Recent results on unimodular triangulations of lattice polytopes

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What? Who? Why?
Lattice polytope $P$ in $\mathbb{R}^d := \text{conv}(S)$, $S \in \mathbb{Z}^d$.

Unimodular simplex := vertices are an affine basis of $\mathbb{Z}^d$.
(Equivalently, normalized volume equal to 1)

(Lattice) subdivision of $P$: “face to face” decomposition into lattice subpolytopes.

(Lattice triangulation) of $P$: same, into simplices.

Unimodular triangulation: triangulation into unimodular simplices.
Proposition

Every lattice polygon has a unimodular triangulation.
Dim 2 versus higher dim

Remark
In dim ≥ 3 there are “empty non-unimodular simplices” ⇒ there are polytopes without unimodular triangulations.
Counting lattice points

The *Ehrhart series* of a lattice polytope $P$ counts how many lattice points lie in $kP$, for $k \in \mathbb{N}$.

It is known that its generating function can be rewritten as

$$
\sum_{k \geq 0} \#(kP \cap \mathbb{Z}^d) \ t^k = \frac{h^*_P(t)}{(1 - t)^{d+1}},
$$

for a certain polynomial $h^*_P$ of degree (at most) $\dim(P)$.
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**Theorem (Ehrhart 1967)**

*If $\mathcal{T}$ is a unimodular triangulation of $P$, then $h^*_P$ equals the $h$-polynomial of $\mathcal{T}$.***
Unimodular triangulations arise in two contexts:

- If $P$ has a unimodular triangulation then the cone $\sigma_P$ generated by $P \times \{1\} \subset \mathbb{R}^{d+1}$ is generated in degree one (We abbreviate this as “$P$ is integrally closed”). In particular, $P \cap \mathbb{Z}^d$ is a Hilbert basis for it.
Unimodular triangulations arise in two contexts:

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- If the normal fan of $P = \{yA = c, y \geq 0\}$ has a unimodular triangulation (using only facet normals as rays) then any integer program over $P$ is totally dual integral.
Commutative algebra

Consider again the cone $\sigma_P$ generated by $P \times \{1\} \subset \mathbb{R}^{d+1}$, and its associated semigroup algebra

$$R_P = \mathbb{K}[\sigma_P \cap \mathbb{Z}^{d+1}]$$

Then $P$ is integrally closed $\Leftrightarrow$ $R_P$ is generated in degree one.

(More generally, the subring of $R_P$ generated in degree one is normal if and only if $P$ is integrally closed in the sub lattice generated by its lattice points).
Algebraic geometry, I

Unimodular triangulations of $P$ correspond to certain (so-called crepant) resolutions of the singular point in the affine toric variety

$$U_P = \text{Spec} \mathbb{K}[\sigma_P^\vee \cap \mathbb{Z}^{d+1}].$$
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The *Stable Reduction Theorem* of Kempf-Knudsen-Mumford-Saint Donat (1973) is based on the following combinatorial result:
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The *Stable Reduction Theorem* of Kempf-Knudsen-Mumford-Saint Donat (1973) is based on the following combinatorial result:

**Theorem (Knudsen-Mumford-Waterman, 1973)**

*For every lattice polytope $P$ there is a dilation factor $c \in \mathbb{N}$ such that $cP$ admits a regular unimodular triangulation.*
Algebraic geometry, II

Consider the projective toric variety associated to $P$:

$$X_P = \text{Proj } \mathbb{K}[\sigma_P \cap \mathbb{Z}^{d+1}] = \text{Proj } R_P.$$ 

The following questions are of interest:
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The following questions are of interest:

- Is the ideal $I_P$ of $X_P$ generated by quadrics?
- Is the embedding $X_P \hookrightarrow \mathbb{P}^{n-1}$ projectively normal?
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**Conjecture**

*Every smooth polytope has a regular unimodular triangulation.*

Here: $P$ smooth $\iff P$ is simple and every vertex normal cone is unimodular $\iff X_P$ is smooth.
Regular triangulations

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A quadratic triangulation is a regular, unimodular, and flag triangulation (flag: $=$ every clique in the graph spans a simplex).
Some examples
A compressed polytope $P$ is a polytope of width one with respect to every facet. That is, for every facet hyperplane $H$ of $P$, all vertices of $P$ not in $H$ lie in the next lattice translation of $H$. 
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We can use this to show that

**Theorem (S. 1996, HPPS 2014+ for flagness)**

If a polytope \( P \) has a (regular, flag) unimodular triangulation \( T \) then every integer dilation \( cP \) of it has one too.
Compresed polytopes

**Sketch of proof.**

Consider the dilation $cT$ of $T$, which subdivides $cP$ into dilated unimodular simplices.
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Slice those simplices by all lattice translates of their facet hyperplanes. This produces a subdivision of $cT$ into compressed polytopes (hypersimplices).
Any pulling refinement of this subdivision is unimodular.
Semidirect product

Join and Cartesian product also preserve existence of unimodular triangulations.
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**Definition**

Let $Q \subset \mathbb{R}^d$ and $P_i \subset \mathbb{R}^{d_i}$ for $i = 1, \ldots, n$ be lattice polytopes, and let $\phi : \mathbb{Z}^d \to \mathbb{Z}^n$ be an integer affine map with $\phi(Q) \subset \mathbb{R}_{\geq 0}$. 
Join and Cartesian product also preserve existence of unimodular triangulations. We generalize both (plus dilations) to the following definition.

**Definition**

Let \( Q \subset \mathbb{R}^d \) and \( P_i \subset \mathbb{R}^{d_i} \) for \( i = 1, \ldots, n \) be lattice polytopes, and let \( \phi : \mathbb{Z}^d \rightarrow \mathbb{Z}^n \) be an integer affine map with \( \phi(Q) \subset \mathbb{R}_{\geq 0} \). The **semidirect product** of \( Q \) and the tuple \((P_1, \ldots, P_n)\) along \( \phi \) is

\[
Q \ltimes_{\phi} (P_1, \ldots, P_n) := \text{conv}_{a \in Q} \left( \{a\} \times \prod \phi_i(a) P_i \right),
\]

where \((\phi_1, \ldots, \phi_n)\) are the coordinates of \( \phi \).
Semidirect product

This includes:

- $\Delta^d \rtimes_{\text{Id}} (P_0, \ldots, P_d)$ is the join of $P_0, \ldots, P_d$, 

$(\ast)$ This is related to the so-called Nakajima polytopes, whose associated toric singularities are l.c.i.
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- \( \Delta^d \rtimes_{\text{Id}} (P_0, \ldots, P_d) \) is the join of \( P_0, \ldots, P_d \),
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- \( \{ \text{pt} \} \ltimes_k \left( P \right) \) is the \( k \)-th dilation of \( P \).
- The chimney \( \text{chim}(Q, \ell, u) \) (Haase-Paffenholz 2007) associated to two functionals \( \ell \leq u \) on \( Q \) is the semidirect product \( Q \ltimes_{u-\ell} I \), where \( I \) is a unimodular segment.
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Semidirect product

**Theorem (HPPS 2014+)**

If $Q, P_1, \ldots, P_n$ admit unimodular triangulations, then every semidirect product $Q \ltimes_\phi (P_1, \ldots, P_n)$ admits one too.
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Remark

Semidirect product is essentially equivalent to nested configurations [Aoki et al. 2008]. Aoki et al. prove the theorem above under the assumption that all factor triangulations are regular.
(Crystallographic) root systems give examples of particularly nice lattices. It seems natural to look at lattice polytopes related to them. We can do this in two ways:

- Polytopes **cut out by roots**: facet normals belong to the root system (**alcoved** polytopes).
- Polytopes with vertex sets contained in the root system.
Alcoved polytopes of type $A$

Payne (2009) has proved that all alcoved polytopes in the classical types $A$, $B$, $C$ and $D$ are integrally closed. This suggests they may all have unimodular triangulations.
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In type $A$ this is easy to show. Remember that.

\[ A_n = \{e_i - e_j : i, j \in [n+1]\} \subset \mathbb{R}^{n+1} \]

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Moreover, all cells in the arrangement are simplices.
Hence:
Alcoved polytopes of type $A$

**Theorem**

Let $P$ be an alcoved polytope of type $A$. The *dicing* triangulation obtained slicing $P$ by all lattice hyperplanes normal to the roots is a flag, regular, unimodular (that is, quadratic) triangulation of $P$. 

*Remark* If $\Delta = \text{conv}\{v_1, \ldots, v_n\}$ is any lattice simplex with its vertices given in a specific order, we can consider the linear map sending its facet normals to the (normals of) the simple roots of type $A_n$, in that order. The preimage of the $A$-dicing gives a canonical triangulation of $c\Delta$, for every $c \in \mathbb{N}$, into simplices of the same volume as $\Delta$. This canonical triangulation will be important in our proof of the KMW theorem.
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Alcoved polytopes in other types

**Theorem (HPPS-2014+)**

Every alcoved polytope \( P \) of type \( B \) has a regular unimodular triangulation
Alcoved polytopes in other types

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*Every alcoved polytope $P$ of type $B$ has a regular unimodular triangulation*

Sketch of proof.

First slice $P$ by the hyperplanes corresponding to the “short roots” of type $B$. This gives a regular subdivision into compressed cells. Any pulling refinement of this is unimodular.

The triangulation in the theorem may need to use simplices that are not alcoved.
Alcoved polytopes in other types

Theorem (HPPS-2014+)

Every alcoved polytope $P$ of type $B$ has a regular unimodular triangulation

Sketch of proof.

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The triangulation in the theorem may need to use simplices that are not alcoved.

For other types:

- In $F_4$ and $E_8$ we have explicit examples of polytopes without r.u.t.’s
- In $C_n$, $D_n$, $E_6$ and $E_7$ we do not know.
Polytopes spanned by roots

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**Theorem**

Let $S$ be a set of roots of type $A_n$. then $\text{conv}(S \cup \{O\})$ always has a regular unimodular triangulation (pulling from the origin).
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**Remark**

When $O \not\in \text{conv}(S)$, $\text{conv}(S)$ may not have any unimodular triangulation (e.g. $S = \{12, 23, 34, 45, 14, 35\}$). Or it may have unimodular triangulations but none of them flag (e.g. $S = \{12, 23, 13, 34, 45, 35\}$).
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In other types much less is known (Ohsugi-Hibi 2001 have some partial results).
Intro

Examples

Dilations

Polytopes related to root systems

Edge polytopes

For a given graph $G$, we denote $P_G$ the convex hull of its edge-vertex incidence vectors. (Put differently, edge-polytopes of graphs are the subpolytopes of the second hyper simplex $\{e_i + e_j : i, j \in [n]\}$).
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- $P_G$ has a unimodular cover iff no induced subgraph consists of two odd cycles (Ohsugi-Hibi, Simis-Vasconcelos-Villarreal, 1998).
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- Even if $P_G$ is integrally closed, it may not have a r.u.t. (Hibi-Ohsugi 1999).
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- $P_G$ has a unimodular cover iff no induced subgraph consists of two odd cycles (Ohsugi-Hibi, Simis-Vasconcelos-Villarreal, 1998).
- Even if $P_G$ is integrally closed, it may not have a r.u.t. (Hibi-Ohsugi 1999).
- The Hibi-Ohsugi example has a non-regular unimodular triangulation (Firla-Ziegler 1999).
Some smooth polytopes

Smooth empty polytopes

It is open whether every smooth (⇔ simple and with unimodular vertex-cones) has a unimodular triangulation ("Oda’s conjecture"). We can prove it if $P$ is empty (no other lattice point than its vertices).
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**Theorem (HPPS-2014+)**

*Every empty smooth polytope is lattice equivalent to a product of unimodular simplices.*

Our proof is inspired by Kaibel-Wolff’s 2000 proof of: “any simple 0/1-polytope is a product of 0/1-simplices”.
Some smooth polytopes

Smooth reflexive polytopes

A reflexive polytope is a lattice polytope $P$ with $O \in \mathcal{J}(P)$ and such that its polar is also a lattice polytope. (Equivalently, a lattice polytope that can be defined as $P = \{ Ax \leq 1 \}$ for an integer matrix $A$).
A reflexive polytope is a lattice polytope \( P \) with \( O \in \mathbb{Z}(P) \) and such that its polar is also a lattice polytope. (Equivalently, a lattice polytope that can be defined as \( P = \{ Ax \leq 1 \} \) for an integer matrix \( A \)).

There are finitely many reflexive polytopes in each dimension, and smooth ones have been enumerated up to \( d = 9 \). The following is known [Haase-Paffenholz 2007]:

**Smooth reflexive polytopes**

- All smooth reflexive \( d \)-polytopes for \( d \leq 8 \) are integrally closed.
- All smooth reflexive polytopes for \( d \leq 6 \) admit a quadratic triangulation.
- All but perhaps one (out of 72256) of the 7-dimensional smooth reflexive polytopes have a r. u. t.
A reflexive polytope is a lattice polytope $P$ with $O \in \int(P)$ and such that its polar is also a lattice polytope. (Equivalently, a lattice polytope that can be defined as $P = \{Ax \leq 1\}$ for an integer matrix $A$).

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The KMW Theorem

There is the following classical theorem of Knudsen, Mumford, and Waterman (1973):

**Theorem**

*Given a polytope* $P$, *there is a factor* $c = c(P) \in \mathbb{N}$ *such that the dilation* $c \cdot P$ *admits a regular unimodular triangulation.*
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This raises some questions:

- Is there a $c(d)$ common to every $d$-polytope?
- What is a (good?) bound on $c(P)$ for a given $P$?
- What is the structure of the set of valid $c(P)$’s of a given $P$? Is it additively closed? (It at least contains all its multiples).
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There are examples where $cP$ has a r.u.t. but $(c + 1)P$ is not integrally closed [Cox-Haase-Hibi-Higashitani 2012].
If we relax the requirement from r.u.t. to weaker properties, the following is known:

- $cP$ is **integrally closed** for every $c \geq d - 1$ [Bruns-Gubeladze-Trung, 1997].
- There is a $c_0(d) \in O(d^6)$ such that $cP$ has a unimodular cover for every $c \geq c_0$ [Bruns-Gubeladze 2002, von Thaden 2008].
In contrast:

- Neither the original KMW proof nor the reworking of it by Bruns and Gubeladze (2009) contains any explicit bound on the $c$ needed for a given $P$. 
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- Neither the original KMW proof nor the reworking of it by Bruns and Gubeladze (2009) contains any explicit bound on the $c$ needed for a given $P$.

- Working out a bound from those proofs is not easy, and would certainly lead to a tower of exponentials of length related to the volume of $P$. 
In contrast:

- Neither the original KMW proof nor the reworking of it by Bruns and Gubeladze (2009) contains any explicit bound on the $c$ needed for a given $P$.
- Working out a bound from those proofs is not easy, and would certainly lead to a tower of exponentials of length related to the volume of $P$.
- The regularity part of the proof is not totally clear (it is omitted in [Bruns-Gubeladze 2009]).
An effective KMW Theorem

We prove the following:

**Theorem (HPPS 2014+)**

If $P$ is a lattice polytope of dimension $d$ and (lattice) volume $\text{Vol}(P)$, then the dilation

$$d! \text{Vol}(P)!d^2 \text{Vol}(P) P$$

has a regular unimodular triangulation.

F. Santos

Unimodular triangulations of lattice polytopes
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We prove the following:

**Theorem (HPPS 2014+)**

*If* $P$ *is a lattice polytope of dimension* $d$ *and (lattice) volume* $\text{Vol}(P)$, *then the dilation*

$$d! \text{Vol}(P)! d^2 \text{Vol}(P) \ P$$

*has a regular unimodular triangulation. More precisely, if* $P$ *has a triangulation $\mathcal{T}$ into* $N$ *$d$-simplices, of volumes* $V_1, \ldots, V_N$, *then the dilation*

$$d! \sum_{i=1}^{N} V_i! ((d+1)!d^d)^{V_i-1} \mathcal{T}$$

*has a regular unimodular refinement.*
Our proof is not substantially different from the previous ones, but uses a better “book-keeping” based on the canonical triangulation of dilations of an ordered simplex:

**Definition**

An ordered simplex $\Delta$ is a simplex with its vertices given in a specified order.
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An **ordered simplex** $\Delta$ is a simplex with its vertices given in a specified order. The **canonical triangulation** of $c\Delta$ is the inverse image of the dicing triangulation of type $A$, under the natural affine map sending $\Delta$ to an alcoved simplex of type $A$. 
Canonical triangulation

Canonical triangulations glue together nicely; if $F$ is a face of $P$, the c.t. of $P$ restricted to $F$ is the c.t. of $F$. In particular:

---

**Lemma**

If $T$ is a triangulation of $P$, canonically refining each simplex of $cT$ produces a triangulation of $cP$ in which:

- Volume of simplices is preserved. (Each simplex in the final triangulation has the volume of the simplex of $T$ that it refines).
- Regularity and flagness are preserved.

(Remark: as a corollary we recover that "if $P$ has a quadratic triangulation then $cP$ has one, for every $c \in \mathbb{N}$".)
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Canonical refinement of a dilated simplex

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Reducing the volume of a single dilated simplex

Let \( \Delta \) be a non-unimodular simplex. Let \( \Lambda_\Delta \) be the lattice spanned by its vertices (rather, the linear lattice parallel to it...), so that
\[
\text{Vol}(\Delta) = \left| \mathbb{Z}^2 / \Lambda_\Delta \right|.
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A box point for a simplex $\Delta$ is non-degenerate if it is not in the (affine span) of any proper face of $\Delta$. 

![Diagram showing a dilated simplex with box points indicated]
Reduction the volume in a single simplex

**Lemma (Elementary volume reduction)**

If $\mathcal{T}$ is a lattice triangulation on an ordered set of vertices and $F = \{v_0, \ldots, v_k\}$ is a non-unimodular face with a non-degenerate box point $m = (m_0, \ldots, m_k) \in \mathbb{Z}^d \setminus \Lambda_F$, then for every integer $c \in (k + 1)\mathbb{N}$, $c \cdot \text{Star}(F; \mathcal{T})$ has a refinement $\mathcal{T}_m$ such that:

1. The volume of every full-dimensional simplex $\Delta'$ in $\mathcal{T}_m$ is strictly less than the volume of simplex $\Delta$ for which $\Delta' \subset c \Delta$.
2. $\mathcal{T}_m$ induces the canonical triangulation on the boundary $c \cdot \partial \text{Star}(F; \mathcal{T})$.
3. $\mathcal{T}_m$ is a regular refinement of $\mathcal{T}$, so if $\mathcal{T}$ is regular then $\mathcal{T}_m$ is regular.
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Reducing the volume in a single simplex
Reducing the volume in a single simplex

70 HAASE, PAFFENHOLZ, PIECHNIK, AND SANTOS

Figure 4.4. star(F; T) (left) and the subdivision of c·star(F; T) into Cayley polytopes via concentric dilated copies of ∂star(F; T) (right).
Reducing the volume in several simplices at a time

Remarks:

- If we have non-degenerate box-points $m_1, \ldots, m_N$ for a family of simplices $F_1, \ldots, F_N$ with disjoint stars, the reduction lemma can be applied simultaneously to all of them, to reduce the volumes in all stars simultaneously.
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- This happens, for example, for all simplices of prime volume:

**Corollary**

Let $T$ be a triangulation of a lattice polytope $P$ and assume that the maximal volume $V$ among all simplices in $T$ is a prime. Then $(d + 1)!T$ can be refined to a triangulation with all simplices of volume $< V$.

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...so, if every number was a prime, \( (d + 1)!V \mathcal{T} \) would have a unimodular refinement...
Reducing the volume in all simplices iteratively

What we can still do is apply the reduction lemma over and over, hoping that eventually we get rid off all simplices of maximal volume $V$, then go to those of volume $V - 1$, etc.
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If we do not process all simplices of volume $V$ at the same time, in the unprocessed ones what we get is the canonical refinement of the dilation, which creates a lot of new simplices of volume $V$. 
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If we do not process all simplices of volume $V$ at the same time, in the unprocessed ones what we get is the canonical refinement of the dilation, which creates a lot of new simplices of volume $V$. The number of simplices of volume $V$ will actually increase, not decrease.
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F. Santos

Unimodular triangulations of lattice polytopes
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Knudsen-Mumford-Waterman and Bruns-Gubeladze solve this via the use of “rational structures” or “local lattices”… which leads to a tower of exponentials.
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Knudsen-Mumford-Waterman and Bruns-Gubeladze solve this via the use of “rational structures” or “local lattices”. . . . which leads to a tower of exponentials. We solve it by taking advantage of some properties of canonical triangulations.
Definition

An ordered $k$-simplex is a simplex with a specified order in its vertices. Two ordered simplices $\Delta = \text{conv}\{p_0, \ldots, p_k\}$ and $\Delta' = \text{conv}\{p'_0, \ldots, p'_k\}$ are called $A$-equivalent if

$$\{p_i - p_{i-1} : i = 1 \ldots k\} = \{p'_i - p'_{i-1} : i = 1 \ldots k\}$$
Canonical refinement, revisited

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**Lemma (A-equivalence)**

1. All the simplices in the canonical triangulation of $c\Delta$ are $A$-equivalent to $\Delta$.
2. If two simplices $\Delta$ and $\Delta'$ are $A$-equivalent then the $A$-dicing defined by $\Delta$ and by $\Delta'$ are the same, modulo a translation.
Part (2) of the previous lemma allows us to consider a box point for a simplex $\Delta$ as a box point for any other $A$-equivalent simplex $\Delta'$ (by the unique, modulo $L_\Delta$ translation sending one $A$-dicing to the other).
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The crucial property that we need is:

**Lemma**

Let $\Delta$ and $\Delta'$ be two $A$-equivalent simplices in a triangulation $\mathcal{T}$, and let $m$ be a box point for both (in the above sense). Let $F$ and $F'$ be the faces of $\Delta$ and $\Delta'$ for which $m$ is non-degenerate. Then, either $F = F'$ or they have disjoint stars.
Thus:

Corollary

The elementary volume reduction can be applied simultaneously to all simplices of a given A-equivalence class.
Canonical refinement, revisited

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The elementary volume reduction can be applied simultaneously to all simplices of a given A-equivalence class.

**Corollary**

Let $T$ be a triangulation of a lattice polytope $P$ and let $V$ be the maximal volume $V$. Let $N$ be the number of A-equivalence classes of maximal simplices of volume $V$ in $T$. Then, $(d + 1)!^N T$ can be refined to a triangulation with all simplices of volume $< V$.  

F. Santos  Unimodular triangulations of lattice polytopes
An effective KMW Theorem

An algorithm

To get a unimodular refinement of \( cP \) for some constant \( c \):

1. Construct any lattice triangulation \( T \) of \( P \). Let \( V \) be the maximal volume among its simplices and \( N \) the number of \( A \)-equivalence classes of them.

2. While \( N > 0 \), apply the reduction lemma to all the simplices in one of the \( A \)-equivalence classes of volume \( V \). This reduces by (at least) one the number of them.

3. At the end of step 2 all simplices have volume bounded by a \( V' < V \). Iterate.
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Remark: in all steps regularity of the triangulation can be preserved.
Each elementary reduction substitutes an equivalence class of $A$ simplices of a certain volume $V$ to at most $(d + 1)d!^d$ classes of simplices of volume at most $V - 1$. 
An effective KMW Theorem

Analysis of the algorithm

- Each elementary reduction substitutes an equivalence class of $A$ simplices of a certain volume $V$ to at most $(d + 1)d!^d$ classes of simplices of volume at most $V - 1$.
- Setting up the corresponding recursion, to unimodularly refine a dilation of a particular $A$-class of volume $V$, at most $V! \left((d + 1)!c^d\right)^{V-1}$ $A$-equivalence classes of intermediate simplices are used.
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Theorem (HPPS 2014+)

If a lattice polytope $P$ of dimension $d$ has a triangulation into $N$ d-simplices, of volumes $V_1, \ldots, V_N$, then the dilation of factor

$$d! \left( \sum_{i=1}^{N} V_i! \left( (d+1)! (d!)^d \right)^{V_i-1} \right)^T$$

has a regular unimodular refinement.
Dimension 3

In dimension three the following is known:

- For every lattice 3-polytope $P$, $2P$ has a unimodular cover (Ziegler 1997, Kantor-Sarkaria 2003).

Theorem (SZ 2013)
For every $c \geq 4$ except perhaps $c = 5$ and for every lattice 3-polytope $P$, $cP$ has a unimodular triangulation. 

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THANK YOU