right|^2\mathrm{d}s\,,
\end{aligned}$$

and hence, as σ_m^h is bounded by c from below,

$$\begin{aligned}
\|f^h-I_m/|e_m^h|\|_{L^2(e_m^h)}^2 = \|\psi\|_{L^2(e_m^h)}^2 &\le \frac{1}{c}\int_{Q^h}\left|\psi(X_m^h(s))\sigma_m^h(s)\right|^2\mathrm{d}s \\
&= \frac{1}{c}\,h^{n-1}\int_Q\left|\psi(X_m^h(sh))\sigma_m^h(sh)\right|^2\mathrm{d}s \\
&\le Ch^2\int_{Q^h}\left|\nabla_s\Big(\psi(X_m^h(s))\sigma_m^h(s)\Big)\right|^2\mathrm{d}s\,.
\end{aligned}$$

Due to (12), σ_m^h is uniformly bounded in $C^1(\overline{Q}_h)$ with respect to h, and we can continue by using (13) to obtain

$$\begin{aligned}
\|f^h-I_m/|e_m^h|\|_{L^2(e_m^h)}^2 &\le Ch^2\|\psi\circ X_m^h\|_{H^1(Q^h)}^2 \le Ch^2\|\psi\|_{H^1(e_m^h)}^2 \\
&\le Ch^2\Big(\|f^h-I_m/|e_m^h|\|_{L^2(e_m^h)}^2+\|f^h\|_{H^1(e_m^h)}^2\Big),
\end{aligned}$$

and hence, assuming that $Ch^2 \leq 1/2$ and using (18), we conclude that

$$\|f^h - I_m/|e_m^h|\|_{L^2(e_m^h)} \leq Ch\|f^h\|_{H^1(e_m^h)} \leq Ch^{3/2-n/2}\|I\|_{\mathbb{C}^M} . \tag{20}$$

3. Due to (14), for any $x \in e_m^h$ there exists a smooth curve $\Gamma \subset e_m^h$ connecting x and x_m such that

$$|\Gamma| \leq Ch.$$

Indeed, one can construct such a Γ by taking the line segment between the points $(X_m^h)^{-1}(x)$ and $(X_m^h)^{-1}(x_m)$ in Q^h, and mapping it back onto e_m^h with X_m^h. As a consequence,

$$\big|\varphi(x) - \varphi(x_m)\big| \leq Ch\|\varphi\|_{C^1(\partial\Omega)} ,$$

and hence, by virtue of (15) there holds

$$\|\varphi - \varphi(x_m)\|^2_{L^2(e_m^h)} \leq Ch^2|e_m^h|\|\varphi\|^2_{C^1(\partial\Omega)} \leq Ch^{n+1}\|\varphi\|^2_{C^1(\partial\Omega)} . \tag{21}$$

4. According to (16), the quadrature formula

$$\int_{e_m^h} \varphi \, \mathrm{d}S = \int_{Q^h} (\varphi \circ X_m^h)\, \sigma_m^h \, \mathrm{d}s \approx \varphi(x_m)|e_m^h| \tag{22}$$

is exact whenever $\varphi \circ X_m^h$ is a polynomial of degree less or equal to one. Because Q^h is convex, we can expand

$$(\varphi \circ X_m^h)(s) = \varphi(x_m) + s \cdot \nabla_s(\varphi \circ X_m^h)(0) + r(s) ,$$

where

$$|r(s)| \leq C|s|^2\|\varphi\|_{C^2(\partial\Omega)}$$

because of (12). Hence,

$$\left|\int_{e_m^h} \varphi \, \mathrm{d}S - \varphi(x_m)|e_m^h|\right| = \left|\int_{Q^h} r(s)\sigma_m^h(s)\, \mathrm{d}s\right| \leq Ch^{n+1}\|\varphi\|_{C^2(\partial\Omega)} . \tag{23}$$

5. Inserting the three estimates (20), (21), and (23), together with (15) into (19), we finally arrive at

$$\big|\langle \nu \cdot \sigma\nabla(u^h - u), \varphi\rangle\big| \leq Ch^2\|I\|_{\mathbb{C}^M}\|\varphi\|_{C^2(\partial\Omega)} .$$

Now, if $\varepsilon > 0$ then $H^{(n+3)/2+\varepsilon}(\partial\Omega)$ is continuously embedded in $C^2(\partial\Omega)$ according to the Sobolev embedding theorem, cf. Hebey [10], and hence, there is $C_\varepsilon > 0$ such that

$$\big|\langle \nu \cdot \sigma\nabla(u^h - u), \varphi\rangle\big| \leq C_\varepsilon h^2\|I\|_{\mathbb{C}^M}\|\varphi\|_{H^{(n+3)/2+\varepsilon}(\partial\Omega)} .$$

Because $C^\infty(\partial\Omega)$ is dense in $H^{(n+3)/2+\varepsilon}(\partial\Omega)$ (see, e.g., [16, Chapter 1, Section 7.3]), we deduce that

$$\|\nu \cdot \sigma\nabla(u^h - u)\|_{H^{-(n+3)/2-\varepsilon}(\partial\Omega)} \leq C_\varepsilon h^2\|I\|_{\mathbb{C}^M} ,$$

which completes the proof. $\square$

Due to the regularity properties of elliptic partial differential equations, the approximation of Lemma 3.2 gets stronger if one concentrates on the behavior of the corresponding potentials at some distance from the boundary $\partial\Omega$. This statement is made concrete by the following corollary.

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Corollary 3.3. *Let $\Omega_0 \subset \mathbb{R}^n$ be a nonempty domain such that $\overline{\Omega}_0 \subset \Omega$. Then, there holds that*

$$\|u^h - u\|_{H^1(\Omega_0)/\mathbb{C}} \leq Ch^2 \|I\|_{\mathbb{C}^M},$$

and

$$\|u^h\|_{H^1(\Omega_0)/\mathbb{C}} + \|u\|_{H^1(\Omega_0)/\mathbb{C}} \leq C\|I\|_{\mathbb{C}^M},$$

where $C = C(\Omega_0) > 0$ is independent of h.

Proof. Since $u^h - u$ satisfies the conductivity equation for the admittance σ of (1), and since σ is smooth in some neighborhood of $\partial\Omega$, it follows from Lemma 3.2 and the continuous dependence on the boundary data for the Neumann problem (cf. [9, (A.5)]) that

$$\|u^h - u\|_{H^{-n/2-\varepsilon}(\Omega)/\mathbb{C}} \leq C_\varepsilon h^2 \|I\|_{\mathbb{C}^M} \tag{24}$$

for some $C_\varepsilon > 0$.

Using a similar interior regularity argument as in the proof of Lemma 2.1, we see that $u^h - u$ satisfies the Neumann problem

$$\nabla \cdot \sigma \nabla (u^h - u) = 0 \quad \text{in } \Omega_0, \qquad \nu \cdot \sigma \nabla (u^h - u) = g \quad \text{on } \partial\Omega_0 \tag{25}$$

for any smooth domain Ω_0 with connected complement such that $\operatorname{supp} \kappa \subset \Omega_0$ and $\overline{\Omega}_0 \subset \Omega$, and for some mean-free g with $\|g\|_{L^2(\partial\Omega_0)} \leq C(\Omega_0) h^2 \|I\|_{\mathbb{C}^M}$. Notice that a more general Ω_0 can be enclosed by a domain with these properties. Hence, the claim about $u^h - u$ follows from the continuous dependence on the Neumann data in (25); see, e.g, the variational techniques in [7].

The estimate for u is obtained in the exactly same manner, with (9) playing the role of (24). Finally, the claim about u^h follows from the triangle inequality. □

We can now deduce Theorem 2.2 from Corollary 3.3 by a duality argument.

Proof of Theorem 2.2. Let $I \in \mathbb{C}^M_\diamond$ be arbitrary and choose an auxiliary domain Ω_0 such that $\operatorname{supp} \kappa \subset \Omega_0$ and $\overline{\Omega}_0 \subset \Omega$. We fix the ground level of potential, i.e., choose a representative of a quotient equivalence class, so that

$$J := \overline{W^h - W} = \overline{((R^h - R^h_0) - A)I} \in \mathbb{C}^M_\diamond, \tag{26}$$

where $W_m = w(x_m)$ is defined as in (10). We denote by $(v^h, V^h) \in \mathcal{H}$ the solution of the complete electrode problem (3) for this newly defined electrode current pattern $J = [J_m]_{m=1}^M$. The variational formulation for this problem in [20, Proposition 3.1] gives

$$\begin{aligned} \sum_{m=1}^M J_m W^h_m &= \int_\Omega \sigma \nabla v^h \cdot \nabla w^h \,\mathrm{d}x + \sum_{m=1}^M \frac{1}{z_m} \int_{e^h_m} (v^h - V^h_m)(w^h - W^h_m)\,\mathrm{d}S \\ &= -\int_{\Omega_0} \kappa \nabla u^h_0 \cdot \nabla v^h \,\mathrm{d}x, \end{aligned} \tag{27}$$

where the second step is a consequence of the very same variational formulation for the pairs (u^h, U^h) and (u^h_0, U^h_0), respectively, together with the definition of (w^h, W^h).

Similarly, let $v \in H^{(4-n)/2-\varepsilon}(\Omega)/\mathbb{C}$ solve the forward problem

$$\nabla \cdot \sigma \nabla v = 0 \quad \text{in } \Omega, \qquad \nu \cdot \sigma \nabla v = g \quad \text{on } \partial\Omega,$$

with the point current pattern

$$g = \sum_{m=1}^{M} J_m \, \delta_{x_m}.$$

We introduce a mean-free sequence $(g_k) \subset C^\infty(\partial\Omega)$ that converges towards g in the topology of $H^{(1-n)/2-\varepsilon}(\partial\Omega)$ (cf., e.g., [16, Chapter 1, Section 7.3]). As in the proof of Corollary 3.3, it follows from interior regularity arguments that the solutions $(v_k) \subset H^1(\Omega)/\mathbb{C}$ of

$$\nabla \cdot \sigma \nabla v_k = 0 \quad \text{in } \Omega, \qquad \nu \cdot \sigma \nabla v_k = g_k \quad \text{on } \partial\Omega,$$

fulfill

$$\lim_{k\to\infty} \|v_k - v\|_{H^1(\Omega_0)/\mathbb{C}} = 0.$$

Since

$$\int_{\partial\Omega} g_k w \,\mathrm{d}S = \int_\Omega \sigma \nabla v_k \cdot \nabla w \,\mathrm{d}x = -\int_{\Omega_0} \kappa \nabla u_0 \cdot \nabla v_k \,\mathrm{d}x,$$

and since w is smooth on $\partial\Omega$ (cf. Lemma 2.1), we obtain that

$$\sum_{m=1}^{M} J_m W_m = \lim_{k\to\infty} \int_{\partial\Omega} g_k w \,\mathrm{d}S = -\int_{\Omega_0} \kappa \nabla u_0 \cdot \nabla v \,\mathrm{d}x. \tag{28}$$

Combining (26), (27) and (28), we deduce that

$$\begin{aligned} \|W^h - W\|_{\mathbb{C}^M}^2 &= \int_{\Omega_0} (\kappa \nabla u_0 \cdot \nabla v - \kappa \nabla u_0^h \cdot \nabla v^h) \,\mathrm{d}x \\ &\le C\Big(\|u_0 - u_0^h\|_{H^1(\Omega_0)/\mathbb{C}} \|v\|_{H^1(\Omega_0)/\mathbb{C}} \\ &\qquad + \|u_0^h\|_{H^1(\Omega_0)/\mathbb{C}} \|v - v^h\|_{H^1(\Omega_0)/\mathbb{C}} \Big) \\ &\le Ch^2 \|W^h - W\|_{\mathbb{C}^M} \|I\|_{\mathbb{C}^M}, \end{aligned}$$

where the last step follows by applying Corollary 3.3 to each of the four distributions $u_0 - u_0^h$, v, u_0^h and $v - v^h$. In consequence, division by $\|W^h - W\|_{\mathbb{C}^M}$ completes the proof. □

Concerning Remark 2.4 we note that the special definition (16) of x_m as the weighted center of mass of e_m^h is only used in the fourth part of the proof of Lemma 3.2: Assuming merely that x_m belongs to e_m^h reduces the accuracy of the quadrature formula (22), so that it is exact only for constants. Accordingly, this decreases the exponent of h on the right-hand side of (23) by one, which carries over to (19) and, eventually, to the conclusion of Lemma 3.2. This first order convergence rate in h then transports trivially to Corollary 3.3 and Theorem 2.2.

4. Electrode Dipoles and Backscatter Data

In this section we restrict our attention to two space dimensions, i.e., $n = 2$, and to the case where there are only two small electrodes attached close to each other on $\partial\Omega$. For any fixed $y \in \partial\Omega$ we let $X \in C^\infty(\mathbb{R}; \mathbb{R}^2)$ be a counterclockwise $|\partial\Omega|$-periodic parameterization of $\partial\Omega$ with respect to arc length, such that $X(0) = y$, and

$$\partial\Omega = \big\{ X(s) \;\big|\; -|\partial\Omega|/2 \le s < |\partial\Omega|/2 \big\}.$$

Then we define a pair of electrodes centered around y via

$$e_+^h = \{X(s) \mid h/2 < s < 3h/2\},$$

and

$$e_-^h = \{X(s) \mid X(-s) \in e_+^h\},$$

where $h > 0$ is the length of the electrodes. The idea is to drive $1/(2h)$ units of current from e_+^h to e_-^h and measure the resulting potential difference.

In this setting, the forward solution of this problem is the unique pair $(u^h, U^h) \in H^1(\Omega) \oplus \mathbb{C}$ that satisfies weakly

$$\begin{aligned} &\nabla \cdot \sigma \nabla u^h = 0 && \text{in } \Omega, \\ &\nu \cdot \sigma \nabla u^h = 0 && \text{on } \partial\Omega \setminus (\overline{e}_+^h \cup \overline{e}_-^h), \\ &u^h + z_\pm \nu \cdot \sigma \nabla u^h = \pm U^h && \text{on } e_\pm^h, \\ &\int_{e_\pm^h} \nu \cdot \sigma \nabla u^h \,\mathrm{d}S = \pm 1/(2h), \end{aligned} \tag{29}$$

where we have fixed the ground level of potential in an obvious way. Let $(u_0^h, U_0^h) \in H^1(\Omega) \oplus \mathbb{C}$ be the solution of (29) when σ of (1) is replaced by the smooth background admittance σ_0. As in Section 3 we set $(w^h, W^h) = (u^h - u_0^h, U^h - U_0^h)$, and define

$$b^h = W^h/h.$$

Our goal is to prove that b^h can be approximated by the corresponding *backscatter data* introduced in [8, 9][1]. Such data are defined via the following variant of the point electrode forward problem introduced in Section 2.2:

$$\nabla \cdot \sigma \nabla u = 0 \quad \text{in } \Omega, \qquad \nu \cdot \sigma \nabla u = -\delta_y' \quad \text{on } \partial\Omega, \tag{30}$$

where the (mean-free) dipole current $\delta_y' \in H^{-3/2-\varepsilon}(\partial\Omega)$, $\varepsilon > 0$, is defined by virtue of

$$\langle \delta_y', v \rangle = -\frac{\partial v(X(s))}{\partial s}\Big|_{s=0} \tag{31}$$

for $v \in H^{3/2+\varepsilon}(\partial\Omega)$. It follows, e.g., from the material in [9, Appendix] that (30) has a unique solution $u \in H^{-\varepsilon}(\Omega)/\mathbb{C}$ for any $\varepsilon > 0$ satisfying

$$\|u\|_{H^{-\varepsilon}(\Omega)/\mathbb{C}} \le C, \tag{32}$$

where $C = C(\Omega, \sigma, \epsilon) > 0$ is independent of y. We denote by u_0 the reference potential, i.e., the solution of (30) with σ replaced by σ_0, and set $w = u - u_0$. Then, the backscatter data of electrical impedance tomography at y is defined to be

$$b = -\langle \delta_y', w \rangle.$$

Take note that b is a well defined number because the dipole δ_y' does not see the ground level of potential and, furthermore,

$$\|w\|_{H^r(\partial\Omega)/\mathbb{C}} \le C, \tag{33}$$

for any $r \in \mathbb{R}$ and $C = C(r) > 0$ that can be chosen independently of $y \in \partial\Omega$. This estimate follows by repeating the argumentation of Lemma 2.1 and noticing

[1] We mention that it has been shown in [8] that these backscatter data (as a function of the point $y \in \partial\Omega$) uniquely define a simply connected insulating obstacle within Ω, if the background admittance σ_0 is constant.

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that the appearing constants can be chosen so that they depend on κ, σ_0 and the geometry of Ω, but not on y.

The main result of this section is as follows:

Theorem 4.1. *There holds*

$$|b^h - b| \leq Ch,$$

where $C > 0$ is independent of $h > 0$ and $y \in \partial\Omega$.

The rest of this section is devoted to proving Theorem 4.1. We start by presenting the counterpart of Corollary 3.3 in this new setting.

Lemma 4.2. *Let u^h and u be given by (29) and (30), respectively. Furthermore, let $\Omega_0 \subset \mathbb{R}^2$ be a nonempty domain such that $\overline{\Omega}_0 \subset \Omega$. Then there holds that*

$$\|u^h - u\|_{H^1(\Omega_0)/\mathbb{C}} \leq Ch,$$

and

$$\|u^h\|_{H^1(\Omega_0)/\mathbb{C}} + \|u\|_{H^1(\Omega_0)/\mathbb{C}} \leq C,$$

where $C = C(\Omega_0) > 0$ is independent of h and $y \in \partial\Omega$.

Proof. The leading idea of this proof is the same as in Lemma 3.2 and Corollary 3.3: We first show that $\nu \cdot \sigma\nabla u^h|_{\partial\Omega}$ provides an approximation of δ'_y in some weak Sobolev norm, after which the assertion follows by an interior regularity argument. It is straightforward to see that the constants in the estimates below can be chosen so that they depend on σ and the geometry of Ω, but not on $y \in \partial\Omega$.

A simple calculation utilizing the boundary conditions of (29) shows that

$$\begin{aligned}\int_{\partial\Omega} \nu \cdot \sigma\nabla u^h \varphi \,\mathrm{d}S = \frac{1}{2h^2}\int_{e^h_+} \varphi \,\mathrm{d}S + \int_{e^h_+} (\nu \cdot \sigma\nabla u^h - 1/(2h^2))(\varphi - \varphi(y))\,\mathrm{d}S \\ - \frac{1}{2h^2}\int_{e^h_-} \varphi \,\mathrm{d}S + \int_{e^h_-} (\nu \cdot \sigma\nabla u^h + 1/(2h^2))(\varphi - \varphi(y))\,\mathrm{d}S\end{aligned}$$

for $\varphi \in C^\infty(\partial\Omega)$. As $\pm 1/(2h^2)$ is the mean of $\nu \cdot \sigma\nabla u^h$ over $e^h_\pm$, the Poincaré inequality and Lemma 3.1 provide the estimate

$$\|\nu \cdot \sigma\nabla u^h \mp 1/(2h^2)\|_{L^2(e^h_\pm)} \leq Ch\|\nu \cdot \sigma\nabla u^h\|_{H^1(e^h_\pm)} \leq Ch^{-1/2}.$$

Furthermore, as in part 3 of Lemma 3.2, we have that for all $x \in e^h_\pm$,

$$|\varphi(x) - \varphi(y)| \leq Ch\|\varphi\|_{C^1(\partial\Omega)},$$

so that we get from the Sobolev embedding theorem

$$\|\varphi - \varphi(y)\|_{L^2(e^h_\pm)} \leq Ch^{3/2}\|\varphi\|_{C^1(\partial\Omega)} \leq Ch^{3/2}\|\varphi\|_{H^{3/2+\varepsilon}(\partial\Omega)}$$

for any $\varepsilon > 0$ and some $C = C_\varepsilon > 0$. Combining the above estimates, it follows from the Schwarz inequality that

$$\left|\int_{\partial\Omega} \nu \cdot \sigma\nabla u^h \varphi \,\mathrm{d}S - \frac{1}{2h^2}\int_{e^h_+} \varphi \,\mathrm{d}S + \frac{1}{2h^2}\int_{e^h_-} \varphi \,\mathrm{d}S\right| \leq Ch\|\varphi\|_{H^{3/2+\varepsilon}(\partial\Omega)},$$

and hence, the triangle inequality gives

$$\begin{aligned}|\langle \nu \cdot \sigma\nabla(u^h - u), \varphi\rangle| \leq \left|\frac{\partial\varphi(X(s))}{\partial s}\Big|_{s=0} - \frac{1}{2h^2}\int_{e^h_+} \varphi \,\mathrm{d}S + \frac{1}{2h^2}\int_{e^h_-} \varphi \,\mathrm{d}S\right| \\ + Ch\|\varphi\|_{H^{3/2+\varepsilon}(\partial\Omega)}.\end{aligned} \tag{34}$$

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Using Taylor's theorem around $s = 0$ together with the Sobolev embedding theorem, it is straightforward to deduce that (cf. [8, Appendix])

$$\left| \frac{\partial \varphi(X(s))}{\partial s}\Big|_{s=0} - \frac{1}{2h^2}\int_{e_+^h} \varphi \, \mathrm{d}S + \frac{1}{2h^2}\int_{e_-^h} \varphi \, \mathrm{d}S \right| \leq Ch \|\varphi\|_{H^{5/2+\varepsilon}(\partial\Omega)}.$$

Hence, as $C^\infty(\partial\Omega)$ is dense in $H^{5/2+\varepsilon}(\partial\Omega)$, the estimate

$$\|\nu \cdot \sigma\nabla(u^h - u)\|_{H^{-5/2-\varepsilon}(\partial\Omega)} \leq Ch, \qquad C = C(\varepsilon) > 0, \tag{35}$$

follows by taking the supremum over φ with $\|\varphi\|_{H^{5/2+\varepsilon}(\partial\Omega)} = 1$ in (34). With (35) in the role of Lemma 3.2 and (32) in that of (9), the assertion follows by repeating the argumentation from the proof of Corollary 3.3. □

Now, we have gathered enough material to prove Theorem 4.1. The techniques used below are in essence the same as in the proof of Theorem 2.2.

Proof of Theorem 4.1. Let us fix $y \in \partial\Omega$, but note that all constants in the following estimates can be chosen independently of y. Moreover, we choose an auxiliary domain Ω_0 such that $\operatorname{supp}\kappa \subset \Omega_0$ and $\overline{\Omega}_0 \subset \Omega$.

The definition of b^h and the variational formulation of the forward problem (29) gives (cf. [20, Proposition 3.1])

$$\begin{aligned}
b^h &= 2(1/(2h))W^h \\
&= \int_\Omega \sigma\nabla u^h \cdot \nabla w^h \, \mathrm{d}x + \frac{1}{z_+}\int_{e_+^h} (u^h - U^h)(w^h - W^h) \, \mathrm{d}S \\
&\quad + \frac{1}{z_-}\int_{e_-^h} (u^h + U^h)(w^h + W^h) \, \mathrm{d}S \\
&= -\int_{\Omega_0} \kappa \nabla u_0^h \cdot \nabla u^h \, \mathrm{d}x. \qquad (36)
\end{aligned}$$

On the other hand, after approximating $-\delta_y'$ by a sequence of smooth mean-free functions (g_k) in the topology of $H^{-3/2-\varepsilon}(\partial\Omega)$, exactly the same line of reasoning as in the second paragraph of the proof of Theorem 2.2 indicates that

$$b = -\langle \delta_y', w\rangle = \lim_{k\to\infty} \int_{\partial\Omega} g_k w \, \mathrm{d}S = -\int_{\Omega_0} \kappa\nabla u_0 \cdot \nabla u \, \mathrm{d}x.$$

Combining this with (36) results in

$$|b^h - b| = \left| \int_{\Omega_0} (\kappa\nabla u_0 \cdot \nabla u - \kappa \nabla u_0^h \cdot \nabla u^h) \, \mathrm{d}x \right| \leq Ch,$$

where the last step follows with the same rationale that has been used for the last estimate in the proof of Theorem 2.2, with Lemma 4.2 playing the role of Corollary 3.3. This completes the proof. □

References

[1] D. C. Barber and B. H. Brown, *Applied potential tomography*, J. Phys. E: Sci. Instrum., 17 (1984), pp. 723–733.
[2] R. H. Bayford, *Bioimpedance Tomography (Electrical Impedance Tomography)*, Annu. Rev. Biomed. Eng., 8 (2006), pp. 63–91.
[3] L. Borcea, *Electrical impedance tomography*, Inverse Problems, 18 (2002), pp. R99–R136.
[4] L. Borcea, *Addendum to 'Electrical impedance tomography'*, Inverse Problems, 19 (2003), pp. 997–998.

[5] M. Cheney, D. Isaacson and J. C. Newell, *Electrical Impedance Tomography*, SIAM Rev., 41 (1999), pp. 85–101.
[6] K.-S. Cheng, D. Isaacson, J. S. Newell and D. G. Gisser, *Electrode models for electric current computed tomography*, IEEE Trans. Biomed. Eng., 36 (1989), pp. 918–924.
[7] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, Vol. 2, Springer, Berlin, 1988.
[8] M. Hanke, N. Hyvönen and S. Reusswig, *An inverse backscatter problem for electric impedance tomography*, SIAM J. Math. Anal., 41 (2009), pp. 1948–1966.
[9] M. Hanke, N. Hyvönen, and S. Reusswig, *Convex backscattering support in electric impedance tomography*, submitted.
[10] E. Hebey, *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities*, Courant Lecture Notes in Mathematics, Vol. 5, Ney York, Courant Institute of Math. Sciences, 2000.
[11] D. Holder, *Electrical Impedance Tomography: Methods, History and Applications*, IOP Publishing, Bristol, UK, 2005.
[12] N. Hyvönen, *Complete electrode model of electrical impedance tomography: Approximation properties and characterization of inclusions*, SIAM J. Appl. Math., 64 (2004), pp. 902–931.
[13] N. Hyvönen, *Approximating idealized boundary data of electric impedance tomography by electrode measurements*, Math. Models Methods Appl. Sci., 19 (2009), pp. 1185–1202.
[14] A. Lechleiter, N. Hyvönen, and H. Hakula, *The factorization method applied to the complete electrode model of impedance tomography*, SIAM J. Appl. Math., 68 (2008), pp. 1097–1121.
[15] W. R. B. Lionheart, *EIT reconstruction algorithms: pitfalls, challenges and recent developments*, Physiol. Meas., 25 (2004), pp. 125–142.
[16] J.-L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, Vol. I, Springer, Berlin, 1972.
[17] R. L. Parker, *The inverse problem of resistivity sounding*, Geophysics, 142 (1984), pp. 2143–2158.
[18] A. Ramirez, W. Daily, B. Binley, D. LaBreque and D. Roelant, *Detection of leaks in underground storage tanks using electrical resistance methods*, J. Environ. Eng. Geophys., 1 (1996), pp. 189–203.
[19] A. Ramirez A, W. Daily, D. LaBreque, E. Owen and D. Chesnut, *Monitoring an underground steam injection process using electrical resistance tomography*, Water Resources Res., 29 (1993), pp. 73–87.
[20] E. Somersalo, M. Cheney, and D. Isaacson, *Existence and uniqueness for electrode models for electric current computed tomography*, SIAM J. Appl. Math., 52 (1992), pp. 1023–1040.

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