Symplectic representations and
Riemann surfaces with many automorphisms

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Introduction. In this note I will describe a constructive method to calculate the representation of the automorphism group $Aut(X)$ on the first homology group $H_1(X)$ where X is a Riemann surface with many automorphisms. Therefore we describe a combinatorial method to calculate a basis of $H_1(X)$, the matrices describing the action of $Aut(X)$ on this basis and the intersection matrix S of this basis. Clearly this is equivalent to the calculation of a symplectic representation of $Aut(X)$ of degree $2g$ where $g \geq 1$ is the genus of X. There are various applications for which the explicit calculation of these matrices are of interest. One application is to get by a further splitting in two g–dimensional representation the action of $Aut(X)$ on the abelian differetials of the first kind. In some cases there is also the possibility to calculate period matrices of the surface as fixed points of the action of the symplectic representation on the Siegel upper half plane, cf. [Bol,Schi]. We obtain the following generalization of a result in [Str]:

Theorem 1. The representation of $Aut(X)$ for X with many automorphisms and genus $g \geq 1$ on $H_1(X)$ is a subrepresentation of the regular representation.

The proof of this result is fully constructive and the algorithm is implemented in GAP [Gap] and can be obtained from the author. Lemma 1 and Lemma 2 also appear in [Str] and are included for the convenience of the reader.

Part I

Let G be a finite group generated by two elements g_0, g_1, such that the inequality

$$q := \frac{1}{k_0} + \frac{1}{k_1} + \frac{1}{k_\infty} \leq 1 \tag{1}$$

is satisfied, where $k_i := ord(g_i)$, $i = 0, 1, \infty$ and $g_\infty := (g_0 g_1)^{-1}$. This is the natural situation one can find for Riemann surfaces with many automorphisms, i.e. for any regular hypermap, see [JoSi, JSt, Wo]. We therefore want to describe the construction of a Riemann surface X admitting G as a (sub)group of conformal automorphisms. Let

$$\Delta_{k_0,k_1,k_\infty} := \langle \gamma_0, \gamma_1, \gamma_\infty \mid \gamma_0^{k_0} = \gamma_1^{k_1} = \gamma_\infty^{k_\infty} = \gamma_0\gamma_1\gamma_\infty = id \rangle$$

be a Fuchsian or plane triangle group. For the following we fix the geometric action of γ_i, $i = 0, 1, \infty$. We want them to be anticlockwise rotations with rotation angles $2\pi/k_i$ around the non accidental vertices of a canonical fundamental domain of the triangle group $\Delta_{k_0,k_1,k_\infty}$. We then define an homomorphism $\varphi : \Delta_{k_0,k_1,k_\infty} \longrightarrow G$, $\gamma_i \longmapsto g_i$, $i = 0, 1, \infty$. The kernel of this epimorphism $ker(\varphi) =: \Gamma$ defines a compact Riemann surface X taking $X := \Gamma\backslash\mathfrak{H}$ for $q < 1$ or $X := \Gamma\backslash\mathbb{C}$ for $q = 1$. The genus g of X can be easily calculated by

$$2g - 2 := |G|(1 - q). \tag{2}$$

It is well known that $Aut(X) \supseteq \Delta_{k_0,k_1,k_\infty}/\Gamma$. In fact $Aut(X) = \Delta_{k_0,k_1,k_\infty}/\Gamma$ is the generic situation as $Aut(X) \supset \Delta_{k_0,k_1,k_\infty}/\Gamma$ holds only if $g = 1$ or ifΓ is normally contained in some triangle group which also contains $\Delta_{k_0,k_1,k_\infty}$ and this is quite rare.

We now give a construction of the representation of $\Delta_{k_0,k_1,k_\infty}/\Gamma$ on the first homology group $H_1(X)$ as a subrepresentation of the regular representation. This is a generalization of the construction in [Str] because no further restrictions to the group G is necessary. Let $V := \langle e_g \mid g \in G\rangle$ be a $|G|$–dimensional $\mathbb{C}$–vector space. A basis $\{e_g \mid g \in G\}$ is labeled by the group elements of G. We define two linear representations ρ, λ on V. ρ stands for multiplication from the right and for any group element $h \in G$ we define $\rho(h)$ on our basis $\{e_g \mid g \in G\}$ by:

$$\rho(h) : e_g \longmapsto e_{gh}, \quad g \in G.$$

With λ we describe the analogous situation with multiplication from the left:

$$\lambda(h) : e_g \longmapsto e_{hg}, \quad g \in G.$$

Lemma 1 The eigenspaces $V_i := Eig(\rho(g_i), 1)$ of $\rho(g_i)$, $i = 0, 1, \infty$ to the eigenvalue 1 have dimension $|G|/k_i$. The space V_i is λ–invariant.

Proof. Let $\underline{x} = \sum \mu_g e_g$ an element of V_i. This can be expressed in the condition (A_i)

$$(A_i): \ \rho(g_i)\underline{x} = \underline{x}$$

or equivalently

$$(A_i): \underline{x} \in V_i \Longleftrightarrow \mu_g = \mu_{gg_i} \text{ for all } g \in G.$$

Then it is easy to see that V_i is generated by the orbit of $e_{g_i} + \ldots e_{g_i^{k_i}}$ under λ. As an induced representation V_i has the decomposition

$$\bigoplus_{\tilde{g} \in G/H_i} \langle e_{\tilde{g}g_i} + \ldots + e_{\tilde{g}g_i^{k_i}}\rangle,$$

where $H_i := \langle g_i\rangle$, $i = 0, 1, \infty$ is the subgroup of generated by g_i. The character of this induced representation is denoted by $\chi_i := Ind(1, H_i)$. □

Lemma 2 The following statements are true for $i \neq j$, $i, j = 0, 1, \infty$
i) $V_i \cap V_j = \langle \sum_{g \in G} e_g\rangle$,
ii) $dim(V_i + V_j) = |G|/k_i + |G|/k_j - 1$,
iii) The character of the restriction $res(\lambda, V_i + V_j)$ of λ to $V_i + V_j$ is $\chi_i + \chi_j - 1$.

Proof. It is enough to prove i) as ii) and iii) are simple consequences of i). Let $\underline{x} = \sum \mu_g e_g \in V_i \cap V_j$. This implies that the conditions (A_i) and (A_j) hold simultaneously. We therefore find for $\underline{x}$:

$$\mu_g = \mu_{gg_i} \text{ for all } g \in G$$
$$\mu_g = \mu_{gg_j} \text{ for all } g \in G.$$

As $G = \langle g_i, g_j\rangle$ we find $\mu_g = \mu_{id}$ for all $g \in G$. This proves the statements. □

Lemma 3: Let E be the $|G| \times |G|$ identity matrix. Then the following statements are true:
i) $dim((E - \rho(g_1))V_0) = |G|/k_0 - 1$,
ii) $(E - \rho(g_1))V_0 \perp V_1 + V_\infty$,
iii) $dim((E - \rho(g_1))V_0 + V_1 + V_\infty) = |G|(1/k_0 + 1/k_1 + 1/k_\infty) - 2$,
iv)$((E - \rho(g_1))V_0 + V_1 + V_\infty)^\perp$ has dimension $2g$. λ restricted $((E - \rho(g_1))V_0 + V_1 + V_\infty)^\perp$ is isomorphic to the representation of $Aut(X)$ on $\mathbb{C} \otimes H_1(X)$ and has the character $2 - \chi_0 - \chi_1 - \chi_\infty$.

Proof. i) The kernel of $(E - \rho(g_1))$ is V_1. From Lemma 2i) we know $V_0 \cap V_1 = \langle \sum e_g\rangle$.

This shows the statement.
ii) V_0 has a basis

$$\left\{ \sum_{s=1}^{k_0} e_{\tilde{g}g_0^s} \mid \tilde{g} \,\epsilon\, G/\langle g_0 \rangle \right\}.$$

The image of V_0 under $(E - \rho(g_1))$ is generated by

$$\left\{ \sum_{s=1}^{k_0} e_{\tilde{g}g_0^s} - \sum_{s=1}^{k_0} e_{\tilde{g}g_0^s g_1} \mid \tilde{g} \,\epsilon\, G/\langle g_0 \rangle \right\}.$$

We prove orthogonality to V_∞ first. Let $\sum_{s=1}^{k_0} e_{\tilde{g}g_0^s} - \sum_{s=1}^{k_0} e_{\tilde{g}g_0^s g_1}$, $\sum_{r=1}^{k_\infty} e_{gg_\infty^r}$ be any two generating vectors of V_0, resp. V_∞.

$$\begin{aligned}
\langle \sum_{s=1}^{k_0} e_{\tilde{g}g_0^s} - \sum_{s=1}^{k_0} e_{\tilde{g}g_0^s g_1}, \sum_{r=1}^{k_\infty} e_{gg_\infty^r} \rangle &= \sum_{s=1}^{k_0}\sum_{r=1}^{k_\infty} \langle e_{\tilde{g}g_0^s}, e_{gg_\infty^r} \rangle - \sum_{s=1}^{k_0}\sum_{r=1}^{k_\infty} \langle e_{\tilde{g}g_0^s g_1}, e_{gg_\infty^r} \rangle \\
&= \sum_{s=1}^{k_0}\sum_{r=1}^{k_\infty} \langle e_{\tilde{g}g_0^s}, e_{gg_\infty^r} \rangle - \sum_{s=1}^{k_0}\sum_{r=1}^{k_\infty} \langle e_{\tilde{g}g_0^{s-1} g_\infty^{-1}}, e_{gg_\infty^r} \rangle \\
&= \sum_{s=1}^{k_0}\sum_{r=1}^{k_\infty} \langle e_{\tilde{g}g_0^{s-1}}, e_{gg_\infty^r} \rangle - \sum_{s=1}^{k_0}\sum_{r=1}^{k_\infty} \langle e_{\tilde{g}g_0^{s-1} g_\infty^{-1}}, e_{gg_\infty^{(r-1)}} \rangle
\end{aligned}$$

The first equation is a consequence of the group relation $g_0 g_1 g_\infty = id$. In the last equation we have changed the summation index in the second sum from r to $r-1$. The entries in the occuring sums are always equal to 0 or 1. We find

$$\begin{aligned}
\langle e_{\tilde{g}g_0^{s-1}}, e_{gg_\infty^r} \rangle = 1 &\iff \tilde{g}g_0^s = gg_\infty^r \iff \tilde{g}g_0^{s-1} g_\infty^{-1} = gg_\infty^{r-1} \\
&\iff -\langle e_{\tilde{g}g_0^{s-1} g_\infty^{-1}}, e_{gg_\infty^{r-1}} \rangle = -1.
\end{aligned}$$

Therefore we have

$$\langle \sum_{s=1}^{k_0} e_{\tilde{g}g_0^s} - \sum_{s=1}^{k_0} e_{\tilde{g}g_0^s g_1}, \sum_{r=1}^{k_\infty} e_{gg_\infty^r} \rangle = 0.$$

The orthogonality to V_1 follows with the same type of argument. This shows ii).
iii) This is a simple consequence of ii) together with equality (2).
iv) There are two ways of proving the statements of iv). The first method uses a theorem of Chevalley and Weil [ChW] which enables us to calculate how often a given irreducible character χ occurs in $Aut(X)$ on $\mathbb{C} \otimes H_1(X)$. It turns out to be the same as the scalar product$(\chi, 2 - (\chi_0 + \chi_1 + \chi_\infty))$. A detailed proof of this can be found in [Str]. The second way is to construct a basis of $((E - \rho(g_1))V_0 + V_1 + V_\infty)^\perp$ and give it a direct geometric interpretation as a homology basis of the $\mathbb{Z}$ module $H_1(X)$ ie. as closed curves on X. We do this in part II and give also a method to construct the intersection matrix for this basis. iv) follows from this construction.
□

Part II

To explain how to calculate a basis of the space $((E - \rho(g_1))V_0 + V_1 + V_\infty)^\perp$ it is more convenient to work with permutations. Let $(g_1, \dots, g_{|G|})$ be a fixed list of group elements. Associated with this list we define a basis $e_i := e_{g_i}$, $i = 1 \dots |G|$. The regular representations λ, ρ can then be expressed by permutations:

$$\lambda(g) \longleftrightarrow \alpha_g \text{ with } gg_i = g_{\alpha_g(i)} \text{ and}$$

$$\rho(g) \longleftrightarrow \pi_g \text{ with } g_i g = g_{\pi_g(i)}.$$

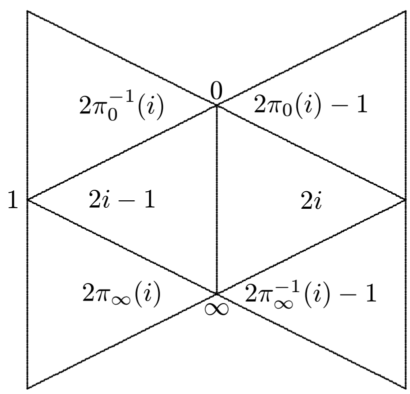
for all $g, g_i \in G$. For the following we only need the group generators g_0, g_1, g_∞ which we have introduced in part I. Note that the indices $0, 1, \infty$ are introduced to refer to the so called *Belyi* functions and have nothing to do with the ordering of the above list where the same group elements appear again with a different label. The triple $(\pi_0, \pi_1, \pi_\infty)$ is called a regular hypermap. The triple $(\alpha_0, \alpha_1, \alpha_\infty)$ is the automorphism group of the regular hypermap. It is known that the regular hypermap carries all the information one needs to construct a connected fundamental domain of the group Γ from a fundamental domain of the triangle group $\Delta_{k_0,k_1,k_\infty}$ [JoSi]. This is equivalent to construct a decomposition of the surface $X = \Gamma\backslash\mathfrak{H}$ into triangles. One can proceed as follows:

1. Take $2|G|$ triangles $\triangle_1, \dots, \triangle_{2|G|}$ and label the vertices of each triangle $\triangle_i$ with $0, 1, \infty$ in positive sense whenever i is odd and in negative sense when i is even. In order to construct a fundamental domain we have to identify the edges of these triangles in saying for instance the triangle i and the triangle j have a common edge of type $(0, 1)$. Another way to give the same information is to start a walk around the vertex 0 of the triangle i in say positive sense and list all the triangles we are passing by in order of appearance $i, i_1, i_2, \dots, i_k$. Remark that an even triangle is always followed by an odd triangle and an odd triangle by an even.

2. The situation we find for regular hypermaps allows us to start to construct regular $2k_0$–gons in putting all the triangles together with a common vertex 0. The order of appearance is given by the permutation π_0. In positive sense we meet the triangles

$$[2i-1, 2i, 2\pi_0(i)-1, 2\pi_0, 2\pi_0^2(i)-1, 2\pi_0^2(i), \dots, 2\pi_0^{k_0-1}(i)-1, 2\pi_0^{k_0-1}(i)].$$

3. We now have to deal with $|G|/k_0$ many $2k_0$–gons to construct our fundamental domain. The only remaining edges are of type $(1, \infty)$ as all vertices of type 0 are mid points of the $2k_0$–gons. The permutation π_1 encodes all the information we need to identify those edges which are to be identified to obtain either a connected fundamental domain or even the surface itself. To construct the fundamental domain we would start with any $2k_0$–gon. An anticlockwise walk around a vertex of type 1 of say an even triangle $2i$ in this $2k_0$–gon brings us to the triangle $2\pi_1(i)-1$. So we get an identification with another $2k_0$–gon and we glue them along this $(1, \infty)$–edge together which leaves us with one remaining $2k_0$–gon less. It is quite clear that we can repeat this procedure as long as we have remaining $2k_0$–gons and that we finish with a connected fundamental domain of our group Γ. But we have not used all the information we can get from π_1 as it also reveals all the side identifications of the so constructed fundamental domain. In the following picture we show all the adjacent triangles of the triangle pair $\triangle_{2i-1}, \triangle_{2i}$ and it is obvious that this is enough information to construct the surface X.



Algorithmically the gluing together process can be described by inserting lists like

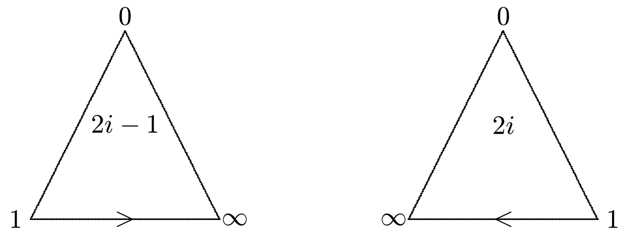
$$[2\pi_\infty^{-1}(i)-1, 2\pi_\infty^{-1}(i), 2\pi_0(\pi_\infty^{-1}(i))-1, \dots]$$

which represents a $2k_0$–gon into a starting list $[2i-1, 2i, \dots]$. The result is

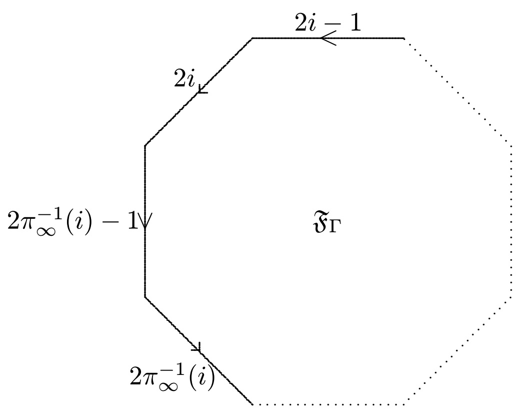
$$[2i-1, 2i, [2\pi_\infty^{-1}(i)-1, 2\pi_\infty^{-1}(i), 2\pi_0(\pi_\infty^{-1}(i))-1 \dots] \dots]. \tag{3}$$

Repeating this process until we have placed all the $2k_0$–lists we will end up with a list $\mathcal{L}$ with entries either natural numbers or lists of natural numbers and this characterization holds at each level. This simply means that every entry of each list which is a list contains natural numbers and lists of natural numbers. This list $\mathcal{L}$ clearly is a precise description of the fundamental domain $\mathfrak{F}_\Gamma$.

4. Now we associate to each triangle $\triangle_i$ a directed edge c_i joining 1 and ∞.

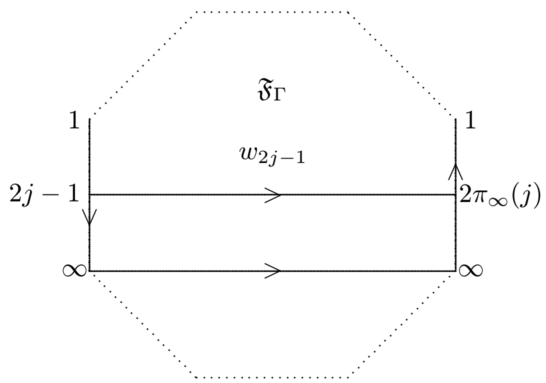


With the help of this directed edges we get a new interpretation of the list $\mathcal{L}$. Reading the indices appearing in $\mathcal{L}$ from the left to the right and ignoring the different levels of the list we obtain a walk around the fundamental domain $\mathfrak{F}_\Gamma$. Therefore we need to go $-c_i$ when i appears and when i is even and c_i if i appears and i is odd. If we realize this walk on the surface X we have a 0–homotopic closed curve which can be thought to be the border of the fundamental domain $\mathfrak{F}_\Gamma$ cancelling subsequently edges if they differ only by their orientation. According to the list (3) a part of the border of the fundamental domain $\mathfrak{F}_\Gamma$ would look as follows.



5. In order to describe a basis of $H_1(X)$ we consider all the side identification which we obtain through π_∞. We know that the $(1,\infty)$–edge of the triangle $\triangle_{2j-1}$ has to be identified with the $(1,\infty)$–edge of the triangle $\triangle_{2\pi_\infty(j)}$. Therefore we find

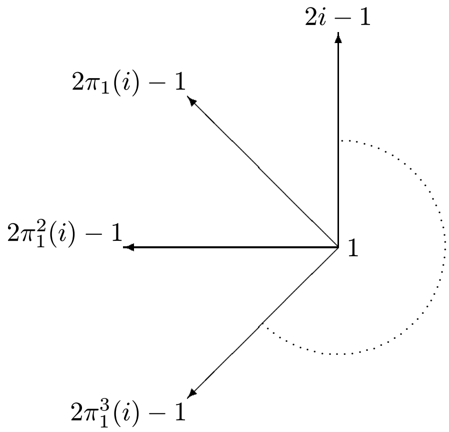
a closed curve w on X which is represented by a curve w_{2j-1} cutting through $\mathfrak{F}_\Gamma$ and connecting the midpoints of the $(1,\infty)$–edges of both of the triangles. The convention to start at a mid point of a $(1,\infty)$–side is not necessary. One also could start at the ∞–point of the triangle $\triangle_{2j-1}$ as shown by the picture below as both curves are freely homotopic. But then it is obvious that this curve w_{2j-1} is homotopically equivalent to a curve $\sum \pm c_i$ along the border of $\mathfrak{F}_\Gamma$. Which way to go can be easily read off by the list $\mathcal{L}$. We just have to go all the c_i with i standing in $\mathcal{L}$ in between $2j-1$ and $2\pi_\infty(j)$. It is also obvious that any closed curve on X can be deformed such that it meets the border of $\mathfrak{F}_\Gamma$ only transversally. Therefore $\{w_{2j-1} | 1 = 1, \ldots, |G|\}$ contains a basis of $H_1(X)$.



6. If we wish to choose a basis of $H_1(X)$ in $\{w_{2j-1} | 1 = 1, \ldots, |G|\}$ we best consider the intersection matrix $\tilde{S} = (w_{2i-1} \cdot w_{2j-1})_{i,j=1,\ldots,|G|}$. Like every intersection matrix $\tilde{S}$ is skew symmetric. From our construction it is clear that w_{2i-1} and w_{2j-1} meet at most once. Therefore the entries of S are either 0 or ± 1. (This answers a question of Rodriguez and Riera [RiRo]. They observed that in all known examples of explicitly constructed intersection matrices the entries only consist of $0, \pm 1$. But this automatically happens if you choose to construct a homology basis in the described manner, i.e. as paths along the border of a fundamental domain.) Again the list $\mathcal{L}$ gives all the information one needs to calculate the intersection number as the list gives an ordering of the border of $\mathfrak{F}_\Gamma$. The whole task of choosing a basis therefore is to choose a submatrix S of $\tilde{S}$ with maximal rank which clearly is $2g$ where g is the genus of X. Therefore we can define a basis $v_k := w_{2j_k-1}$, $k = 1, \ldots, 2g$ where $S = (w_{2j_k-1} \cdot w_{2j_l-1})_{k,l=1,\ldots,2g} = (v_k \cdot v_l)_{k,l=1,\ldots,2g}$.

7. We now return to the notations of part I to give the connection of the construction above to the space $((E - \rho(g_1))V_0 + V_1 + V_\infty)^\perp$. We map the curves c_i, $i = 1, \ldots, 2|G|$ into the space $\langle e_g | g \epsilon G \rangle$. This will be done by the definition $c_{2i-1} \leftrightarrow e_{g_i}$. Note that $c_{2i-1} = c_{2\pi_\infty(i)}$. By this definition we can map $H_1(X)$ into the space $\langle e_g | g \epsilon G \rangle$. An interesting space is V_1. A generating element of V_1 is of type say $e_{g_i} + e_{g_i g_1} + \cdots + e_{g_i g_1^{k_1 - 1}}$. This element is the image of a directed star $c_{2i-1} + c_{2\pi_1(i)-1} + \cdots +$

$c_{2\pi_1^{k_1-1}(i)-1}$.



These are all directed edges which start at the point of type 1 of the triangle $\triangle_{2i-1}$. Let w be the image of $w \,\epsilon\, H_1(X)$ under the above embedding. The scalar product $\langle w, e_{g_i} + e_{g_i g_1} + \cdots + e_{g_i g_1^{k_1-1}} \rangle$ simply counts how often any of the edges of the directed star $c_{2i-1} + c_{2\pi_1(i)-1} + \cdots + c_{2\pi_1^{k_1-1}(i)-1}$ are used counting a going in negatively and a going out with a positive sign. Therefore $\langle w, e_{g_i} + e_{g_i g_1} + \cdots + e_{g_i g_1^{k_1-1}} \rangle = 0$ as w is closed. We see that w is an element of $V_1^{\perp}$. The same argument shows that $w \,\epsilon\, V_\infty^{\perp}$. Therefore the images of the above constructed basis $v_1, \ldots, v_{2g} \,\epsilon\, H_1(X)$ automatically fall in $(V_1 + V_\infty)^{\perp}$. The generating elements of $(E - \rho(g_1))V_0$ are of type $e_{g_i} - e_{g_i g_1} + e_{g_i g_0} - e_{g_i g_0 g_1} + \ldots + e_{g_i g_0^{k_0-1}} - e_{g_i g_0^{k_0-1} g_1}$. This is the directed border of a $2k_0$–gon from which we have bild up the fundamental domain $\mathfrak{F}_\Gamma$. Therefore the associated closed curve in $H_1(X)$ is 0. The basis $v_1, \ldots, v_{2g}$ project therefore uniquely and well defined to the vector space elements $v_1, \ldots, v_{2g}$ in $((E - \rho(g_1))V_0 + V_1 + V_\infty)^{\perp}$ with a well defined intersection matrix S. This finally shows iv) of Lemma 3 and Theorem 1.

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