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JUSTIFICATION OF REGULARIZATIONS FOR THE PARABOLIC-ELLIPTIC EDDY CURRENT EQUATION

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ABSTRACT

Transient excitation currents generate electromagnetic fields which, in turn, induce electric currents in proximal conductors. For slowly varying fields, this can be described by the eddy current equation.

In many applications, the considered domain consists of both, conducting and non-conducting regions. Thus the equation is of parabolic-elliptic type: In insulating regions the field instantaneously adapts to the excitation (quasi stationary elliptic behavior), while in conducting regions this adaptation takes some time due to the induced eddy currents (parabolic behavior).

The eddy current equation can be made fully parabolic by setting the conductivity in the insulated region to a small but positive value $\varepsilon > 0$. The aim of this work is to rigorously justify this parabolic regularization. We show, that the parabolic regularization leads to a well-posed problem and that, for $\varepsilon \to 0$, its solutions converge against the solution of the parabolic-elliptic eddy current equation. We also consider an elliptic regularization and show an analogous result there.

INTRODUCTION

Transient excitation currents $J(x,t)$ generate electromagnetic fields, $E(x,t)$ and $H(x,t)$, which can be described by Maxwell's equations

$$\begin{aligned} \operatorname{curl} H &= e\partial_t E + \sigma E + J, \\ \operatorname{curl} E &= -\mu \partial_t H, \end{aligned}$$

where the curl-operator acts on the three spatial coordinates, ∂_t denotes the time-derivative, and (under the assumption of linear and isotropic time-independent material laws) $\sigma(x)$, $e(x)$ and $\mu(x)$ are the conductivity, permittivity and permeability of the considered domain.

For slowly varying electromagnetic fields, the displacement currents $e\frac{\partial E}{\partial t}$ can be neglected, cf. Alonso [1999], Ammari et al. [2000] and Pepperl [2005]. This leads to the *transient eddy current equation*

$$\partial_t(\sigma E) + \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} E\right) = I \qquad (1)$$

with $I = -\partial_t J$. In a typical application the domain under consideration consists of both, conducting regions ($\sigma(x) > 0$) and non-conducting regions ($\sigma(x) = 0$). The consequence is, that equation (1) is of parabolic-elliptic type. The physical interpretation is that the time-scales are different: In the insulating regions, the field instantaneously adapts to the excitation (quasi stationary behavior), while in the conducting regions this adaptation takes some time (due to eddy currents induced by the varying electromagnetic fields). Moreover, it is easily shown, that equation only determines $\operatorname{curl} E$ and σE, cf. Theorem 2.5 below.

To overcome this non-uniqueness and also for computational reasons (cf., e.g., Lang and Teleaga [2008]), it seems natural to regularize the problem by setting the conductivity to a small value $\varepsilon > 0$ in the non-conducting region. In that way, the eddy current equation is made full parabolic:

$$\partial_t(\sigma_\varepsilon E_\varepsilon) + \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} E_\varepsilon\right) = I$$

with

$$\sigma_\varepsilon = \begin{cases} \sigma(x) & \sigma(x) > 0 \\ \varepsilon & \sigma(x) = 0 \end{cases},$$

and is uniquely solvable (cf. Section 4). The aim of this work is to rigorously justify this regularization: We show, that

$$\sigma_\varepsilon E_\varepsilon \to \sigma E \quad \text{and} \quad \operatorname{curl} E_\varepsilon \to \operatorname{curl} E$$

as ε approaches zero, where E denotes a solution of (1). Note that for a scalar parabolic-elliptic equation (that appears, e.g., as a two-dimensional version of the eddy current equation), this result was shown in [Gebauer, 2007].

Moreover, we will also justify an elliptic regularization for (1). It is similar to the one presented by Nicaise and Tröltzsch [2013] and motivated by (but not equivalent to, cf. Section 5)

$$\partial_t(\sigma E_\varepsilon) + \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} E_\varepsilon\right) + \varepsilon E_\varepsilon = I.$$

Usually, the transient eddy current equation is treated by imposing a gauge condition. In that case, several well-posed variational formulations have been proposed for (1), cf., e.g. Bachinger et al. [2005]; Acevedo et al. [2009]; Kolmbauer [2011]; Nicaise and Tröltzsch [2013], but these approaches concentrate on solving the equation with a fixed conducting region. Accordingly, the variational formulations, with their underlying solution spaces, depend on the support of the conductivity.

For several applications such as inverse problems, sensitivity considerations, or, as in our case, the regularization of the equation, it is useful to have a variational formulation that does not depend on the conducting domain. In case of unbounded domains, the authors derive in [Arnold and Harrach, 2012] a variational formulation for (1), that is unified with respect to σ. In this work, we carry these results over to the case of bounded domains: We propose a well-posed variational formulation, that is somehow flexible with respect to the conductivity. Moreover, its solution not only solves the variational problem but also yields a solution of the eddy current equation: It represents the solutions up to gradient fields, that vanish on the conductor. In this paper, we use this theory to justify the parabolic and the elliptic regularization.

This paper is organized as follows. After introducing the necessary notations in Section 1, we carry over the results of [Arnold and Harrach, 2012] about the well-definedness of (1) in Section 2. Section 3 contains our variational formulation and the solvability of (1). In Section 4 we justify the parabolic regularization: We show the convergence of the solutions when the fully positive conductivity approaches zero in a part of the domain. We finish this paper by presenting a similar result for an elliptic regularization in Section 5. A conclusion can be found in Section 6.

1 NOTATION

Let $T > 0$ and $O \subset \mathbb{R}^3$ be a simply connected bounded domain with Lipschitz boundary Σ and outer unit normal ν. We denote by $L^\infty_+(O)$ the space of $L^\infty(O)$-functions with positive (essential) infima.

$\mathcal{D}(O)$, respectively, $\mathcal{D}(O\times]0,T[)$ denote the space of C^∞-functions which are compactly supported in O, respectively, $O\times]0,T[$. We also use the notation $\mathcal{D}(O\times[0,T[)$ for the space of restrictions of functions from $\mathcal{D}(O\times]-\infty,T[)$ to $O\times[0,T]$.

Beside $L^2(O)$, we use the spaces

$$\begin{aligned} H(\operatorname{curl}) &:= \{E \in L^2(O)^3 \mid \operatorname{curl} E \in L^2(O)^3\},\\ H_0(\operatorname{curl}) &:= \{E \in H(\operatorname{curl}) \mid \nu \times E|_\Sigma = 0\},\\ H^1(O) &:= \{E \in L^2(O) \mid \nabla E \in L^2(O)^3\}, \end{aligned}$$

which are Hilbert spaces with respect to their graph norms.

We denote the dual space of a space H by H'. We frequently use the dual pairing between $H(\operatorname{curl})'$ and $H(\operatorname{curl})$, which we denote by

$$\langle G, E\rangle \quad \text{for } G \in H(\operatorname{curl})',\ E \in H(\operatorname{curl}).$$

We also write $O_T := O\times]0,T[$ and $L^2(O_T)$ instead of $L^2(O\times]0,T[)$, and usually omit the arguments x and t and only use them where we expect them to improve readability.

For a Banach space X, $C(0,T,X)$ and $L^2(0,T,X)$ denote the spaces of vector-valued functions

$$E:\ [0,T] \to X,$$

which are continuous on $[0,T]$, respectively, square integrable, cf., e.g., [Dautray and Lions, 2000c, XVIII, §1].

2 THE EDDY CURRENT PROBLEM

We consider the space $L^2(0,T,H_0(\operatorname{curl}))$ as a proper space to look for a solution of (1).

Let us assume that $\mu \in L^\infty_+(O)$ and either

$$\sigma \in L^\infty_+(O)$$

or

$$\begin{aligned} \sigma \in L_C := \{\sigma \in L^\infty(O) \mid\ & \sigma \in L^\infty_+(\Omega) \text{ with } \Omega := \operatorname{supp}\sigma \subsetneq O,\\ & \text{and } \Omega = \cup_{i=1}^s \Omega_i,\ s \in \mathbb{N},\ \Omega_i \text{ bounded Lipschitz}\\ & \text{domains, } \overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset,\ i \neq j, \text{ and}\\ & O \setminus \overline{\Omega} \text{ is connected}\}. \end{aligned}$$

Throughout this paper, we denote the support of σ by Ω.

We assume that we are given $E_\iota \in L^2(O)^3$ with $\operatorname{div}(\sigma E_\iota) = 0$ and the excitation

$$I \in L^2(0,T,H(\operatorname{curl})') \text{ with } \operatorname{div} I = 0.$$

Then, for $E \in L^2(0,T,H_0(\operatorname{curl}))$ the eddy current equation (1) posed on $O\times]0,T[$ is well-defined in a distributional sense and equivalent to

$$\begin{aligned} -\int_0^T \int_O \sigma E \cdot \partial_t \Phi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_O \frac{1}{\mu} \operatorname{curl} E \cdot \operatorname{curl}\Phi \,\mathrm{d}x\,\mathrm{d}t \\ = \int_0^T \langle I, \Phi\rangle \,\mathrm{d}t \quad \text{for all } \Phi \in \mathcal{D}(O\times]0,T[)^3. \quad (2) \end{aligned}$$

The assertions of this section are proven in [Arnold and Harrach, 2012, Sect. 2] for unbounded domains. The proofs are analogously.

We first establish, that every solution of (1) has well-defined initial values. Therefore we introduce the space

$$\mathcal{W}_\sigma := \{E \in L^2(0,T,H_0(\mathrm{curl})) \mid (\sigma E)^\cdot \in L^2(0,T,H_0(\mathrm{curl})')\},$$

where $(\sigma E)^\cdot$ denotes the time-derivative of $\sigma E \in L^2(O_T)^3$ in the sense of vector-valued distributions with respect to the canonical injection $L^2(O)^3 \hookrightarrow H_0(\mathrm{curl})'$.

Lemma 2.1. *If $E \in \mathcal{W}_\sigma$, then $\sqrt{\sigma}E \in C(0,T,L^2(O)^3)$. Additionally, for $E,F \in \mathcal{W}_\sigma$ the following integration by parts formula holds:*

$$\int_0^T \langle (\sigma E)^\cdot, F\rangle \,\mathrm{d}t + \int_0^T \langle (\sigma F)^\cdot, E\rangle \,\mathrm{d}t = \int_O \sigma\left(E(T)\cdot F(T) - E(0)\cdot F(0)\right) \mathrm{d}x. \quad (3)$$

Lemma 2.2 (Initial values). *If $E \in L^2(0,T,H_0(\mathrm{curl}))$ solves* (1)*, then $E \in \mathcal{W}_\sigma$ and thus has well-defined initial values $\sqrt{\sigma}E(0) \in L^2(O)^3$.*

For $t \in]0,T[$ a.e., $(\sigma E)^\cdot(t) \in H_0(\mathrm{curl})'$ is given by

$$\langle (\sigma E)^\cdot(t), F\rangle = \langle I(t), F\rangle - \int_O \frac{1}{\mu} \mathrm{curl}\, E(t) \cdot \mathrm{curl}\, F \,\mathrm{d}x \quad (4)$$

for all $F \in H_0(\mathrm{curl})$.

Corollary 2.3. *The following problem is well-defined: Find $E \in L^2(0,T,H_0(\mathrm{curl}))$ that solves*

$$\partial_t(\sigma(x)E(x,t)) + \mathrm{curl}\left(\frac{1}{\mu(x)}\mathrm{curl}\, E(x,t)\right) = I(x,t) \quad \text{in } O\times]0,T[, \quad (5)$$

$$\sqrt{\sigma(x)}E(x,0) = \sqrt{\sigma(x)}E_\iota(x) \quad \text{in } O. \quad (6)$$

Now, we give an equivalent variational formulation:

Lemma 2.4 (Standard variational formulation). *The following problems are equivalent:*

a) Find $E \in L^2(0,T,H_0(\mathrm{curl}))$ that solves (5)–(6)*.*
b) Find $E \in \mathcal{W}_\sigma$ that solves (6) *and*

$$\int_0^T \langle (\sigma E)^\cdot, F\rangle \,\mathrm{d}t + \int_0^T \int_O \frac{1}{\mu}\mathrm{curl}\, E \cdot \mathrm{curl}\, F \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \langle I, F\rangle \,\mathrm{d}t \quad (7)$$

for all $F \in L^2(0,T,H_0(\mathrm{curl}))$.

c) Find $E \in L^2(0,T,H_0(\mathrm{curl}))$ that solves

$$-\int_0^T \int_O \sigma E \cdot \partial_t \Phi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_O \frac{1}{\mu}\mathrm{curl}\, E \cdot \mathrm{curl}\, \Phi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \langle I, \Phi\rangle \,\mathrm{d}t + \int_O \sigma E_\iota \cdot \Phi(0)\,\mathrm{d}x$$

for all $\Phi \in \mathcal{D}(O\times[0,T[)^3$.

Theorem 2.5 (Uniqueness). *Equations* (5)–(6) *uniquely determine* $\mathrm{curl}\, E$ *and* $\sqrt{\sigma}E$*.*

Moreover, if $E \in L^2(0,T,H_0(\mathrm{curl}))$ solves (5)–(6)*, then every function $F \in L^2(0,T,H_0(\mathrm{curl}))$ with $\mathrm{curl}\, F = \mathrm{curl}\, E$ and $\sqrt{\sigma}F = \sqrt{\sigma}E$ also solves* (5)–(6)*.*

3 A VARIATIONAL SOLUTION THEORY

Unfortunately, the non-uniqueness implies, that none of the variational formulations in Lemma 2.4 is well-posed. Our approach is as follows. We keep this non-uniqueness and try to determine the unique part of the solutions - that is the divergence-free part. Therefore, we write

$$E = \tilde{E} + \nabla u$$

with a divergence free field $\tilde{E}$, and a gradient field ∇u. The crucial point is to consider $\nabla u = \nabla u_{\tilde{E}}$ as a continuous linear function of $\tilde{E}$, cf. Lemma 3.1. This allows us to rewrite the eddy current equations (5)–(6) as a variational equation for $\tilde{E}$, which is uniformly coercive on the space of divergence free functions and thus uniquely determines the field $\tilde{E}$. Note, that $\tilde{E}$ does not solve the eddy current equations.

This section is similar to Section 3 of [Arnold and Harrach, 2012] for the case of unbounded domains.

Lemma 3.1. *There is a continuous linear map*

$$L^2(O)^3 \to H_0(\mathrm{curl}\, 0) := \{E \in H_0(\mathrm{curl}) \mid \mathrm{curl}\, E = 0\},$$
$$E \mapsto \nabla u_E,$$

with

$$\mathrm{div}(\sigma(E + \nabla u_E)) = 0 \quad \text{in } O. \quad (8)$$

Proof. Let $E \in L^2(O)^3$.

We first consider the case $\Omega = O$. Due to Poincare's inequality (cf., e.g., [Dautray and Lions, 2000a, IV, §7, Prop. 2]), the fact, that σ is positively bounded from below on O, and Lax-Milgram's Theorem (cf., e.g., [Renardy and Rogers, 2004, §8, Thm. 8.14]), there exists a unique $u_E \in H_0^1(O)$ that solves

$$\int_O \sigma \nabla u \cdot \nabla v \,\mathrm{d}x = -\int_O \sigma E \cdot \nabla v \,\mathrm{d}x \quad \text{for all } v \in H_0^1(O),$$

and u_E depends continuously on $E \in L^2(O)^3$.

Now, let $\Omega \subsetneq O$. Again, as σ is positively bounded from below on Ω, we obtain as above a unique $u_E \in H^1_\square(\Omega)$ that solves

$$\int_\Omega \sigma \nabla u \cdot \nabla v \, \mathrm{d}x = -\int_\Omega \sigma E \cdot \nabla v \, \mathrm{d}x \quad \text{for all } v \in H^1(\Omega),$$

where $H^1_\square(\Omega) := \left\{ v \in H^1(\Omega) \,|\, \int_{\Omega_i} v \, \mathrm{d}x = 0, i = 1, \ldots, s \right\}$, and u_E depends continuously on $E|_\Omega$. We extend u_E to an element of $H^1_0(O)$ by solving $\Delta u = 0$ on $O \setminus \overline{\Omega}$ with $u|_{\partial\Omega} = u_E|_{\partial\Omega}$ for $u \in H^1(O \setminus \overline{\Omega})$ with $u|_\Sigma = 0$. Again, Lax-Milgram's Theorem provides a unique solution, which depends continuously on $u_E|_{\partial\Omega}$ and thus on E. Let u_E, again, denote its extension.

In both cases $u_E \in H^1_0(O)$, $\nabla u_E \in H_0(\mathrm{curl}\, 0)$ and the mapping $E \mapsto \nabla u_E$ is well-defined, linear and continuous with a continuity constant that depends on the lower and upper bounds of σ. Moreover, (8) is fulfilled. □

For the rest of this paper, let ∇u_E denote the image of E under this mapping. Obviously, there are different possibilities to construct this map, but $\sqrt{\sigma}\nabla u_E$ is uniquely determined by the condition (8). Moreover, it holds that

$$\|\sqrt{\sigma}\nabla u_E\|_{L^2(O)^3} \leq \|\sqrt{\sigma}E\|_{L^2(O)^3}. \tag{9}$$

Note that ∇u_E depends nonlinearly on σ. Also continuous dependence on σ for fixed E must not be true. We will discuss a special case in Section 4.

Now we use this Lemma to show a variational formulation for (5)–(6). We define the bilinear form

$$\begin{aligned} a : L^2(0,T,H_0(\mathrm{curl})) \times H^1(0,T,H_0(\mathrm{curl})) \to \mathbb{R} : \\ a(E,\Phi) := -\int_0^T \int_O \sigma(E + \nabla u_E) \cdot \dot{\Phi} \, \mathrm{d}x \, \mathrm{d}t \\ + \int_0^T \int_O \frac{1}{\mu} \mathrm{curl}\, E \cdot \mathrm{curl}\, \Phi \, \mathrm{d}x \, \mathrm{d}t, \end{aligned} \tag{10}$$

and, motivated by Lemma 2.4c), the linear form l:

$$\begin{aligned} l : H^1(0,T,H_0(\mathrm{curl})) \to \mathbb{R} \\ l(\Phi) := \int_0^T \langle I, \Phi \rangle \, \mathrm{d}t + \int_O \sigma E_\iota \cdot \Phi(0) \, \mathrm{d}x. \end{aligned}$$

To get around the non-uniqueness, cf. Theorem 2.5, we consider the Hilbert space

$$W_0 := \{ E \in H_0(\mathrm{curl}) \,|\, \mathrm{div}\, E = 0 \}$$

equipped with the norm $\|\mathrm{curl}\cdot\|_{L^2(O)^3}$, which is equivalent to the graph norm, cf. [Girault and Raviart, 1986, Lemma 3.4]. Especially, there is a constant C_O only depending on O such that

$$\|E\|_{L^2(O)^3} \leq C_O \|\mathrm{curl}\, E\|_{L^2(O)^3}.$$

Let $H^1_{T0}(0,T,W_0) := \{ \Psi \in H^1(0,T,W_0) \,|\, \Psi(T) = 0 \}$.

Theorem 3.2 (Equivalence). *If $\tilde{E} \in L^2(0,T,W_0)$ solves*

$$a(\tilde{E},\Phi) = l(\Phi) \quad \textit{for all } \Phi \in H^1_{T0}(0,T,W_0), \tag{11}$$

then $\tilde{E} + \nabla u_{\tilde{E}} \in L^2(0,T,H_0(\mathrm{curl}))$ solves (5)–(6).

Proof. Obviously, $a(\cdot,\nabla\phi)$ as well as $l(\nabla\phi)$ vanish for gradient fields $\nabla\phi \in H^1(0,T,H_0(\mathrm{curl}))$, $\phi \in H^1(0,T,H^1(O))$. (For the latter, recall that $\mathrm{div}\, I = 0$ and $\mathrm{div}(\sigma E_\iota) = 0$.) Now we use the following simple decomposition: Every $\Phi \in \mathcal{D}(O)^3$ can be written as

$$\Phi = \Psi + \nabla\phi, \tag{12}$$

with $\Psi \in W_0$, $\phi \in H^1_0(O)$. From that and the linearity of a and l it follows, that (for any $\tilde{E} \in L^2(0,T,W_0)$)

$$a(\tilde{E},\Phi) = l(\Phi)$$

holds for all $\Phi \in \mathcal{D}(O \times [0,T[)^3$, if it holds for all $\Phi \in H^1_{T0}(0,T,W_0)$. Lemma 2.4 yields the assertion. □

We now show, that (11) is well-posed. We will use the Lions-Lax-Milgram Theorem.

Lemma 3.3 (Lions-Lax-Milgram Theorem). *Let $\mathcal{H}$ be a Hilbert space and V be a normed (not necessarily complete) vector space. Let $a : \mathcal{H} \times V \to \mathbb{R}$ be a bilinear form satisfying the following properties:*

a) For every $\Phi \in V$, the linear form $E \mapsto a(E,\Phi)$ is continuous on $\mathcal{H}$.
b) There exists $\alpha > 0$ such that

$$\inf_{\|\Phi\|_V = 1} \sup_{\|E\|_{\mathcal{H}} \leq 1} |a(E,\Phi)| \geq \frac{1}{\alpha}.$$

Then for each continuous linear form $l \in V'$, there exists $E_l \in \mathcal{H}$ such that

$$a(E_l,\Phi) = l(\Phi) \ \textit{ for all } \Phi \in V \ \textit{ and } \ \|E_l\|_{\mathcal{H}} \leq \alpha \|l\|_{V'}.$$

The proof can be found, for example, in [Showalter, 1997, III.2, Thm. 2.1, Cor. 2.1].

Theorem 3.4 (Existence). *There is a unique solution $\tilde{E} \in L^2(0,T,W_0)$ of (11). $\tilde{E}$ depends continuously on I and $\sqrt{\sigma}E_\iota$ and with $\alpha = \max(\|\mu\|_\infty, 2)$ it holds, that*

$$\|\tilde{E}\|_{L^2(0,T,W_0)} \leq \alpha \max(C_O\|I\|_{L^2(0,T,H(\mathrm{curl})')}, \|\sqrt{\sigma}E_\iota\|_{L^2(O)^3}). \quad (13)$$

$\tilde{E} + \nabla u_{\tilde{E}}$ solves the eddy current equations (5)–(6) and any other solution $E \in L^2(0,T,H_0(\mathrm{curl}))$ of (5)–(6) fulfills

$$\mathrm{curl}\, E = \mathrm{curl}\, \tilde{E}, \quad \sqrt{\sigma}E = \sqrt{\sigma}(\tilde{E} + \nabla u_{\tilde{E}}). \quad (14)$$

$\mathrm{curl}\, E$ *and $\sqrt{\sigma}E$ depend continuously on I and $\sqrt{\sigma}E_\iota$:*

$$\begin{aligned}\|\mathrm{curl}\, E\|_{L^2(0,T,L^2(O)^3)} \leq \\ \alpha \max(C_O\|I\|_{L^2(0,T,H(\mathrm{curl})')}, \|\sqrt{\sigma}E_\iota\|_{L^2(O)^3}), \\ \|\sqrt{\sigma}E\|_{L^2(O_T)^3} \leq 2C_O\|\sqrt{\sigma}\|_\infty \|\mathrm{curl}\, E\|_{L^2(0,T,L^2(O)^3)}.\end{aligned}$$

Proof. To apply Lions-Lax-Milgram Theorem we use the Hilbert space $\mathcal{H} := L^2(0,T,W_0)$ and equip its subspace $V := H^1_{T0}(0,T,W_0)$ with the norm

$$\|\Phi\|_V^2 := \|\Phi\|^2_{L^2(0,T,W_0)} + \|\sqrt{\sigma}(\Phi + \nabla u_\Phi)(0)\|^2_{L^2(O)^3}.$$

Then, it is straightforward to show, that for fixed $\Phi \in V$ the linear form $E \mapsto a(E,\Phi)$ is continuous on $\mathcal{H}$ and that $l \in V'$ with

$$\|l\|_{V'} \leq \max(C_O\|I\|_{L^2(0,T,H(\mathrm{curl})')}, \|\sqrt{\sigma}E_\iota\|_{L^2(O)^3}).$$

Moreover, for $\Phi \in V$, Lemma 3.1 and the integration by parts formula (3) yield that

$$a(\Phi,\Phi) \geq \frac{1}{2}\|\sqrt{\sigma}(\Phi + \nabla u_\Phi)(0)\|^2_{L^2(O)^3} + \frac{1}{\|\mu\|_\infty}\|\Phi\|^2_{L^2(0,T,W_0)}, \quad (15)$$

which implies, that

$$\inf_{\|\Phi\|_V=1} \sup_{\|E\|_{\mathcal{H}}\leq 1} |a(E,\Phi)| \geq \frac{1}{\alpha}.$$

Now, Lemma 3.3 yields the existence of an $\tilde{E} \in \mathcal{H}$ that fulfills (11) and depends continuously on l.

Theorem 3.2 yields that $\tilde{E} + \nabla u_{\tilde{E}} \in L^2(0,T,H_0(\mathrm{curl}))$ is a solution of the eddy current equations (5)–(6).

To show uniqueness, let $\tilde{E}_1, \tilde{E}_2 \in L^2(0,T,W_0)$ be two solutions of (11). Then, $\tilde{E}_1 + \nabla u_{\tilde{E}_1}, \tilde{E}_2 + \nabla u_{\tilde{E}_2} \in L^2(0,T,H_0(\mathrm{curl}))$ both solve the eddy current equations (5)–(6) and Theorem 2.5 implies $\tilde{E}_1 = \tilde{E}_2$.

The remaining assertions follow similarly from Theorem 2.5. ☐

Corollary 3.5. *Let $(\sigma_n)_{n\in\mathbb{N}} \subset L_C \cup L^\infty_+(O)$ be a bounded sequence and $\tilde{E}_n$, $n \in \mathbb{N}$, be the corresponding unique solutions of (11). Then the sequences*

$$\begin{aligned}(\tilde{E}_n)_{n\in\mathbb{N}} \subset L^2(0,T,W_0), \\ (\sqrt{\sigma_n}\tilde{E}_n)_{n\in\mathbb{N}},\ (\sqrt{\sigma_n}\nabla u_{\tilde{E}_n})_{n\in\mathbb{N}} \subset L^2(O_T)^3\end{aligned}$$

are bounded. The bounds depend on the bound of $(\sigma_n)_{n\in\mathbb{N}}$.

In particular, for any sequence $(E_n)_{n\in\mathbb{N}} \subset L^2(0,T,H_0(\mathrm{curl}))$ of corresponding solutions of equations (5)–(6) the sequences

$$(\mathrm{curl}\, E_n)_{n\in\mathbb{N}},\ (\sqrt{\sigma_n}E_n)_{n\in\mathbb{N}} \subset L^2(O_T)^3$$

are bounded.

Remark 3.6. *Several results about the dependence of the solution on the conductivity can be obtained. In particular the solution's sensitivity with respect to the eddy current equation changing from elliptic to parabolic type is studied by the authors in [Arnold and Harrach, 2012, Sect. 4] in the case of unbounded domains. The results can be directly carried over to the bounded setting.*

4 PARABOLIC REGULARIZATION

In this section we keep $\sigma \in L_C$, $E_\iota \in L^2(O)^3$ with $\mathrm{div}(\sigma E_\iota) = 0$ and I as in Section 2 fixed and analyze the solution(s) behavior corresponding to

$$\sigma_\varepsilon = \begin{cases} \sigma & x \in \Omega \\ \varepsilon & x \in O \setminus \overline{\Omega} \end{cases},$$

if the positive real number ε approaches zero. Obviously, we have $\lim_{\varepsilon\to 0}\sigma_\varepsilon = \sigma$ in $L^\infty(O)$. In that way, the eddy current equation is made fully parabolic:

$$\partial_t(\sigma_\varepsilon E_\varepsilon) + \mathrm{curl}\left(\frac{1}{\mu}\mathrm{curl}\, E_\varepsilon\right) = I. \quad (16)$$

Our main result is Theorem 4.4, where we show, that the relevant parts of the solutions of (16), i.e. $\mathrm{curl}\, E_\varepsilon$ and $\sigma_\varepsilon E_\varepsilon$, converge against the corresponding unique parts of the solutions of the eddy current equation

$$\partial_t(\sigma E) + \mathrm{curl}\left(\frac{1}{\mu}\mathrm{curl}\, E\right) = I$$

if ε tends to zero. Therefore, we use the variational formulation (11) and show that its (unique) solutions converge (cf. Theorem 4.3).

Let us first remark, that, as $\sigma_\varepsilon \in L^\infty_+(O)$, the theory of Sections 2 and 3 (with appropriate initial conditions) holds.

Especially, (16) is uniquely solvable, and the unique solution is given by $\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon}$, where $\tilde{E}_\varepsilon \in L^2(0,T,W_0)$ is the unique solution of (11) with $\sigma = \sigma_\varepsilon$ and $\nabla u_{\tilde{E}_\varepsilon}$ is its image under the mapping from Lemma 3.1 with $\sigma = \sigma_\varepsilon$.

We start with the analysis of the mapping from Lemma 3.1,

$$L^2(O)^3 \to H_0(\mathrm{curl}\, 0), \quad E \mapsto \nabla u_{E,\varepsilon}$$

such that $\operatorname{div}(\sigma_\varepsilon(E + \nabla u_{E,\varepsilon})) = 0$, if ε tends to zero. Here, we indicate the nonlinear dependence of u_E on σ_ε by $u_{E,\varepsilon}$.

Lemma 4.1. *Let $(F_\varepsilon) \subset L^2(O)^3$ be bounded with $F_\varepsilon \rightharpoonup F \in L^2(O)^3$ as $\varepsilon \to 0$. Let $(u_{F_\varepsilon,\varepsilon}) \subset H_0^1(O)$ denote the corresponding unique elements from Lemma 3.1, that solve*

$$\int_O \sigma_\varepsilon \nabla u_{F_\varepsilon,\varepsilon} \cdot \nabla v \,\mathrm{d}x = -\int_O \sigma_\varepsilon F_\varepsilon \cdot \nabla v \,\mathrm{d}x \quad \text{for all } v \in H_0^1(O)$$

and let $u_{F,\sigma} \in H_0^1(O)$ be the corresponding element from Lemma 3.1 (that is unique by construction). Then

a) $\|\sqrt{\sigma_\varepsilon} F_\varepsilon\|_{L^2(O\setminus\overline{\Omega})^3} \to 0$, $\sqrt{\sigma_\varepsilon} F_\varepsilon \rightharpoonup \sqrt{\sigma} F$ *in* $L^2(O)^3$ *and* $(\sqrt{\sigma_\varepsilon} \nabla u_{F_\varepsilon,\varepsilon}) \subset L^2(O)^3$ *is bounded.*

b) $\sigma_\varepsilon \nabla u_{F_\varepsilon,\varepsilon} \rightharpoonup \sigma \nabla u_{F,\sigma} \in L^2(O)^3$.

Especially, for fixed $F \in L^2(O)^3$ it holds, that $\sigma_\varepsilon F \to \sigma F$ in $L^2(O)^3$ and $\sigma_\varepsilon \nabla u_{F,\varepsilon} \to \sigma \nabla u_{F,\sigma}$ in $L^2(O)^3$.

Proof. Let $\phi \in L^2(O)^3$.

a) Obviously, it holds, that

$$\|\sqrt{\sigma_\varepsilon} F_\varepsilon\|_{L^2(O\setminus\overline{\Omega})^3} = \sqrt{\varepsilon} \|F_\varepsilon\|_{L^2(O\setminus\overline{\Omega})^3} \to 0,$$

moreover

$$\begin{aligned}&(\sqrt{\sigma_\varepsilon} F_\varepsilon - \sqrt{\sigma} F, \phi)_{L^2(O)^3} \\ &= \sqrt{\varepsilon}(F_\varepsilon, \phi)_{L^2(O\setminus\overline{\Omega})^3} + (F_\varepsilon - F, \sqrt{\sigma}\phi)_{L^2(\Omega)^3} \to 0,\end{aligned}$$

and with

$$\|\sqrt{\sigma_\varepsilon} \nabla u_{F_\varepsilon,\varepsilon}\|_{L^2(O)^3} \le \|\sqrt{\sigma_\varepsilon} F_\varepsilon\|_{L^2(O)^3}$$

we obtain, that $(\sqrt{\sigma_\varepsilon} \nabla u_{F_\varepsilon,\varepsilon})$ is bounded in $L^2(O)^3$.

b) First we show that every subsequence of $(\sqrt{\sigma_\varepsilon} \nabla u_{F_\varepsilon,\varepsilon})$ has a subsequence that converges weakly against $\sqrt{\sigma} \nabla h$ for some $h \in H_0^1(O)$. In a second step we show that all these weak limits coincide.

As $(\sqrt{\sigma_\varepsilon} \nabla u_{F_\varepsilon,\varepsilon}) \subset L^2(O)^3$ is bounded, every subsequence is bounded, and Alaoglu's Theorem, cf., e.g., [Renardy and Rogers, 2004, Thm. 6.62], yields that every subsequence contains subsequence (that we still indicate by ε for the ease of notation), again, that converges weakly against some $a \in L^2(O)^3$:

$$\sqrt{\sigma_\varepsilon} \nabla u_{F_\varepsilon,\varepsilon} \rightharpoonup a \in L^2(O)^3.$$

We then also have

$$\sqrt{\sigma_\varepsilon} \nabla u_{F_\varepsilon,\varepsilon}|_\Omega = \sqrt{\sigma} \nabla u_{F_\varepsilon,\varepsilon}|_\Omega \rightharpoonup a|_\Omega \in L^2(\Omega)^3$$

and therefore

$$\nabla u_{F_\varepsilon,\varepsilon}|_\Omega \rightharpoonup \frac{a|_\Omega}{\sqrt{\sigma}} \in L^2(\Omega)^3.$$

The orthogonal decomposition

$$\nabla H^1(\Omega) \oplus^\perp H_0(\operatorname{div} 0, \Omega) = L^2(\Omega)^3,$$

cf. [Dautray and Lions, 2000b, IX, §3, Prop. 1], where $H_0(\operatorname{div} 0, O) = \{E \in L^2(O)^3 \,|\, \operatorname{div} E = 0,\ \nu \cdot E|_\Sigma = 0\}$, yields then $\frac{a|_\Omega}{\sqrt{\sigma}} \in \nabla H^1(\Omega)$ and hence there is some $h \in H^1(\Omega)$ with

$$\frac{a|_\Omega}{\sqrt{\sigma}} = \nabla h.$$

Obviously, ∇h is uniquely determined, but h is not. To overcome this, we fix h by the choice $h \in H^1_\square(\Omega)$ as in Lemma 3.1 and extend it to an element of $H_0^1(O)$ by solving $\Delta h = 0$ on $O \setminus \overline{\Omega}$. Then it still holds that

$$\sqrt{\sigma} \nabla u_{F_\varepsilon,\varepsilon} \rightharpoonup \sqrt{\sigma} \nabla h \ \text{ in } \ L^2(O)^3$$

and hence

$$\begin{aligned}&(\sigma_\varepsilon \nabla u_{F_\varepsilon,\varepsilon} - \sigma \nabla h, \phi)_{L^2(O)^3} = \\ &\quad (\sigma \nabla u_{F_\varepsilon,\varepsilon} - \sigma \nabla h, \phi)_{L^2(\Omega)^3} + \sqrt{\varepsilon}(\sqrt{\varepsilon} \nabla u_{F_\varepsilon,\varepsilon}, \phi)_{L^2(O\setminus\overline{\Omega})^3} \\ &\quad \to 0,\end{aligned}$$

i.e. $\sigma_\varepsilon \nabla u_{F_\varepsilon,\varepsilon} \rightharpoonup \sigma \nabla h$ in $L^2(O)^3$.

To conclude, that all these weak limits are identical, we show

$$\sigma \nabla h = \sigma \nabla u_{F,\sigma}.$$

For every $v \in H_0^1(O)$, Part 1) yields

$$\begin{aligned}0 &= \int_O \sigma_\varepsilon \nabla u_{F_\varepsilon,\varepsilon} \cdot \nabla v \,\mathrm{d}x + \int_O \sigma_\varepsilon F_\varepsilon \cdot \nabla v \,\mathrm{d}x \\ &\to \int_\Omega \sigma \nabla h \cdot \nabla v \,\mathrm{d}x + \int_\Omega \sigma F \cdot \nabla v \,\mathrm{d}x \quad \text{if } \varepsilon \to 0,\end{aligned}$$

and therefore also the right hand side vanishes for every $v \in H_0^1(O)$. Accordingly, $\sigma\nabla h = \sigma\nabla u_{F,\sigma}$ and $\nabla h|_\Omega = \nabla u_{F,\sigma}|_\Omega$.
Altogether, the second assertion follows. □

The next step is to show, that the solutions of the variational equation (11) converge.

To obtain meaningful initial values for (16), we modify the initial value $E_\iota \in L^2(O)^3$ to make its product with σ_ε divergence-free by $E_\iota + \nabla u_{E_\iota,\varepsilon}$. The precedent Lemma then yields $\sigma_\varepsilon(E_\iota + \nabla u_{E_\iota,\varepsilon}) \to \sigma E_\iota$ in $L^2(O)^3$ and the right hand side of (11), $l_\varepsilon : H^1(0,T,H_0(\mathrm{curl})) \to \mathbb{R}$, obviously fulfills

$$\begin{aligned} l_\varepsilon(\Phi) &:= \int_0^T \langle I, \Phi\rangle \,\mathrm{d}t + \int_O \sigma_\varepsilon(E_\iota + \nabla u_{E_\iota,\varepsilon}) \cdot \Phi(0)\,\mathrm{d}x \\ &\to \int_0^T \langle I, \Phi\rangle \,\mathrm{d}t + \int_O \sigma E_\iota \cdot \Phi(0)\,\mathrm{d}x = l(\Phi) \end{aligned}$$

for every $\Phi \in H^1(0,T,H_0(\mathrm{curl}))$.

Corresponding to σ_ε let $\tilde{E}_\varepsilon \in L^2(0,T,W_0)$ denote the unique solution of

$$a_\varepsilon(\tilde{E}_\varepsilon, \Phi) = l_\varepsilon(\Phi) \quad \text{for all } \Phi \in H^1_{T0}(0,T,W_0), \tag{17}$$

that is the variational problem (11) with $\sigma = \sigma_\varepsilon$. The bilinear form $a_\varepsilon : L^2(0,T,H_0(\mathrm{curl})) \times H^1(0,T,H_0(\mathrm{curl})) \to \mathbb{R}$ is then given by

$$\begin{aligned} a_\varepsilon(E,\Phi) = &-\int_0^T \int_O \sigma_\varepsilon(E + \nabla u_{E,\varepsilon}) \cdot \dot{\Phi}\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T \int_O \frac{1}{\mu} \mathrm{curl}\, E \cdot \mathrm{curl}\,\Phi\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

The next lemma shows that the solutions converge weakly towards the solution $\tilde{E} \in L^2(0,T,W_0)$ of (11) (that corresponds to $\varepsilon = 0$).

Lemma 4.2. *It holds, that $\tilde{E}_\varepsilon \rightharpoonup \tilde{E}$ in $L^2(0,T,W_0)$, $\sqrt{\sigma_\varepsilon}\tilde{E}_\varepsilon \rightharpoonup \sqrt{\sigma}\tilde{E}$ and $\sigma_\varepsilon \nabla u_{\tilde{E}_\varepsilon,\varepsilon} \rightharpoonup \sigma\nabla u_{\tilde{E},\sigma}$ in $L^2(O_T)^3$.*

Proof. The precedent Lemma yields that it suffices to show that $\tilde{E}_\varepsilon \rightharpoonup \tilde{E}$. To show this, we use the same technique: From Corollary 3.5 we know that $(\tilde{E}_\varepsilon) \subset L^2(0,T,W_0)$ is bounded. Again, Alaoglu's Theorem yields that every subsequence contains a subsequence (that we still denote by $(\tilde{E}_\varepsilon)$ for ease of notation) that converges weakly against some $\tilde{E}' \in L^2(0,T,W_0)$. In the following we show, that all these weak limits are identical to $\tilde{E}$.

The previous Lemma yields

$$\sqrt{\sigma_\varepsilon}\tilde{E}_\varepsilon \rightharpoonup \sqrt{\sigma}\tilde{E}' \text{ in } L^2(O_T)^3$$

and

$$\sigma_\varepsilon \nabla u_{\varepsilon,\tilde{E}_\varepsilon} \rightharpoonup \sigma\nabla u_{\tilde{E}'} \in L^2(O_T)^3.$$

Moreover, $\tilde{E}_\varepsilon \rightharpoonup \tilde{E}'$ in $L^2(0,T,W_0)$ implies that $\mathrm{curl}\,\tilde{E}_\varepsilon \rightharpoonup \mathrm{curl}\,\tilde{E}'$ in $L^2(O_T)^3$, so that for every $\Phi \in H^1_{T0}(0,T,W_0)$ the left hand side $a_\varepsilon(\tilde{E}_\varepsilon,\Phi)$ of (11) with $\sigma = \sigma_\varepsilon$ converges against $a(\tilde{E}',\Phi)$:

$$\begin{aligned} a_\varepsilon(\tilde{E}_\varepsilon,\Phi) = &-\int_0^T \int_O \sigma_\varepsilon(\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\varepsilon}) \cdot \dot{\Phi}\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T \int_O \frac{1}{\mu} \mathrm{curl}\,\tilde{E}_\varepsilon \cdot \mathrm{curl}\,\Phi\,\mathrm{d}x\,\mathrm{d}t \\ &\to a(\tilde{E}',\Phi) \end{aligned}$$

As $l_\varepsilon(\Phi) \to l(\Phi)$, $\tilde{E}'$ solves (11) and thus uniqueness provides $\tilde{E} = \tilde{E}'$. □

Theorem 4.3. *It holds, that $\tilde{E}_\varepsilon \to \tilde{E}$ in $L^2(0,T,W_0)$, $\sqrt{\sigma_\varepsilon}\tilde{E}_\varepsilon \to \sqrt{\sigma}\tilde{E}$ and $\sqrt{\sigma_\varepsilon}\nabla u_{\tilde{E}_\varepsilon,\varepsilon} \to \sqrt{\sigma}\nabla u_{\tilde{E},\sigma}$ in $L^2(O_T)^3$.*

Proof. Using the fact, that $\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\varepsilon}$ solves (16) with initial values $\sqrt{\sigma_\varepsilon}(E_\iota + \nabla u_{E_\iota,\varepsilon})$, the integration by parts formula (3) and Lemma 2.4b) we obtain for every ε, that

$$\begin{aligned} &\|\mu^{-\frac{1}{2}}\,\mathrm{curl}\,\tilde{E}_\varepsilon\|^2_{L^2(O_T)^3} + \frac{1}{2}\|\sqrt{\sigma_\varepsilon}(\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\varepsilon})(T)\|^2_{L^2(O)^3} \\ &= \frac{1}{2}\int_O \sigma_\varepsilon(E_\iota + \nabla u_{E_\iota,\varepsilon})^2\,\mathrm{d}x + \int_0^T\int_O \frac{1}{\mu}\mathrm{curl}\,\tilde{E}\cdot\mathrm{curl}\,\tilde{E}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^T \langle(\sigma(\tilde{E} + \nabla u_{\tilde{E},\sigma}))^\cdot, \tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\varepsilon}\rangle\,\mathrm{d}t \\ &\le \limsup_{\varepsilon\to 0}\Bigg[\frac{1}{2}\int_O \sigma_\varepsilon(E_\iota + \nabla u_{E_\iota,\varepsilon})^2\,\mathrm{d}x \\ &\quad + \int_0^T\int_O \frac{1}{\mu}\mathrm{curl}\,\tilde{E}\cdot\mathrm{curl}\,\tilde{E}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^T \langle(\sigma(\tilde{E} + \nabla u_{\tilde{E},\sigma}))^\cdot, \tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\varepsilon}\rangle\,\mathrm{d}t\Bigg] \\ &= \frac{1}{2}\int_O \sigma\,|E_\iota + \nabla u_{E_\iota}|^2\,\mathrm{d}x + \int_0^T\int_O \frac{1}{\mu}\mathrm{curl}\,\tilde{E}\cdot\mathrm{curl}\,\tilde{E}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^T \langle(\sigma(\tilde{E} + \nabla u_{\tilde{E},\sigma}))^\cdot, \tilde{E} + \nabla u_{\tilde{E},\sigma}\rangle\,\mathrm{d}t \\ &= \|\mu^{-\frac{1}{2}}\,\mathrm{curl}\,\tilde{E}\|^2_{L^2(O_T)^3} + \frac{1}{2}\|\sqrt{\sigma}(\tilde{E} + \nabla u_{\tilde{E},\sigma})(T)\|^2_{L^2(O)^3} \end{aligned}$$

and hence

$$\begin{aligned} &\limsup_{\varepsilon\to 0}\Bigg[\|\mu^{-\frac{1}{2}}\,\mathrm{curl}\,\tilde{E}_\varepsilon\|^2_{L^2(O_T)^3} \\ &\qquad + \frac{1}{2}\|\sqrt{\sigma_\varepsilon}(\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\varepsilon})(T)\|^2_{L^2(O)^3}\Bigg] \\ &\le \|\mu^{-\frac{1}{2}}\,\mathrm{curl}\,\tilde{E}\|^2_{L^2(O_T)^3} + \frac{1}{2}\|\sqrt{\sigma}(\tilde{E} + \nabla u_{\tilde{E},\sigma})(T)\|^2_{L^2(O)^3}, \end{aligned}$$

which, together with $\tilde{E}_\varepsilon \rightharpoonup \tilde{E}$ and the other results of

Lemma 4.2 yields

$$\begin{aligned}&\lim_{\varepsilon\to 0}\Big[\|\mu^{-\frac{1}{2}}\operatorname{curl}(\tilde E_\varepsilon-\tilde E)\|^2_{L^2(O_T)^3}\\&+\frac{1}{2}\|\sqrt{\sigma_\varepsilon}(\tilde E_\varepsilon+\nabla u_{\tilde E_\varepsilon,\varepsilon})(T)-\sqrt{\sigma}(\tilde E+\nabla u_{\tilde E,\sigma})(T)\|^2_{L^2(O)^3}\Big]\\&\le 2\|\mu^{-\frac{1}{2}}\operatorname{curl}\tilde E\|^2_{L^2(O_T)^3}-2(\mu^{-\frac{1}{2}}\operatorname{curl}\tilde E,\mu^{-\frac{1}{2}}\operatorname{curl}\tilde E)_{L^2(O_T)^3}\\&+\|\sqrt{\sigma}(\tilde E+\nabla u_{\tilde E,\sigma})(T)\|^2_{L^2(O)^3}\\&-(\sqrt{\sigma}(\tilde E+\nabla u_{\tilde E,\sigma})(T),\sqrt{\sigma}(\tilde E+\nabla u_{\tilde E,\sigma})(T))_{L^2(O)^3}\\&=0.\end{aligned}$$

Hence the first assertion follows from

$$\begin{aligned}\lim_{\varepsilon\to 0}\|\tilde E_\varepsilon-\tilde E\|^2_{L^2(0,T,W_0)}&=\lim_{\varepsilon\to 0}\|\operatorname{curl}(\tilde E_\varepsilon-\tilde E)\|^2_{L^2(O_T)^3}\\&\le\|\mu\|_\infty\limsup_{\varepsilon\to 0}\|\mu^{-\frac{1}{2}}\operatorname{curl}(\tilde E_\varepsilon-\tilde E)\|^2_{L^2(O_T)^3}=0\end{aligned}$$

and the third assertion follows from

$$\begin{aligned}&\lim_{\varepsilon\to 0}\|\sqrt{\sigma_\varepsilon}(\tilde E_\varepsilon+\nabla u_{\tilde E_\varepsilon,\varepsilon})(T)-\sqrt{\sigma}(\tilde E+\nabla u_{\tilde E,\sigma})(T)\|^2_{L^2(O)^3}\\&\quad=0.\end{aligned}$$

The second assertion now follows immediately with

$$\begin{aligned}&\lim_{\varepsilon\to 0}\|\sqrt{\sigma}_\varepsilon\tilde E_\varepsilon-\sqrt{\sigma}\tilde E\|^2_{L^2(O_T)^3}\\&=\lim_{\varepsilon\to 0}(\|\sqrt{\sigma}(\tilde E_\varepsilon-\tilde E)\|^2_{L^2(\Omega_T)^3}+\|\sqrt{\varepsilon}\tilde E_\varepsilon\|^2_{L^2(O\setminus\overline{\Omega}_T)^3})\\&\le C_O(\|\sigma\|_\infty\lim_{\varepsilon\to 0}\|\tilde E_\varepsilon-\tilde E\|^2_{L^2(0,T,W_0)}+\lim_{\varepsilon\to 0}\varepsilon\|\tilde E_\varepsilon\|^2_{L^2(0,T,W_0)})\\&=0.\end{aligned}$$

□

Now we can formulate our main result. Corresponding to σ_ε, let $E_\varepsilon\in L^2(0,T,H_0(\operatorname{curl}))$ denote the unique solution of (16) with initial values $\sqrt{\sigma_\varepsilon}(E_\iota+\nabla u_{E_\iota,\varepsilon})$. For $\varepsilon=0$, let $E\in L^2(0,T,H_0(\operatorname{curl}))$ denote any solution of (5)–(6).

Theorem 4.4. *It holds, that* $\operatorname{curl}E_\varepsilon\to\operatorname{curl}E$ *and* $\sqrt{\sigma_\varepsilon}E_\varepsilon\to\sqrt{\sigma}E$ *in* $L^2(O_T)^3$ *and that* $(\sigma_\varepsilon E_\varepsilon)^\cdot\to(\sigma E)^\cdot$ *in* $L^2(0,T,H_0(\operatorname{curl})')$.

Proof. It holds, that $\sqrt{\sigma_\varepsilon}E_\varepsilon=\sqrt{\sigma_\varepsilon}(\tilde E_\varepsilon+\nabla u_{\tilde E_\varepsilon,\varepsilon})$, $\operatorname{curl}E_\varepsilon=\operatorname{curl}\tilde E_\varepsilon$ and $\operatorname{curl}E=\operatorname{curl}\tilde E$, so that the precedent Lemma provides the first and the second assertion.

From the explicit form (4) of $(\sigma_\varepsilon E_\varepsilon)^\cdot$ given in Lemma 2.2, we obtain for all $F\in L^2(0,T,H_0(\operatorname{curl}))$

$$\begin{aligned}&\left|\int_0^T\langle(\sigma_\varepsilon E_\varepsilon)^\cdot-(\sigma E)^\cdot,F\rangle\,\mathrm{d}t\right|\\&=\left|\int_0^T\int_O\frac{1}{\mu}\operatorname{curl}(E-E_\varepsilon)\cdot\operatorname{curl}F\,\mathrm{d}x\,\mathrm{d}t\right|\to 0.\end{aligned}$$

This yields $(\sigma_\varepsilon E_\varepsilon)^\cdot\to(\sigma E)^\cdot$ in $L^2(0,T,H_0(\operatorname{curl})')$. □

5 ELLIPTIC REGULARIZATION

We finish this paper by justifying an elliptic regularization of the variational problem (11).

Again, we keep $\sigma\in L_C\cup L_+^\infty(O)$, E_ι and I fixed and modify the variational equation (11) by setting the bilinear-form $a_\varepsilon: L^2(0,T,H_0(\operatorname{curl}))\times H^1(0,T,H_0(\operatorname{curl}))\to\mathbb{R}$ to

$$\begin{aligned}a_\varepsilon(E,\Phi)&:=a(E,\Phi)+\varepsilon(E,\Phi)_{L^2(O_T)^3}\\&=-\int_0^T\int_O\sigma(E+\nabla u_{E,\sigma})\cdot\dot\Phi\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\int_0^T\int_O\frac{1}{\mu}\operatorname{curl}E\cdot\operatorname{curl}\Phi\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_O\varepsilon E\cdot\Phi\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

for some $\varepsilon>0$. Now, a_ε is (with respect to the space variable) coercive on the whole space $H_0(\operatorname{curl})$.

We consider the variational problem of finding $\tilde E_\varepsilon\in L^2(0,T,W_0)$ that solves

$$a_\varepsilon(\tilde E_\varepsilon,\Phi)=l(\Phi)\quad\text{for all }\Phi\in H^1_{T0}(0,T,W_0)\tag{18}$$

and study the solutions behavior if ε tends to zero.

It has to be stated clearly, that, in contrast to the parabolic regularization, we do not have any assertion about the solutions of the related (but not equivalent) regularized eddy current problem

$$\partial_t(\sigma E_\varepsilon)+\operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl}E_\varepsilon\right)+\varepsilon E_\varepsilon=I.\tag{19}$$

This is due to the fact, that a solution of (18) does not naturally imply a solution of (19), as it is the case for the original problem, cf. Theorem 3.2 and the parabolic regularization in Section 4. Anyway, for some applications, the variational equation might be of interest on itself.

In the following we show, that the solutions of (18) converge against the solution of (11), if ε tends to zero. Therefore, let us shortly answer the question of well-posedness. Obviously, the problem to find $\tilde E_\varepsilon\in L^2(0,T,W_0)$ that solves (18) for all $\Phi\in H^1_{T0}(0,T,W_0)$ still fits into the framework of the proof of the first part of Theorem 3.4 and hence there is a solution. Moreover, it can be shown, that, if $\tilde E_\varepsilon\in L^2(0,T,W_0)$ is such a solution, then $\tilde E_\varepsilon+\nabla u_{\tilde E_\varepsilon}\in\mathcal{W}_{\sigma_\varepsilon}$ (cf. Lemma 2.2 and the proof of Lemma 2.3 in [Arnold and Harrach, 2012]). Therefore, the integration by parts formula (3) holds and a result similar to Lemma 2.4. Using this, one easily sees, that $\tilde E_\varepsilon$ is unique.

Theorem 5.1. *Let* $\tilde E\in L^2(0,T,W_0)$ *denote the unique solution of* (11) *and* $\tilde E_\varepsilon\in L^2(0,T,W_0)$ *denote the unique solution of* (18)*. Then it holds, that* $\tilde E_\varepsilon\to\tilde E$ *in* $L^2(0,T,W_0)$ *if* $\varepsilon\to 0$.

Proof. First of all the coercivity and continuity constants in Theorem 3.4 are the same for both, the regularized and the original problem. Therefore, Theorem 3.5 yields that

$\tilde{E}_\varepsilon$ is bounded. Moreover, it obviously holds for all $F \in L^2(0,T,W_0)$, that

$$0 = l(F) - l(F) = a_\varepsilon(\tilde{E}_\varepsilon, F) - a(\tilde{E}, F) = a(\tilde{E}_\varepsilon - \tilde{E}, F) + \varepsilon(\tilde{E}_\varepsilon, F)_{L^2(O_T)^3}.$$

By use of a similar formulation as in Lemma 2.4b) we obtain with $\alpha = \max(\|\mu\|_\infty, 2)$, that

$$\|\tilde{E}_\varepsilon - \tilde{E}\|^2_{L^2(0,T,W_0)} \leq \frac{\varepsilon}{\alpha}(\tilde{E}_\varepsilon, \tilde{E}_\varepsilon - \tilde{E})_{L^2(O_T)^3} \leq \alpha\varepsilon C_O^2 \|\tilde{E}_\varepsilon\|_{L^2(0,T,W_0)} \|\tilde{E}_\varepsilon - \tilde{E}\|_{L^2(0,T,W_0)}$$

and hence

$$\|\tilde{E}_\varepsilon - \tilde{E}\|_{L^2(0,T,W_0)} \leq \alpha\varepsilon C_O^2 \|\tilde{E}_\varepsilon\|_{L^2(0,T,W_0)}.$$

The assertion follows from the fact, that $\|\tilde{E}_\varepsilon\|_{L^2(0,T,W_0)}$ is bounded. $\square$

In addition, one can in the same way as in Section 4, that $\sigma(\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon}) \to \sigma(\tilde{E} + \nabla u_{\tilde{E}})$ and $(\sigma(\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon}))^\cdot \to (\sigma(\tilde{E} + \nabla u_{\tilde{E}}))^\cdot$.

6 CONCLUSION

We have considered the transient eddy current equation in a bounded domain consisting of a conducting and a non-conducting part, which are described by the conductivity coefficient. A consequence is, that the equation is of parabolic-elliptic type and does not determine its solutions uniquely in the non-conducting part.

We have presented a variational solution theory, that is uniquely solvable and whose solution represents all solutions of the eddy current equation. This solution theory treats the conductivity merely as a parameter, especially it does not depend on the conducting region. We have used this theory to show a parabolic and an elliptic regularization for the equation.

A natural way to regularize the equation is to set the conductivity to a small positive value ε in the non-conducting part. Then the resulting equation is fully parabolic and leads to a well-posed problem. We have justified this regularization by proving the convergence of its solutions against the solution of the original parabolic-elliptic equation if ε tends to zero.

We have also showed an adequate result for an elliptic regularization.

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