SEMIDEFINITE NETWORK GAMES: MULTIPLAYER MINIMAX AND COMPLEMENTARITY PROBLEMS

CONSTANTIN ICKSTADT, THORSTEN THEOBALD, ELIAS TSIGARIDAS, AND ANTONIOS VARVITSIOTIS

Abstract. Network games provide a powerful framework for modeling agent interactions in networked systems, where players are represented by nodes in a graph, and their payoffs depend on the actions taken by their neighbors. Extending the framework of network games, in this work we introduce and study semidefinite network games. In this model, each player selects a positive semidefinite matrix with trace equal to one, known as a density matrix, to engage in a two-player game with every neighboring node. The player's payoff is the cumulative payoff acquired from these edge games. Network semidefinite games are of interest because they provide a simplified framework for representing quantum strategic interactions. Initially, we focus on the zero-sum setting, where the sum of all players' payoffs is equal to zero. We establish that in this class of games, Nash equilibria can be characterized as the projection of a spectrahedron. Furthermore, we demonstrate that determining whether a game is a semidefinite network game is equivalent to deciding if the value of a semidefinite program is zero. Beyond the zero-sum case, we characterize Nash equilibria as the solutions of a semidefinite linear complementarity problem.

1. Introduction

In normal form games, agents use probability distributions to select one action from a finite set, while their utility functions are multilinear. In a broader context, continuous games enable agents to choose from an infinite set of pure strategies, often constrained to a compact set. Computationally, continuous games involve calculations with polynomial roots instead of just manipulating rational numbers, and their solvability relies on algorithmic techniques that go beyond linear programming.

Among the numerous applications of continuous games, there has been significant interest in quantum games. Quantum game theory has recently emerged as a powerful tool for analyzing the interactions of quantum-enabled agents, which can process and exchange information in accordance with the laws of quantum mechanics. The existing literature on quantum games covers a broad range of topics, including tractable representations of quantum strategies [12, 7], complexity aspects [2], supremacy of quantum resources in classical games [9, 22], and more recently, no-regret learning [15, 21, 20].

The foundational concept of a Nash equilibrium [23] has been extended to the quantum regime, leading to the notion of a Quantum Nash Equilibrium (QNE) [29]. Defined as an unentangled state that retains stability against unilateral quantum deviations, QNEs offer a plausible solution to a strategic quantum interaction. The

Date: December 9, 2023.

investigation into the computational complexity of QNEs has yielded both positive and negative results. On one hand, [14] demonstrated that QNEs in zero-sum games can be efficiently computed using semidefinite programming. In contrast, [2] showed that finding QNEs in general games is hard. Motivated by this, our work aims to investigate:

Is it possible to expand the range of settings where quantum Nash equilibria can be efficiently computed?

Model and results. The simplest model of a two-player quantum game is a semidefinite (SDP) game [14]. In a two-player SDP game, the strategy sets of the two players are described by positive semidefinite matrices with trace equal to one. We refer the reader to Section 2 for preliminaries and additional notation.

Building on [14], we introduce and study *semidefinite network games*, where (quantum-enabled) agents are interconnected in an undirected graph G and each edge corresponds to a two-player SDP game. In this setting, each player selects a single density matrix, which is used for playing all the games associated with its neighboring nodes, and their payoff is the cumulative payoff acquired from these edge games.

As the strategy space of each player is convex and compact, there always exists an equilibrium point, see, e.g., [10]. In this work we establish the following three results:

(1) In a zero-sum semidefinite network game, Nash equilibria can be found efficiently using semidefinite programming.
(2) A zero-sum semidefinite network game can be recognized using semidefinite programming.
(3) Beyond the zero-sum case, semidefinite Nash equilibria are solutions to a semidefinite Linear Complementarity problem.

Previous and related work. For two-player zero-sum SDP games, optimal strategies can be efficiently computed using semidefinite programming, see [16] and in the context of real SDP games [14]. Furthermore, it is shown in [14] that two-player zero-sum SDP games are closely related to semidefinite programming, extending Dantzig's seminal work on the near equivalence of bimatrix games and linear programming. SDP games fall within the class of continuous games, e.g., [1, 25], where the strategy space is a convex compact set and the payoffs are convex functions (in this case multilinear).

The results in this work are motivated by similar results for classical games played on networks. The network version of normal form games was introduced by Janovskaja [17] under the name *polymatrix game* and the connection to linear programming first appeared in [4, 3]. In particular, Brègman and Fokin [4] transform the polymatrix game to a two-player zero-sum polyhedral game, which in turn corresponds to an exponential size linear program. Daskalakis and Papadimitriou [8] consider (the computation of Nash equilibria of) polymatrix games, where each edge corresponds to a zero sum game. To compute the Nash equilibria they carefully round the optimal solution of a linear program. Cai and Daskalakis [6], among other results, directly relate the equilibria of a polymatrix zero sum game to a linear program and Cai et al. [5] show how to compute the equilibria of a zero sum polymatrix game using a polynomial size linear program.

2. The model

2.1. **Semidefinite games.** In a two-player SDP game, the strategy sets of the two players are described by density matrices X_1 and X_2, acting on complex Euclidean spaces $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Recall that a density matrix is a positive semidefinite matrix with trace equal to one. Consequently, the state of the joint system is in the product (i.e., unentangled) state $X_1 \otimes X_2$. Then, the players give their registers to a third party who measures the joint system and determines their utilities. The payoff at a strategy profile (X_1, X_2) is a bilinear function, i.e.,

$$u_1(X_1, X_2) = \langle R_1, X_1 \otimes X_2 \rangle \quad \text{and} \quad u_2(X_1, X_2) = \langle R_2, X_1 \otimes X_2 \rangle,$$

for Hermitian matrices $R_i \in \mathrm{Herm}(\mathcal{A}_i)$ and $\langle \cdot, \cdot \rangle$ corresponds to the Euclidean inner product of two matrices. Note that as R_i are Hermitian, it is ensured that the inner products $\langle R_i, X_1 \otimes X_2 \rangle$ result in real numbers.

2.2. **Semidefinite network games.** A semidefinite network game is a non-cooperative N-player game taking place on an undirected graph $G = ([N], E)$. The strategy space of player $i \in [N]$ is the set of density matrices acting on a register $\mathcal{A}_i$, i.e., $\mathcal{X}_i = \mathrm{D}(\mathcal{A}_i)$, where $\mathcal{A}_i$ is a finite-dimensional complex Euclidean space, i.e., isomorphic to $\mathbb{C}^k$ for some $k \geq 1$, and D is the space of density matrices, see (3.1). The set of strategy profiles $\mathcal{X} = \prod_{i=1}^N \mathcal{X}_i$ is the Cartesian product of the individual strategy spaces. Thus, any strategy profile $X \in \mathcal{X}$ is simply a collection of density matrices, one for every node of the underlying graph, i.e., $X = (X_1, \ldots, X_N)$, where $X_i \in \mathcal{X}_i$.

In the strategy profile $(X_1, \ldots, X_N) \in \mathcal{X}$, player i is using $X_i \in \mathcal{X}_i$ to play each two-player game corresponding to an edge edge $(i, j) \in E$. The payoffs of players i and j in the edge game corresponding to $(i, j) \in E$ is respectively given by

$$p_{i,j}(X_i, X_j) = \langle R_{ij}, X_i \otimes X_j \rangle \quad \text{and} \quad p_{j,i}(X_i, X_j) = \langle R_{ji}, X_i \otimes X_i \rangle,$$

for some Hermitian matrices $R_{ij}, R_{ji} \in \mathrm{Herm}(\mathcal{A}_i \otimes \mathcal{A}_j)$. The payoff of player i is the sum of the payoffs accrued from all the edge games in which they participate, i.e.,

$$p_i(X) = \sum_{(i,j) \in E} p_{i,j}(X_i, X_j).$$

The game is called zero-sum if $\sum_{i=1}^N p_i(X) = 0$, for all $X \in \mathcal{X}$. Finally, a profile $(X_1, \ldots, X_N) \in \mathcal{X}$ is a Nash Equilibrium if for each player i, the density matrix X_i is a best response to $X_{-i} = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_N)$, i.e.,

$$X_i \in \arg\max\{p_i(Y_i, X_{-i}) : Y_i \in \mathcal{X}_i\}, \quad \forall i \in [N]. \tag{NE}$$

Example 2.1. Let $G = ([N], E)$ be a connected graph with $|E| \geq 1$ and for any edge $e \in E$ let the two-player SDP game Γ_e associated with edge e be a constant-sum game. Then, the semidefinite network game is a constant-sum game, i.e., there exists a constant $C \in \mathbb{R}$ such that $\sum_{i=1}^N p_i(X) = C$ for every $X \in \mathcal{X}$. This game can be transformed into a zero-sum network SDP game as follows. For any $i \in [N]$, consider the neighbors j of i and construct, for $e = (i, j)$, from Γ_e the game Γ'_e by considering

$(p')_{i,j}(X_i, X_j) = p_{i,j}(X_i, X_j) - \frac{1}{N \deg i}$, where $\deg i$ is the degree of the node i in G. The game Γ'_e can be formulated as a two-player SDP game.

The resulting network game satisfies $\sum_{i=1}^{N} p'_i(X) = \sum_{i=1}^{N} \sum_{(i,j)\in E} (p')_{i,j}(X_i, X_j) = (\sum_{i=1}^{N} \sum_{(i,j)\in E} (p_{i,j}(X_i, X_j)) - C = 0$, and hence, Γ' is a zero-sum SDP game. If G is a non-connected graph with $|E| \geq 1$, then the normalization step to a zero-sum game can be carried out on an arbitrary component with at least one edge.

Example 2.2. Let $\hat{\Gamma}$ be a zero-sum network matrix game. By embedding the pure strategies of the edge games into the diagonal entries of the payoff tensors of a corresponding SDP game, we obtain a zero-sum SDP game Γ. To each edge game of Γ, adding a two-player zero-sum SDP game preservers the property of a zero-sum SDP game.

For example, let $\hat{\Gamma}$ be the following security game considered in [6]. Let G be a complete bipartite graph, where the nodes refer to evaders and inspectors. Every evader and every inspector can choose one of several given exits, which correspond to the pure strategies. If an evader's exit is not chosen by an inspector, then the evader obtains one unit. For every evader whose exit is inspected, the inspector obtains one unit. This is a constant-sum network matrix game, which can be turned into a zero-sum game and viewed as a semidefinite network game as explained before. Adding to each payoff tensor of an edge game the payoff tensor of an arbitrary zero-sum semidefinite game with the same dimensions retains the zero-sum property of the SDP game.

3. Linear Algebra Techniques

3.1. **Linear algebra.** First we give a brief introduction to the linear algebraic notions used in this work. Our notation closely follows [28].

Let $\mathcal{A}$ be a finite-dimensional complex Euclidean space. A Hermitian operator $A \in \operatorname{Herm}(\mathcal{A})$ is *positive semidefinite* (PSD), denoted $A \succeq 0$, if all its eigenvalues are nonnegative. The set of PSD operators acting on $\mathcal{A}$ is denoted by $\operatorname{Pos}(\mathcal{A})$. The set of density matrices acting on $\mathcal{A}$ is the set of PSD matrices with trace equal to one, i.e.,

$$\mathrm{D}(\mathcal{A}) = \Big\{ X \in \operatorname{Herm}(\mathcal{A}) : X \succeq 0 \text{ and } \operatorname{tr}(X) = 1 \Big\}. \tag{3.1}$$

The set of density matrices is convex, compact, and its extreme points are rank-one density matrices, also known as *pure states*, i.e., matrices of the form $X = xx^\dagger$, where $\|x\| = 1$, e.g., see [28]. For any Hermitian matrix $A \in \operatorname{Herm}(\mathcal{A})$ we have that

$$\begin{aligned} \lambda_{\max}(A) &= \max\{\langle A, X\rangle : X \in \mathrm{D}(\mathcal{A})\}, \\ \lambda_{\min}(A) &= \min\{\langle A, X\rangle : X \in \mathrm{D}(\mathcal{A})\}. \end{aligned} \tag{3.2}$$

This well-known fact follows from the Rayleigh-Ritz characterization of the maximum (resp. minimum) eigenvalue of a Hermitian matrix A, i.e.,

$$\lambda_{\max}(A) = \max\{x^\dagger A x : \|x\| = 1\},$$

combined with the characterization of the extreme points of $\mathrm{D}(\mathcal{A})$ given above, Finally, for any $A \in \mathrm{Herm}(\mathcal{A})$ we repeatedly use that $A \preceq t\,\mathbb{I}$ if and only if all the eigenvalues of A are at most t.

Let $\mathcal{A} = \mathbb{C}^n$ and $\mathcal{B} = \mathbb{C}^m$ be arbitrary fixed complex Euclidean spaces. By $L(\mathcal{A})$ we denote the space of linear maps $A : \mathcal{A} \to \mathcal{A}$ and similarly for $L(\mathcal{B})$. We use A to denote both the operator and its corresponding matrix (if we choose some coordinate system). We define the *partial trace*, $\mathrm{tr}_{\mathcal{A}} = \mathrm{tr} \otimes \mathbb{I}_{\mathcal{B}}$, as the map

$$\begin{array}{rcccl} \mathrm{tr}_{\mathcal{A}} & : & L(\mathcal{A}\otimes\mathcal{B}) & \to & L(\mathcal{B}) \\ & & A\otimes B & \mapsto & \mathrm{tr}_{\mathcal{A}}(A\otimes B) = \mathrm{tr}(A)\,B. \end{array} \tag{3.3}$$

The adjoint operator of $\mathrm{tr}_{\mathcal{A}}$ is

$$\begin{array}{rcccl} \mathrm{tr}_{\mathcal{A}}^{\dagger} & : & L(\mathcal{B}) & \to & L(\mathcal{A}\otimes\mathcal{B}) \\ & & B & \mapsto & \mathrm{tr}_{\mathcal{A}}^{\dagger}(B) = \mathbb{I}_{\mathcal{A}} \otimes B. \end{array} \tag{3.4}$$

We denote by $T(\mathcal{A},\mathcal{B})$ the space of linear maps $\Phi : L(\mathcal{A}) \to L(\mathcal{B})$, known as superoperators. There is a well-known bijection between $T(\mathcal{A},\mathcal{B})$ and $L(\mathcal{B}\otimes\mathcal{A})$ given by

$$\begin{array}{rcccl} J & : & T(\mathcal{A},\mathcal{B}) & \to & L(\mathcal{B}\otimes\mathcal{A}) \\ & & \Phi & \mapsto & J(\Phi) = \displaystyle\sum_{i\in[n],j\in[m]} \Phi(E_{i,j}) \otimes E_{i,j}, \end{array} \tag{3.5}$$

where $E_{i,j} = e_i e_j^{\dagger}$ is the operator that sends the canonical basis element e_j to the canonical basis element e_i. The corresponding matrix $J(\Phi)$ is called the Choi representation of Φ, see, e.g., [28]. Moreover, the action of $\Phi \in T(\mathcal{A},\mathcal{B})$ (on an element $A \in L(\mathcal{A})$) can be recovered from its Choi representation $J(\Phi)$ through

$$\Phi(A) = \mathrm{tr}_{\mathcal{A}}\left(J(\Phi)\,(\mathbb{I}_{\mathcal{B}} \otimes A^{\top})\right), \tag{3.6}$$

e.g., see [28, Equation 2.66]. We can express many properties of a superoperator Φ in terms of the Choi representation $J(\Phi)$. The following two properties, e.g. see Theorem 2.25 and Corollary 2.27 in [28], are related to our study:

- $J(\Phi)$ is a Hermitian matrix if and only if $\Phi : L(\mathcal{A}) \to L(\mathcal{B})$ is Hermitian preserving, i.e., $\Phi(X)$ is Hermitian whenever X is Hermitian.
- $J(\Phi)$ is PSD if and only if Φ is completely positive, i.e., for any integer $k \geq 1$ the map $\mathbb{I}_k \otimes \Phi$ maps PSD matrices to PSD matrices.

Semidefinite programming. Consider complex Euclidean spaces $\mathcal{A},\mathcal{B}$, operators $C \in \mathrm{Herm}(\mathcal{A}), B \in \mathrm{Herm}(\mathcal{B})$ and a Hermitian preserving superoperator $\Phi \in T(\mathcal{A},\mathcal{B})$. The triple (C, B, Φ) specifies a pair of primal/dual semidefinite programs (SDPs):

$$\begin{array}{rlrl} \sup & \langle C, X\rangle & \inf & \langle B, Y\rangle \\ \text{s.t.} & \Phi(X) = B, & \text{s.t.} & \Phi^{\dagger}(Y) \succeq C, \\ & X \in \mathrm{Pos}(\mathcal{A}), & & Y \in \mathrm{Herm}(\mathcal{B}). \end{array} \tag{3.7}$$

Here, $\langle \cdot, \cdot \rangle$ is the Frobenius scalar product, $\langle A, B\rangle := \mathrm{tr}(A\,B) = \sum_{i,j} a_{ij} b_{ij}$. SDPs represent a broad generalization of linear programming, offering significant expressive capabilities and efficient algorithms for their solution. The feasible region of an SDP is called a spectrahedron and the projection of a spectrahedron is called an SDP-representable set. There has been significant recent interest in determining whether a convex set, e.g., a polyhedron, is SDP-representable, e.g,, see [26].

3.2. **Operator representation of semidefinite network games.** The following operator representation of a semidefinite network game will be a crucial tool in the subsequent sections.

Lemma 3.1. *Consider a semidefinite network game with payoff functions*

$$p_{i,j}(X_i, X_j) = \langle R_{ij}, X_i \otimes X_j \rangle,$$

where $R_{ij} \in \mathrm{Herm}(\mathcal{A}_i \otimes \mathcal{A}_j)$*. If* $\Phi_{ij} : L(\mathcal{A}_j) \to L(\mathcal{A}_i)$ *is the linear map whose Choi representation is* R_{ij}*, then*

$$p_{i,j}(X_i, X_j) = \langle X_i, \Phi_{ij}({X_j}^\top) \rangle. \tag{3.8}$$

We refer to the maps Φ_{ij} *as the payoff operators.*

Proof. Note that

$$\begin{aligned} p_{i,j}(X_i, X_j) &= \langle R_{ij}, X_i \otimes X_j \rangle = \langle X_i \otimes \mathbb{I}_{\mathcal{A}_j}, R_{ij}(\mathbb{I}_{\mathcal{A}_i} \otimes X_j) \rangle \\ &= \langle \mathrm{tr}^\dagger_{\mathcal{A}_j}(X_i), R_{ij}(\mathbb{I}_{\mathcal{A}_i} \otimes X_j) \rangle = \langle X_i, \mathrm{tr}_{\mathcal{A}_j}(R_{ij}(\mathbb{I}_{\mathcal{A}_i} \otimes X_j)) \rangle = \langle X_i, \Phi_{ij}(X_j^\top) \rangle, \end{aligned}$$

where the second equality follows by the cyclicity of trace, the third equality from (3.4), and the last equality follows from (3.6). ☐

Remark 3.2. In the following sections, when referring to (3.8), we simplify notation by omitting the transpose. This may be interpreted in two ways: either as the agents choosing $X_i^\top$ or as including the transpose in the definition of the payoff operator.

For each player i, let $N(i)$ be the open neighborhood of i in the network. Moreover, for each strategy profile $X = (X_1, \ldots, X_N) \in \mathcal{X}$ we set $X_{N(i)} = (X_j : j \in N(i))$ as the vector of strategies of all neighbors of i. To keep our derivations compact we define the linear map $\Phi_i : \bigoplus_{j \in N(i)} L(\mathcal{A}_j) \to L(\mathcal{A}_i)$ where

$$\Phi_i(X_{N(i)}) = \sum_{j \in N(i)} \Phi_{ij}(X_j). \tag{3.9}$$

Although each Φ_i only depends on the density matrices chosen by the neighbors of i, it will be convenient to view Φ_i also as a function from $\bigoplus_{i=1}^N L(\mathcal{A}_i)$. Combining Lemma 3.1 with the definition of Φ_i, we can write the payoff of player i as a linear function of their strategy X_i explicitly, i.e.,

$$p_i(X) = \sum_{j:(i,j) \in E} p_{i,j}(X_i, X_j) = \sum_{j:(i,j) \in E} \langle X_i, \Phi_{ij}(X_j) \rangle = \langle X_i, \Phi_i(X_{N(i)}) \rangle. \tag{3.10}$$

Finally, we define the direct sum of the linear maps Φ_i, i.e.,

$$\Phi : \bigoplus_{i=1}^N L(\mathcal{A}_i) \to \bigoplus_{i=1}^N L(\mathcal{A}_i) \quad \text{where } (X_1, \ldots, X_N) \mapsto (\Phi_1(X), \ldots, \Phi_N(X)). \tag{3.11}$$

As the game is zero-sum, for each strategy profile $X = (X_1, \ldots, X_N) \in \mathcal{X}$ we have

$$\sum_i p_i(X) = \sum_i \langle X_i, \Phi_i(X) \rangle = 0. \tag{3.12}$$

Combining (3.11) and (3.9), (3.12) can be also written as

$$\langle X, \Phi(X) \rangle = 0 \text{ for all } X \in \mathcal{X}. \tag{3.13}$$

Next, set $\Psi = \Phi^\dagger$ and note that

$$\begin{aligned} &\Psi : \bigoplus_{i=1}^N L(\mathcal{A}_i) \to \bigoplus_{i=1}^N L(\mathcal{A}_i) \\ &\Psi(Y_1, \ldots, Y_N) \mapsto (\Psi_1(Y), \ldots, \Psi_N(Y)) \quad \text{where} \quad \Psi_i(Y) = \sum_{j \in N(i)} \Phi_{ji}^\dagger(Y_j). \end{aligned} \tag{3.14}$$

Indeed, on the one hand we have

$$\langle \Psi(Y_1, \ldots, Y_N), (X_1, \ldots, X_N) \rangle = \sum_i \langle \Psi_i(Y), X_i \rangle$$

and on the other hand

$$\begin{aligned} &\langle \Psi(Y_1, \ldots, Y_N), (X_1, \ldots, X_N) \rangle = \langle (Y_1, \ldots, Y_N), \Phi(X_1, \ldots, X_N) \rangle \\ = \; &\sum_i \langle Y_i, \Phi_i(X) \rangle = \sum_i \sum_{j \in N(i)} \langle Y_i, \Phi_{ij}(X_j) \rangle = \sum_i \langle \sum_{j \in N(i)} \Phi_{ji}^\dagger(Y_j), X_i \rangle. \end{aligned}$$

Finally, as $\Psi = \Phi^\dagger$, (3.13) implies that

$$\langle \Psi(X), X \rangle = 0 \text{ for all } X \in \mathcal{X}. \tag{3.15}$$

We next show that we can assume that the payoff operators are completely positive.

Lemma 3.3. *For any semidefinite network game there exists another semidefinite network game with the same Nash equilibria and completely positive payoff operators Φ_{ij}.*

Proof. Consider a new semidefinite network game where we replace each payoff matrix R_{ij} with $R_{ij} + c\,\mathbb{I}_{ij}$, where $\mathbb{I}_{ij}$ denotes the identity operator on $\mathcal{A}_i \otimes \mathcal{A}_j$ and c is chosen such that $R_{ij} + c\,\mathbb{I}_{ij} \succeq 0$ for all $(i, j) \in E$. This ensures that the resulting operators are positive semidefinite. In this modified game, the payoffs for each edge are nonnegative and differ by a constant c from the payoffs in the original game, i.e.,

$$\langle R_{ij} + c\,\mathbb{I}_{ij}, X_i \otimes X_j \rangle = \langle R_{ij}, X_i \otimes X_j \rangle + c.$$

Thus, the payoff of player i in the new game differs by $c \deg(i)$ compared to the original payoff. Consequently, the Nash equilibria in the old game and the new game coincide.

Finally, in the new game, the payoff operator Φ_{ij} is the linear map whose Choi representation is $R_{ij} + c\,\mathbb{I}_{ij}$. As the latter matrix is PSD, it follows that Φ_{ij} is completely positive. $\square$

4. Nash equilibria in semidefinite network games

Recall that a strategy profile $(X_1, \ldots, X_N) \in \mathcal{X}$ is a Nash Equilibrium (NE) if for each player i, the density matrix X_i is a best response to X_{-i}. For any profile $X = (X_1, \ldots, X_N) \in \mathcal{X}$ define the exploitability of a agent i with respect to X as the maximum possible gain that they can achieve by deviating from X_i while the strategies of the other players remain fixed, i.e.,

$$e_i(X) = \max_{S_i \in \mathcal{X}_i} \langle S_i, \Phi_i(X_{N(i)}) \rangle - \langle X_i, \Phi_i(X_{N(i)}) \rangle.$$

Then, for any strategy profile $X = (X_1, \ldots, X_N) \in \mathcal{X}$ we have that

$$X \text{ is a NE} \Longleftrightarrow e_i(X) = 0 \text{ for all } \; i \in [N].$$

Finally, noting that exploitability is always nonnegative, we have that

$$X \text{ is a NE} \Longleftrightarrow \sum_i e_i(X) = 0 \text{ for all } \; i \in [N].$$

Next, define

$$w_i(X) = \max_{S_i \in \mathcal{X}_i} \langle S_i, \Phi_i(X_{N(i)}) \rangle.$$

Recall that for a zero-sum game we have $\langle X, \Phi(X) \rangle = 0$ (see (3.13)), and consequently, for any $X \in \mathcal{X}$ we have

$$\sum_i e_i(X) = \sum_i w_i(X).$$

Putting everything together, for a zero-sum semidefinite network game we have that

$$X \text{ is a NE} \Longleftrightarrow \sum_i w_i(X) = 0 \text{ for all } \; i \in [N]. \tag{4.1}$$

This discussion leads to the following characterization of Nash equilibria in the zero-sum case.

Theorem 4.1. *Consider a semidefinite network game Γ with the payoff operators $\{\Phi_{ij}\}_{i,j \in [N]}$ and let $\mathrm{NE}(\Gamma)$ be the set of Nash equilibria. Let $P^*(\Gamma)$ be the set of optimal solutions of the following semidefinite program*

$$(P(\Gamma)) \qquad \begin{aligned} \min_{w_i, X_i} \; & \sum_{i \in [N]} w_i \\ \text{s.t.} \quad & w_i \, \mathbb{I}_{\mathcal{A}_i} \succeq \Phi_i(X_{N(i)}), \\ & \mathrm{tr}(X_i) = 1, \\ & X_i \succeq 0, \\ & w \in \mathbb{R}^N, \; i \in [N], \end{aligned}$$

where $\Phi_i(X_{N(i)}) = \sum_{j \in N(i)} \Phi_{ij}(X_j)$. Then, the set of Nash equilibria is equal to the coordinate projection of the spectrahedron $P^*(\Gamma)$ onto the X coordinates, i.e.,

$$\mathrm{NE}(\Gamma) = \Big\{ X \in \mathcal{X} : \exists w \in \mathbb{R}^N \text{ for which } (w, X) \in P^*(\Gamma) \Big\}.$$

In particular, the set of Nash equilibria of a zero-sum semidefinite network game is SDP-representable.

Proof. First we consider the optimization problem

$$(P'(\Gamma)) \qquad \begin{aligned} &\min_{w,X} \sum_{i\in[N]} w_i \\ &\text{s.t. } \ w_i \geq p_i(S_i, X_{N(i)}), \qquad \forall S_i \in \mathcal{X}_i,\ i\in[N], \\ &\qquad X\in\mathcal{X},\ w=(w_1,\ldots,w_N)\in\mathbb{R}^N. \end{aligned}$$

The proof has two main steps: First, we show that the optimal value of $(P'(\Gamma))$ is equal to zero and second, that it can be reformulated as the SDP in $(P(\Gamma))$. The fact that optimal solutions correspond to Nash equilibria then follows immediately by (4.1).

We claim that $(P'(\Gamma))$ can be expressed as the semidefinite program $(P(\Gamma))$. Using (3.10) we can rewrite $(P'(\Gamma))$ in terms of the linear maps Φ_i (3.9) as

$$(4.2) \qquad \begin{aligned} &\min_{w,X} \sum_{i\in[N]} w_i \\ &\text{s.t. } \ w_i \geq \langle S_i, \Phi_i(X_{N(i)})\rangle, \qquad \forall S_i \in \mathcal{X}_i,\ i\in[N] \\ &\qquad X\in\mathcal{X},\ w=(w_1,\ldots,w_N)\in\mathbb{R}^N. \end{aligned}$$

Clearly, for each player $i\in[N]$ we have that

$$w_i \geq \langle S_i, \Phi_i(X_{N(i)})\rangle \ \forall S_i\in\mathcal{X}_i \iff w_i \geq \max_{S_i}\Big\{\langle S_i, \Phi_i(X_{N(i)})\rangle : \operatorname{tr}(S_i)=1,\ S_i\succeq 0\Big\}.$$

Moreover, using (3.2), for each of the inner optimization problems we see

$$(4.3) \qquad \max_{S_i}\Big\{\langle S_i, \Phi_i(X_{N(i)})\rangle \ :\ \operatorname{tr}(S_i)=1, S_i\succeq 0\Big\} = \lambda_{\max}(\Phi_i(X_{N(i)})).$$

Finally, using (4.3) and noting that

$$\lambda_{\max}(\Phi_i(X_{N(i)})) \leq w_i \iff \Phi_i(X_{N(i)}) \preceq w_i\,\mathbb{I}_{\mathcal{A}_i},$$

the optimization problem (4.2) can be re-written as the semidefinite program $(P(\Gamma))$.

Next we show that the value of the primal $(P(\Gamma))$ is lower bounded by zero. Indeed, every feasible solution (X_i, w_i) of the primal satisfies

$$\sum_{i=1}^N w_i \ \geq\ \sum_{i=1}^N p_i(X_i, X_{N(i)}) \ =\ 0,$$

where the last equality follows from the zero-sum property. Moreover, the primal is strictly feasible, since by considering large enough w_i, we can make the semidefinite inequality become strict. Consequently, by strong duality for linear conic programs, e.g. see [19, Theorem 3.4.1], it follows that we have strong duality and the value of the dual is attained.

To derive the dual of $(P(\Gamma))$, we consider the Lagrangian function,

$$\mathcal{L}(w, X, \Lambda_i, \Lambda_i', \lambda_i) = \sum_{i\in[N]} w_i + \sum_{i\in[N]} \langle \Lambda_i, \Phi_i(X_{N(i)}) - w_i I_i\rangle - \sum_{i\in[N]} \langle \Lambda_i', X_i\rangle \\ + \sum_{i\in[N]} \lambda_i(1 - \langle I_i, X_i\rangle),$$

where I_i denotes the identity matrix of order i. Using the definitions of Φ_i (cf. (3.9)), Ψ_i (cf. (3.14)) and basic properties of the trace and inner product, the Lagrangian can be arranged as

$$\mathcal{L}(w, X, \Lambda_i, \Lambda_i', \lambda_i) = \sum_i w_i(1 - \mathrm{tr}(\Lambda_i)) + \sum_i \sum_{j\in N(i)} \langle \Lambda_i, \Phi_{ij}(X_j)\rangle - \sum_i \langle \Lambda_i', X_i\rangle \\ + \sum_i \lambda_i(1 - \langle I_i, X_i\rangle).$$

Since $\sum_{j\in N(i)}\langle \Lambda_i, \Phi_{ij}(X_j)\rangle = \sum_{j\in N(i)}\langle \Phi_{ij}^{\dagger}(\Lambda_i), X_j\rangle = \sum_{j\in N(i)}\langle X_j, \Psi_{ji}(\Lambda_i)\rangle = \langle X_i, \Psi_i(\Lambda)\rangle$, we obtain

$$\mathcal{L}(w, X, \Lambda_i, \Lambda_i', \lambda_i) = \sum_i w_i(1 - \mathrm{tr}(\Lambda_i)) + \sum_i \langle X_i, \Psi_i(\Lambda) - \Lambda_i' - \lambda_i I_i\rangle + \sum_i \lambda_i.$$

Putting everything together, the dual of $(P(\Gamma))$ is given by

$$(D(\Gamma)) \qquad \begin{aligned} \max_{\lambda_i, \Lambda_i} \ & \sum_i \lambda_i \\ \text{s.t. } & \mathrm{tr}(\Lambda_i) = 1, \\ & \Psi_i(\Lambda) \succeq \lambda_i \mathbb{I}_i, \\ & \Lambda_i \succeq 0. \end{aligned}$$

The program $(D(\Gamma))$ is strictly feasible, because for given Λ_i, choosing the numbers λ_j negative and with sufficiently large absolute values gives feasible solutions.

We next show that $(D(\Gamma))$ has a nonpositive objective value. Indeed, let λ_i, Λ_i be a dual feasible solution. By $\Psi_i(\Lambda) \succeq \lambda_i \mathbb{I}_i$ and (3.2) we have that

$$\lambda_i \le \min_{S_i\in\mathcal{X}_i} \langle \Psi_i(\Lambda_{N(i)}), S_i\rangle. \tag{4.4}$$

This implies

$$\begin{aligned} \sum_i \lambda_i &\le \sum_i \min_{S_i\in\mathcal{X}_i} \langle \Psi_i(\Lambda_{N(i)}), S_i\rangle \\ &= \min_{(S_1,\dots,S_N)\in\mathcal{X}} \langle (\Psi_1(\Lambda), \dots, \Psi_N(\Lambda)), (S_1, \dots, S_N)\rangle \\ &= \min_{(S_1,\dots,S_N)\in\mathcal{X}} \langle \Psi(\Lambda), (S_1, \dots, S_N)\rangle \\ &\le \langle \Psi(\Lambda), \Lambda\rangle = 0, \end{aligned}$$

where the first inequality follows by summing (4.4), the second equality since the optimization is separable in the S_i's, the third equality from the definition of the Ψ map in (3.14), the fourth inequality as $\Lambda_i \in \mathcal{X}_i$, and the final equality from (3.15).

Putting everything together, ($P(\Gamma)$) and ($D(\Gamma)$) are a pair of primal-dual SDPs, where strict duality holds, both are attained, and the (common) value is equal to 0. $\square$

5. Recognizing zero-sum semidefinite network games

In this section we address the question of recognizing whether a semidefinite network game is zero-sum. This is not immediately clear how to do, as the zero-sum condition $\sum_{j=1}^N p_j(X) = 0$ needs to hold for all (infinitely many) strategies $X \in \mathcal{X}$. In the special case where each edge game is a matrix game, the situation can be reduced to the finite set of pure strategies. This was studied in [5] and it was shown there that recognizing the network version of matrix games (called polymatrix game in that paper) can be achieved by solving a finite number of linear programs and that the number of linear programs is polynomial in the number of players and strategies. Hence, recognizing these games can be done in polynomial time.

In the general case of semidefinite network games, we show that recognizing the zero-sum property can be done by deciding whether a certain finite set of semidefinite programs all have the optimal value zero.

For every player i, a strategy $X_i \in \mathcal{X}_i$ and $X_{-i} \in \mathcal{X}_{-i}$, let

$$W(X_i, X_{-i}) = \sum_{j \in V} p_j(X_i, X_{-i}).$$

Note that a game is constant-sum if and only if

$$W(X_i, X_{-i}) = W(Y_i, X_{-i}) \text{ for all } X_i, Y_i \in \mathcal{X}_i.$$

Theorem 5.1. *Consider a semidefinite network game Γ with the payoff operators $\{\Phi_{ij}\}_{i,j\in[N]}$. The game Γ is constant-sum if and only if for all $i \in [N]$, the semidefinite program*

$$\begin{array}{rcll} \min\limits_{X_i, Y_i, w_\ell} \sum\limits_{\ell \in N(i)} w_\ell & & & \\ (\Phi_{i\ell}^\dagger + \Phi_{\ell i})(X_i - Y_i) & \preceq & w_\ell \, \mathbb{I}_l, & (\ell \in N(i)), \\ w_\ell & \in & \mathbb{R}, & (\ell \in N(i)), \\ X_i, Y_i & \in & \mathcal{X}_i & \end{array} \tag{5.1}$$

attains its minimum at zero.

If the game is constant-sum, then we can check whether the game is zero-sum by evaluating the sum of the payoffs at an arbitrarily chosen strategy profile of the players.

Proof. First we claim that the the game Γ is constant-sum if and only if for all $i \in [N]$ and all $X_i, Y_i \in \mathcal{X}_i$, the optimization problem

$$\max\Big\{W(X_i, X_{-i}) - W(Y_i, X_{-i}) : X_{-i} \in \mathcal{X}_{-i}\Big\} \tag{5.2}$$

attains its maximum at zero.

Assume that the game Γ is constant-sum. Then, by the preceding discussion we have that $W(X_i, X_{-i}) = W(Y_i, X_{-i})$ for all $X_i, Y_i \in \mathcal{X}_i$. This means that every feasible solution has value 0, so (5.2) has optimal value zero.

Conversely, if the optimal value of (5.2) is zero, then $W(X_i, X_{-i}) = W(Y_i, X_{-i})$ for all $i \in V$ and $X_i, Y_i \in \mathcal{X}_i$, $X_{-i} \in \mathcal{X}_i$. Then Γ is a constant-sum game.

The characterization (5.2) involves infinitely many SDPs, because of the quantification over X_i, Y_i in the infinite set $\mathcal{X}_i$. Using duality theory, we show that for each player $i \in [N]$, this characterization can be formulated as the SDP (5.1). For this, note that the payoff at any edge game (k, ℓ) where $k, \ell \neq i$ do not appear in the objective function, as all these terms cancel out. Expanding the objective function, we can rewrite it as

$$W(X_i, X_{-i}) - W(Y_i, X_{-i}) = \sum_{\ell \in N(i)} \langle X_\ell, (\Phi_{i\ell}^\dagger + \Phi_{\ell i})(X_i - Y_i) \rangle.$$

So (5.2) can be written as

$$\max_{X_\ell \in \mathcal{X}_\ell} \sum_{\ell \in N(i)} \langle X_\ell, (\Phi_{i\ell}^\dagger + \Phi_{\ell i})(X_i - Y_i) \rangle = \sum_{\ell \in N(i)} \lambda_{\max}(\Phi_{i\ell}^\dagger + \Phi_{\ell i})(X_i - Y_i)$$
$$= \sum_{\ell \in N(i)} \min_{w_l \in \mathbb{R}} \left\{ w_l : \, (\Phi_{i\ell}^\dagger + \Phi_{\ell i})(X_i - Y_i) \preceq w_l \, \mathbb{I}_l \right\}.$$

For a fixed player i, this is equivalent to the SDP (5.1). □

Remark 5.2. While the optimization problems (5.2) are usually considered as tractable, let us point out that these problems are related to the decision problem whether an SDP has a feasible solution (SDFP, semidefinite program feasibility problem). The complexity of SDFP in the Turing machine model is not known. Either SDFP $\in$ NP $\cap$ co$-$NP or SDFP $\notin$ NP $\cup$ co$-$NP. One obstacle to efficiently solving SDFP is that, by an example of Khachiyan described in Ramana's work [24], exponential-size optimal solutions of semidefinite programs can arise. We note that exponential-size optimal solutions in an SDP can also arise if the optimal value is known to be zero. Namely, if SDP_n is a family of semidefinite programs with parameter n which has exponential size optimal solutions in n, then one can construct a family of semidefinite programs with parameter n which has optimal value 0 and exponential size solutions. To this end, just duplicate SDP_n using a new set of variables, called SDP'_n, and consider the SDP whose objective function is the difference of the objective function of SDP_n and SDP'_n and whose feasible region is the Cartesian product of the feasible regions of SDP_n and SDP'_n.

6. Nash equilibria via SDP linear complementarity problems

It is well-established that Nash equilibria of bimatrix and network games can be equivalently formulated as solutions to Linear Complementarity problems over the nonnegative orthant, e.g., see [27] and [13]. In this section, we extend this result and demonstrate that Nash equilibria of SDP network games can also be characterized as solutions to Semidefinite Linear Complementarity problems (SDP-LCP).

An SDP-LCP instance is defined in terms of a linear map $L: \mathrm{Herm}(\mathcal{A}) \to \mathrm{Herm}(\mathcal{A})$ and a matrix $Q \in \mathrm{Herm}(\mathcal{A})$. The objective is to find a Hermitian matrix X that satisfies the following three conditions:

$$X \succeq 0, \quad L(X) + Q \succeq 0, \quad \langle X, L(X) + Q \rangle = 0. \tag{6.1}$$

Algorithms for solving SDP-LCPs typically exploit the specific properties of the linear operator L and the matrix Q, see, e.g., [18, 11] and the references therein.

Theorem 6.1. *The set of Nash equilibria of an SDP network game are the solutions to an SDP-Linear Complementarity Problem.*

Proof. Consider a strategy profile $(X_1, \ldots, X_n)$ that is a Nash equilibrium of an SDP network game. By definition of a Nash equilibrium, for any player $i \in [N]$ we have that X_i is a best response to X_{-i}, i.e.,

$$X_i \in \underset{Y_i \in \mathcal{X}_i}{\mathrm{argmax}} \langle Y_i, \Phi_i(X) \rangle.$$

Considering the optimization problem

$$\max \Big\{ \langle Y_i, \Phi_i(X) \rangle : Y_i \in \mathcal{X}_i \Big\},$$

its dual is given by

$$\min_{\lambda_i, Z_i} \Big\{ \lambda_i : \lambda_i \, \mathbb{I}_{\mathcal{A}_i} - \Phi_i(X) = Z_i \succeq 0 \Big\}.$$

Consequently, by strong duality for SDPs we have that $(X_1, \ldots, X_n)$ is a Nash equilibrium iff there exist $\lambda_1, \ldots, \lambda_N$ such that for all $i \in [N]$ we have

$$\begin{aligned} X_i &\succeq 0, \\ \lambda_i \, \mathbb{I}_{\mathcal{A}_i} - \Phi_i(X) &\succeq 0, \\ \mathrm{tr}(X_i) &= 1, \\ \langle X_i, \lambda_i \, \mathbb{I}_{\mathcal{A}_i} - \Phi_i(X) \rangle &= 0. \end{aligned} \tag{6.2}$$

Lemma 3.3 establishes that we can assume each payoff matrix to be PSD. Therefore, the map Φ_i becomes a completely positive map, implying that $\Phi_i(X)$ is also a PSD matrix. Thus, the generalized inequality $\lambda_i \, \mathbb{I}_{\mathcal{A}_i} - \Phi_i(X) \succeq 0$ implies that $\lambda_i \geq 0$. Now consider the SDP-LCP

$$\begin{aligned} X_i \succeq 0, \ & i \in [N], \\ \mathbb{I}_{\mathcal{A}_i} - \Phi_i(X) \succeq 0, \ & i \in [N], \\ \langle X_i, \mathbb{I}_{\mathcal{A}_i} - \Phi_i(X) \rangle = 0, \ & i \in [N]. \end{aligned} \tag{6.3}$$

and note that if X_i, λ_i are feasible for (6.2) then $\frac{1}{\lambda_i} X_i$ is feasible for (6.3), and conversely, if X is feasible for (6.3) then $\frac{X_i}{\mathrm{tr}(X_i)}, \lambda_i = \frac{1}{\mathrm{tr}(X_i)}$ is feasible for (6.2). Finally, to write (6.2) in the standard form (6.1) we take

$$X \in \mathrm{Pos}(\oplus_i \mathcal{A}_i), \quad L(X) = - \oplus_i \Phi_i(X), \quad Q = \oplus_i \mathbb{I}_{\mathcal{A}_i}. \tag{6.4}$$

This concludes the proof. □

7. Conclusion

We have considered semidefinite network games, where players reside at graph nodes and their results hinge on the actions of neighboring players. The player's strategies entail positive semidefinite matrices, making them suitable for modeling quantum games on networks. When the games are zero-sum, we compute the corresponding Nash equilibria through semidefinite programs. Identifying a semidefinite network game equates to ascertaining a that an semidefinite program has value zero. In cases beyond zero-sum scenarios, Nash equilibria correspond to solutions of a semidefinite linear complementarity problem.

Acknowledgements. ET is partially supported by ANR JCJC GALOP (ANR-17-CE40-0009). AV is supported by the National Research Foundation, Singapore under its QEP2.0 programme (NRF2021-QEP2-02-P05).

References

[1] H. Bohnenblust, S. Karlin, and L. Shapley. Games with continuous, convex pay-off. *Contributions to the Theory of Games*, 1(24):181–192, 1949.
[2] J. Bostanci and J. Watrous. Quantum game theory and the complexity of approximating quantum Nash equilibria. *Quantum*, 6:882, 2022.
[3] L. M. Brègman and I. N. Fokin. Methods for determining equilibrium states in polymatrix games with zero sum. *Optimizatsiya*, 40(57):70–82, 160, 1987. (In Russian).
[4] L. M. Brègman and I. N. Fokin. On separable non-cooperative zero-sum games. *Optimization*, 44(1):69–84, 1998.
[5] Y. Cai, O. Candogan, C. Daskalakis, and C. Papadimitriou. Zero-sum polymatrix games: A generalization of minmax. *Math. Oper. Res.*, 41(2):648–655, 2016.
[6] Y. Cai and C. Daskalakis. On minmax theorems for multiplayer games. In *Proc. 22nd ACM-SIAM Symposium on Discrete Algorithms*, pages 217–234, 2011.
[7] G. Chiribella, G. M. D'Ariano, and P. Perinotti. Theoretical framework for quantum networks. *Physical Review A*, 80(2):022339, 2009.
[8] C. Daskalakis and C. H. Papadimitriou. On a network generalization of the minmax theorem. In *Proc. 36th International Colloqium on Automata, Languages and Programming, ICALP 2009, Rhodes, Greece*, pages 423–434. Springer, 2009.
[9] J. Eisert, M. Wilkens, and M. Lewenstein. Quantum games and quantum strategies. *Physical Review Letters*, 83(15):3077, 1999.
[10] I. L. Glicksberg. A further generalization of the Kakutani fixed theorem, with application to Nash equilibrium points. *Proc. Amer. Math. Society*, 3(1):170–174, 1952.
[11] M. S. Gowda and Y. Song. On semidefinite linear complementarity problems. *Math. Program.*, 88:575–587, 2000.
[12] G. Gutoski and J. Watrous. Toward a general theory of quantum games. In *Proc. 39th Annual ACM Symposium on Theory of Computing*, pages 565–574, 2007.
[13] J. T. Howson Jr. Equilibria of polymatrix games. *Management Science*, 18(5-part-1):312–318, 1972.
[14] C. Ickstadt, T. Theobald, and E. Tsigaridas. Semidefinite games. Preprint, arXiv:2202.12035, 2022.

[15] R. Jain, G. Piliouras, and R. Sim. Matrix multiplicative weights updates in quantum zero-sum games: Conservation laws & recurrence. *Advances in Neural Information Processing Systems*, 35:4123–4135, 2022.

[16] R. Jain and J. Watrous. Parallel approximation of non-interactive zero-sum quantum games. In *Proc. 24th Annual IEEE Conf. Comput. Complexity*, pages 243–253. IEEE, 2009.

[17] E. Janovskaja. Equilibrium points in polymatrix games. *Lithuanian Math. J.*, 8(2):381–384, 1968. (In Russian).

[18] M. Kojima, S. Shindoh, and S. Hara. Interior-point methods for the monotone semidefinite linear complementarity problem in symmetric matrices. *SIAM J. Optim.*, 7(1):86–125, 1997.

[19] M. Laurent and F. Vallentin. Semidefinite optimization. Lecture notes, 2012.

[20] W. Lin, G. Piliouras, R. Sim, and A. Varvitsiotis. Quantum potential games, replicator dynamics, and the separability problem. Preprint, arXiv:2302.04789, 2023.

[21] K. Lotidis, P. Mertikopoulos, and N. Bambos. Learning in quantum games. Preprint, arXiv:2302.02333, 2023.

[22] D. Meyer. Quantum strategies. *Physical Review Letters*, 82:1052, 1999.

[23] J. F. Nash. Equilibrium points in n-person games. *Proc. Natl. Acad. Sci. USA*, 36:48–49, 1950.

[24] M. V. Ramana. An exact duality theory for semidefinite programming and its complexity implications. *Math. Program.*, 77:129–162, 1997.

[25] J. B. Rosen. Existence and uniqueness of equilibrium points for concave n-person games. *Econometrica*, pages 520–534, 1965.

[26] C. Scheiderer. Spectrahedral shadows. *SIAM J. Appl. Algebra Geom.*, 2(1):26–44, 2018.

[27] B. von Stengel. Computing equilibria for two-person games. *Handbook of Game Theory with Economic Applications*, 3:1723–1759, 2002.

[28] J. Watrous. *The Theory of Quantum Information*. Cambridge Univ. Press, 2018.

[29] S. Zhang. Quantum strategic game theory. In *Proc. 3rd Innovations in Theoretical Computer Science Conference*, pages 39–59, 2012.

Constantin Ickstadt, Thorsten Theobald: Goethe-Universität, FB 12 – Institut für Mathematik, Postfach 11 19 32, 60054 Frankfurt am Main, Germany

Elias Tsigaridas: Sorbonne Université, Paris University, CNRS and Inria Paris. IMJ-PRG, 4 place Jussieu, 75252 Paris Cedex 05, France

Singapore University of Technology and Design (ESD Pillar), 8 Somapah Road 487372, Singapore