SOME RECENT DEVELOPMENTS IN SPECTRAHEDRAL COMPUTATION

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Abstract. Spectrahedra are the feasible sets of semidefinite programming and provide a central link between real algebraic geometry and convex optimization. In this expository paper, we review some recent developments on effective methods for handling spectrahedra. In particular, we consider the algorithmic problems of deciding emptiness of spectrahedra, boundedness of spectrahedra as well as the question of containment of a spectrahedron in another one. These problems can profitably be approached by combinations of methods from real algebra and optimization.

1. Introduction

In the last decade tremendous developments around the connections between algebraic geometry, convexity and optimization have brought the geometric concept of a *spectrahedron* into the focus of research activities. A spectrahedron, whose terminology is due to Ramana and Goldman [33], is the feasible region of a semidefinite program. Hence, spectrahedra are a natural generalization of polyhedra (which are the feasible sets of linear programs). Spectrahedra are basic semialgebraic sets and provide a major concept in modern computational real algebraic geometry [4, 15, 30].

Formally, let $\mathcal{S}_k$ be the set of real symmetric $k \times k$-matrices, $\mathcal{S}_k^+ \subseteq \mathcal{S}_k$ be the subset of positive semidefinite matrices, and $\mathcal{S}_k[x]$ be the set of symmetric $k \times k$-matrices with polynomial entries in $x = (x_1, \ldots, x_n)$. For $A_0, \ldots, A_n \in \mathcal{S}_k$, denote by $A(x)$ the *linear (matrix) pencil* $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n \in \mathcal{S}_k[x]$. The set

$$S_A = \{x \in \mathbb{R}^n : A(x) \succeq 0\} \qquad (1.1)$$

is called a *spectrahedron*, where $A(x) \succeq 0$ denotes positive semidefiniteness of the matrix $A(x)$.

Recent work by a number of authors have advanced a theory of spectrahedral computation. In this expository paper, we review some of these developments, equipped with a view towards real and convex algebraic geometry. A particular focus will then be given on the question whether one given spectrahedron is contained in another one.

Precisely, given linear matrix pencils $A(x)$ and $B(x)$ we consider the following problems:

Emptiness: Is S_A empty?
Boundedness: Is S_A bounded?
Containment: Does $S_A \subseteq S_B$ hold?

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Most of the results discussed here come from the work of Helton, Kellner, Klep, McCullough, Schweighofer, Trabandt as well as the author. Rather than to focus on complete coverage, our goal is to provide an insightful window into these research developments. Most proofs are omitted and can be found in the original papers.

The paper is structured as follows. In Section 2, we introduce polyhedra and spectrahedra and highlight some occurrences of spectrahedra in real and convex algebraic geometry. In Section 3, we discuss some fundamental algorithmic problems, in particular the emptiness and boundedness problem. Then, in Section 4, we deal with fundamental aspects of the containment problem. Section 5 is devoted to hierarchical semidefinite approaches to the containment problem.

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2. From polyhedra to spectrahedra

Starting from polyhedra as a classical cornerstone of mathematics (see the monographs of Grünbaum [11] or Ziegler [38]), we then introduce some basic notions of spectrahedra.

2.1. **Polyhedra and polytopes.** For a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$, the set $P = \{x \in \mathbb{R}^n \, : \, b + Ax \geq 0\}$ is called a *polyhedron*. Geometrically, P is the intersection of a finite number of halfspaces ($\mathcal{H}$-presentation of a polyhedron, or, for short, $\mathcal{H}$-*polyhedron*). If the polyhedron P is a bounded set, then P is called a *polytope*. Polytopes can also be represented as the convex hull of finitely many points, $P = \operatorname{conv}\{p^{(1)}, \ldots, p^{(l)}\}$ with $p^{(1)}, \ldots, p^{(l)} \in \mathbb{R}^n$ ($\mathcal{V}$-*presentation of a polytope* or, for short, $\mathcal{V}$-*polytope).*

As an occurrence of polyhedra in real algebraic geometry, let us state Handelman's Theorem [12], which provides a characterization of the positive polynomials on a given polytope. And under a degree restriction it gives a polyhedron of solutions, since all conditions are linear.

Theorem 2.1 (Handelman)**.** *Let $g_1, \ldots, g_m \in \mathbb{R}[x]$ be affine-linear polynomials such that $K = \{x \in \mathbb{R}^n \, : \, g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}$ is non-empty and bounded, that is, a polytope. Any polynomial $p \in \mathbb{R}[x]$ which is strictly positive on K can be written as a finite sum*

$$p \;=\; \sum_{\beta} c_\beta \prod_{j=1}^{m} g_j^{\beta_j} \tag{2.1}$$

with coefficients $c_\beta \geq 0$ ($\beta \in \mathbb{N}_0^m$). For a fixed upper bound t on the degree, where $t \geq \deg p$, the set of solutions $(c_\beta)_{|\beta| \leq t}$ of

$$p \;=\; \sum_{|\beta| \leq t} c_\beta \prod_{j=1}^{m} g_j^{\beta_j}$$

is a polyhedron.

The latter condition can be transformed into an optimization version to find lower bounds for p on K.

Though polytopes and polyhedra are defined by linear inequalities, they have a rich geometric and combinatorial structure. Denote by $V(P)$ the set of vertices (i.e., 0-dimensional faces) of a polytope P, and by $F(P)$ the set of facets (i.e., faces of codimension 1). By McMullen's Upper bound Theorem [28], any n-dimensional polytope with k vertices has at most

$$\binom{k-\lceil\frac{n}{2}\rceil}{\lfloor\frac{n}{2}\rfloor}+\binom{k-1-\lceil\frac{n-1}{2}\rceil}{\lfloor\frac{n-1}{2}\rfloor} \tag{2.2}$$

facets. This bound, which is of inherent importance for polyhedral computation software such as `polymake` [9], is sharp for neighborly polytopes, that is, for polytopes with the property that every set of at most $\lfloor n/2\rfloor$ vertices is the vertex set of a face of P. For example, cyclic polytopes are neighborly. And, dual to the statement, the maximum number of vertices of any n-dimensional polytope with k facets is given by (2.2) as well, with equality for dually neighborly polytopes.

2.2. **Spectrahedra.** We build upon the terminology from the Introduction. Specifically, for $A_0,\ldots,A_n\in\mathcal{S}_k$, let $S_A = \{x\in\mathbb{R}^n : A(x) = A_0+\sum_{i=1}^n x_iA_i\succeq 0\}$ denote the *spectrahedron* as defined in (1.1). The inequality $A_0+\sum_{i=1}^n x_iA_i \succeq 0$ is called a *linear matrix inequality (LMI)*. Since the operator $A(\cdot)$ is linear, any spectrahedron is a convex set.

Example 2.2. Figure 1 shows the example of the elliptope

$$S_A = \left\{x\in\mathbb{R}^3 : \begin{pmatrix}1 & x_1 & x_2\\ x_1 & 1 & x_3\\ x_2 & x_3 & 1\end{pmatrix}\succeq 0\right\}$$

(see, e.g., [26]).

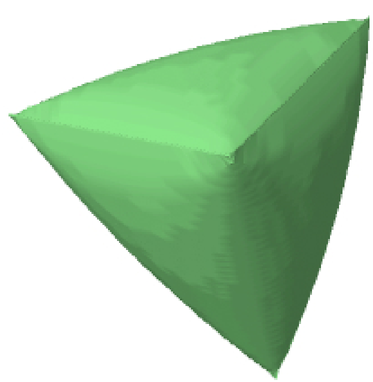

FIGURE 1. Visualization of an elliptope.

Note that every polyhedron $P = \{x \in \mathbb{R}^n \ : \ b + Ax \geq 0\}$ can be regarded as a spectrahedron,

$$P = P_A = \left\{ x \in \mathbb{R}^n \ : \ A(x) = \begin{pmatrix} a_1(x) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_k(x) \end{pmatrix} \succeq 0 \right\}, \tag{2.3}$$

where $a_i(x)$ denotes the i-th entry of the vector $b + Ax$. P_A contains the origin in its interior if and only if the inequalities can be scaled so that b is the all-ones vector $\mathbb{1}_k$ in $\mathbb{R}^k$. In this case, $A(x)$ is called the *normal form* of the polyhedron P_A.

Example 2.3. The unit disc $\{x \in \mathbb{R}^2 \ : x_1^2 + x_2^2 \leq 1\}$ is a spectrahedron. This follows from setting

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and observing that

$$A(x) \ = \ \begin{pmatrix} 1 + x_1 & x_2 \\ x_2 & 1 - x_1 \end{pmatrix}$$

is positive semidefinite if and only if $1 - x_1^2 - x_2^2 \geq 0$.

Every spectrahedron S is a basic closed semialgebraic set. This can be seen by writing $S \ = \ \{x \in \mathbb{R}^n \ : \ p_i(x) \geq 0\,, i \in I\}$ where the $p_i(x)$ are the principal minors of $A(x)$, indexed by the set $I = 2^{\{1,\ldots,k\}} \setminus \emptyset$. A slightly more concise representation is given by the following well-known statement, where I_k denotes the $k \times k$ identity matrix.

Proposition 2.4. *Any spectrahedron $S = S_A$ is a basic closed semialgebraic set. In particular, given the modified characteristic polynomial*

$$t \mapsto \det(A(x) + tI_k) \ =: t^k + \sum_{i=0}^{k-1} p_i(x) t^i \,, \tag{2.4}$$

S has the representation $S \ = \ \{x \in \mathbb{R}^n \, : \, p_i(x) \geq 0,\ 0 \leq i \leq k-1\}$.

Proof. Denoting by $\lambda_1(x), \ldots, \lambda_k(x)$ the eigenvalues of the linear pencil $A(x)$, we observe

$$\det(A(x) + tI_k) \ = \ (t + \lambda_1(x)) \cdots (t + \lambda_k(x))\,.$$

Since $A(x)$ is symmetric, all $\lambda_i(x)$ are real, for any $x \in \mathbb{R}^n$. Comparing the coefficients then shows

$$p_{k-i}(x) \ = \sum_{t_1 < \cdots < t_i} \lambda_{t_1}(x) \cdots \lambda_{t_i}(x)\,, \quad 1 \leq i \leq k\,.$$

Now "$\subseteq$" of the desired representation follows from the fact that positive semidefiniteness of $A(x)$ at a given $x \in \mathbb{R}^n$ implies non-negativity of all eigenvalues $\lambda_1(x), \ldots, \lambda_k(x)$ and thus non-negativity of all $p_i(x)$. Conversely, if for a given $x \in \mathbb{R}^n$ we have $p_i(x) \geq 0$ for all i, then the modified characteristic polynomial has no sign changes. Thus, by Descartes' rule of signs, it has no positive roots, and therefore $A(x)$ is positive semidefinite. $\square$

It is an open question to provide good effective criteria to test whether a given convex semialgebraic set is a spectrahedron or the linear projection of a spectrahedron. Recently, the conjecture that every convex semialgebraic set would be the linear projection of a spectrahedron ("Helton-Nie conjecture") has been disproven by Scheiderer [34].

2.3. **Spectrahedra in real and convex algebraic geometry.** Spectrahedra occur in many places of real and convex algebraic geometry. We point out three connections to the algorithmic problems mentioned in Section 3.

Non-negative polynomials and sums of squares. A polynomial $p = \sum_\alpha c_\alpha x^\alpha \in \mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]$ is called a *sum of squares* (*sos*) if it can be written as a finite sum $\sum_i u_i(x)^2$ with polynomials $u_i \in \mathbb{R}[x]$. The total degree $\deg p$ of an sos-polynomial p is even. Sum of squares polynomials are ubiquitous in real and convex algebraic geometry and provide a fundamental sufficient condition for the property that a polynomial p is non-negative. In order to phrase the sos-property in terms of a spectrahedral property, let y denote the $\binom{n+\deg p/2}{n}$-dimensional vector of all monomials in x up to half of the total degree of p. And for some $m \geq 0$ and $k = \binom{n+\deg p/2}{n}$, let $A(w) = A_0 + \sum_{i=1}^m w_i A_i$ be a matrix pencil spanning the subspace in $\mathcal{S}_k$ defined by the equations

$$c_\alpha \;=\; \sum_{\beta+\gamma=\alpha} z_{\beta,\gamma} \quad \text{for all } \alpha \text{ of total degree at most} \deg p \tag{2.5}$$

in the symmetric matrix of variables $Z = (z_{\beta,\gamma})_{|\beta|,|\gamma| \leq \deg p/2}$.

Proposition 2.5. *A polynomial $p \in \mathbb{R}[x]$ can be written as a sum of squares if and only if the spectrahedron S_A is non-empty.*

Proof. The comparison of the coefficients in (2.5) is satisfied if and only if there exists a matrix Z with $y^T Z y = c$, where c is the coefficient vector of p. Since Z has a Choleski decomposition LL^T if and only if it is positive semidefinite, the claim follows. □

Computation of amoebas. For an ideal $I = \langle f_1, \ldots, f_r \rangle \subseteq \mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_n]$, the *algebraic amoeba* (or *unlog amoeba*) $\mathcal{A}_I$ is the image of its zero set $\mathcal{V}(I)$ under the absolute value map, that is, $\mathcal{A}_I = \{|z| \,:\, z \in \mathcal{V}(I)\}$. Given $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n_{\geq 0}$, the amoeba membership problem asks whether $\lambda \in \mathcal{A}_I$.

For $f \in \mathbb{C}[z]$, let $\Re(f)$ and $\Im(f) \in \mathbb{R}[x, y]$ be given through

$$f(x + iy) \;=\; \Re(f)(x, y) + i\Im(f)(x, y)\,.$$

Now consider the ideal J generated by the set of polynomials

$$\Re(f_j), \Im(f_j) \quad 1 \leq j \leq r, \quad x_k^2 + y_k^2 - \lambda_k^2, \quad 1 \leq k \leq n\,.$$

By the real Nullstellensatz, we have $\lambda \in \mathcal{A}_I$ unless there exists a polynomial $G \in J$ and an sos-polynomial H such that $G + H + 1 = 0$. Given a fixed degree bound, the set of all the certificates satisfying that bound defines a spectrahedron, and thus the amoeba membership problem can be approached through a hierarchy of spectrahedral feasibility problems, see [37].

Non-negative biquadratic forms. Given a biquadratic form

$$F(x,y) \;=\; \sum_{(i,j,s,t)\in\Lambda} b_{ijkl} x_i y_j x_s y_t$$

with $\Lambda = \{(i,j,s,t) \,:\, 1 \le i,s \le k,\ 1 \le j,t \le l\}$ and real coefficients b_{ijkl}, we ask whether F is non-negative. We can assume that the coefficients satisfy the symmetry condition $b_{ijkl} = b_{kjil}$ and $b_{ijkl} = b_{ilkj}$

In order to phrase this question as a containment problem of spectrahedra, set $n = \binom{k+1}{2}$. For notational convenience, we can then identify $x = (x_1, \ldots, x_n)$ with a matrix $X \in \mathcal{S}_k$. Let $A(X) = X$ and $B(X) \in \mathcal{S}_l[X]$ be given by $b_{j,t}(X) = \sum_{1 \le i,s \le k} b_{ijst} x_{is}$, $1 \le j,t \le l$.

Proposition 2.6. *The biquadratic form F is non-negative if and only if the spectrahedron S_A is contained in the spectrahedron S_B.*

Proof. If $\mathcal{S}_A \subseteq S_B$ then any positive semidefinite matrix X satisfies $B(X) \succeq 0$, and thus for every $(x,y) \in \mathbb{R}^k \times \mathbb{R}^l$ we have $F(x,y) = y^T B(xx^T) y \ge 0$. Hence, F is positive semidefinite.

Conversely, let $F(x,y)$ be a positive semidefinite biquadratic form. Since any positive semidefinite matrix X can be written as a finite sum $X = \sum_i x^{(i)} (x^{(i)})^T$ with vectors $x^{(i)} \in \mathbb{R}^k$, linearity implies $y^T B(X) y = \sum_i y^T B(x^{(i)}(x^{(i)})^T) y = \sum_i F(x^{(i)}, y) \ge 0$ for any $y \in \mathbb{R}^l$. Hence, $B(X) \succeq 0$. $\square$

3. Fundamental algorithmic concepts

In the early years, spectrahedra were mainly considered within optimization frameworks. The stronger focus on the geometry of these sets has established new connections to real algebraic geometry and effective computation.

3.1. **Infeasibility certificates.** Given a linear matrix pencil $A(x) \in \mathcal{S}_k[x]$, we study the question whether $S_A = \emptyset$.

Remark 3.1. For polytopes $P_A = \{x \in \mathbb{R}^n \,:\, b + Ax \ge 0\}$, the question whether P_A is non-empty can be phrased as a linear program and thus can be decided in polynomial time for a rational input polytope. Also note that even deciding whether a polytope has an interior point can be decided by a linear program as well (see, e.g., [18, Example 4.3]).

Testing whether $S_A = \emptyset$ can be regarded as the complement of a *semidefinite feasibility problem* (SDFP), which asks whether for a given linear pencil $A(x)$ the spectrahedron S_A is nonempty. While semidefinite programs (with rational input data) can be approximated in polynomial time (see [6]), the complexity of SDFP is open, see [32]. In practice, however, SDFPs can numerically be solved efficiently by semidefinite programming.

In view of the classical Nullstellensätze and Positivstellensätze from real algebraic geometry, it is a natural question how to certify the emptiness of a spectrahedron. For polytopes, the classical Farkas' Lemma (see, e.g., [35, Cor. 7.1e]) characterizes the emptiness of a polytope in terms of an identity of affine functions coming from a geometric cone condition.

Theorem 3.2. *A polyhedron $P = \{x \in \mathbb{R}^n : Ax + b \geq 0\}$ is empty if and only if the constant polynomial -1 can be written as $-1 = \sum_i s_i(Ax+b)_i$ with $s_i \geq 0$; or, equivalently, if -1 can be written as $-1 = c + \sum_i s_i(Ax+b)_i$ with $c \geq 0$, $s_i \geq 0$.*

Let $A(x) \in \mathcal{S}_k[x]$. $A(x)$ is called *feasible* if the spectrahedron S_A is non-empty. Further, $A(x)$ is called *strongly feasible* if $A(x)$ is feasible and there exists an $x \in \mathbb{R}^n$ with $A(x) \succ 0$. In relation to this, the spectrahedron S_A is called *strongly empty* if $A(x)$ it is not strongly feasible.

In order to extend Farkas' Lemma to spectrahedra, denote by C_A the convex cone in $\mathcal{S}_k[x]$ defined by

$$\begin{aligned} C_A &= \{c + \langle A, S\rangle \,:\, c \geq 0,\, S \in \mathcal{S}_k^+\} \\ &= \{c + \sum_i u_i^T A u_i \,:\, c \geq 0,\, u_i \in \mathbb{R}^n\}\,, \end{aligned}$$

where $\langle A, S\rangle = \mathrm{Tr}(AS)$ is the dot product underlying the Frobenius norm and Tr denotes the trace of a matrix. Since $A = A(x)$ is a linear pencil in $\mathcal{S}_k[x]$, every element in C_A is a linear polynomial which is non-negative on the spectrahedron S_A.

Theorem 3.3 (Sturm [36])**.** *Given $A(x) \in \mathcal{S}_k[x]$, the spectrahedron S_A is strongly empty if and only if $-1 \in C_A$.*

An exact characterization for the emptiness of S_A can be established in terms of a quadratic module associated to $A(x)$. Recall that a subset M of a commutative ring R with 1 is called a *quadratic module* if it satisfies the conditions

$$1 \in M,\ M + M \subseteq M \ \text{ and } \ a^2 M \subseteq M \text{ for any } a \in R\,.$$

Given a linear pencil matrix $A = A(x)$, denote by M_A the quadratic module in $\mathbb{R}[x]$

$$M_A = \{s + \langle A, S\rangle \,:\, s \in \Sigma[x],\, S \in \mathbb{R}[x]^{k\times k} \text{ an sos-matrix}\} \tag{3.1}$$

$$= \{s + \sum_i u_i^T A u_i \,:\, s \in \Sigma[x],\, u_i \in \mathbb{R}[x]^k\}\,, \tag{3.2}$$

where $\Sigma[x]$ denotes the subset of sums of squares of polynomials within $\mathbb{R}[x]$ and an sos-matrix is a matrix polynomial of the form $P^T P$ for some matrix polynomial P. Note that if a polynomial $f \in \mathbb{R}[x]$ is contained in M_A then it is non-negative on S_A. Further, denote by $M_A^{(t)}$ the truncated quadratic module

$$\begin{aligned} M_A^{(t)} &= \{s + \langle A, S\rangle \,:\, s \in \Sigma[x] \cap \mathbb{R}[x]_{2t},\, S \in \mathbb{R}[x]_{2t}^{k\times k} \text{ sos-matrix}\} \\ &= \{s + \sum_i u_i^T A u_i \,:\, s \in \Sigma[x]_{2t},\, u_i \in \mathbb{R}[x]_t^k\} \ \subseteq\ \mathbb{R}[x]_{2t+1}\,, \end{aligned}$$

where $\mathbb{R}[x]_t$ denotes the set of polynomials of total degree at most t.

Theorem 3.4 (Klep, Schweighofer [23])**.** *For $A(x) \in \mathcal{S}_k[x]$, the following are equivalent:*

(1) *The spectrahedron S_A is empty.*
(2) $-1 \in M_A$.
(3) $-1 \in M_A^{(2^{\min\{n,k-1\}})}$.

The third of these statements provides the ground for a computational treatment in terms of algebraic certificates for infeasibility. Namely, the question whether such a representation of bounded degree exists can be formulated as a semidefinite feasibility problem.

In order to carry out this formulation as a semidefinite program, set $t = 2^{\min\{n,k-1\}}$. Then the value

$$\max\left\{\gamma \in \mathbb{R} \,:\, -1-\gamma = s + \langle A, S\rangle, s \in \Sigma[x] \cap \mathbb{R}[x]_{2t},\ S \in \mathbb{R}[x]_{2t}^{k\times k} \text{ sos-matrix } \right\}$$

coincides with the value of the semidefinite program

$$\begin{array}{rrcll} & \max \gamma & & & \\ & s.t. \quad -1-\gamma & = & \mathrm{Tr}(P_1 X) + \mathrm{Tr}(Q_1 Y) & \\ (3.3) & 0 & = & \mathrm{Tr}(P_i X) + \mathrm{Tr}(Q_i Y) & \text{for } 2 \le i \le m_w := \binom{n+2t+1}{2t+1}, \\ & X & \succeq & 0,\ Y \succeq 0\,. & \end{array}$$

Here, denoting by $w = w(x)$ and $y = y(x)$ the vectors of monomials in $x_1, \ldots, x_n$ of degrees up to $2t+1$ and t in lexicographic order, Q_i is defined through $y(x)y(x)^T = \sum_{i=1}^{m_w} Q_i w_i(x)$. And, setting $m_y = \binom{n+t}{t}$, the permutation matrix $P \in \mathbb{R}^{km_y \times km_y}$ is given via $P(I_k \otimes y(x)) = y(x) \otimes I_k$, and the matrices P_i are defined through

$$P(I_k \otimes y(x)) \cdot A(x) \cdot (P(I_k \otimes y(x)))^T \;=\; \sum_{i=1}^{m_w} P_i w_i(x) \in \mathbb{R}[x]^{km_y \times km_y}\,.$$

Hence, $-1 \in M_A^{(2\min\{n,k-1\})}$ if and only if the objective value of (3.3) is non-negative. This decision problem is a semidefinite feasibility problem, since the property of a non-negative linear objective function can also be viewed as an additional linear constraint.

Example 3.5. Let

$$A(x) = \begin{pmatrix} 1+x & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & x \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} + x \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $\min\{n, k-1\} = \{1, 3-1\} = 1$, we can assume $y = y(x) = (1, x)^T$. We obtain

$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q_4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and the matrices $P_1, \ldots, P_4$ are

$$P_1 \;=\; \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad P_2 \;=\; \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since the positive semidefinite matrices

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{2} & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

provide a feasible solution of the semidefinite program (3.3) with objective value 0, we see that the spectrahedron S_A is empty. By a Choleski factorization

$$X = LL^T \quad \text{with } L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{6}/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6}/2 & \sqrt{2}/2 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

we can deduce from the semidefinite program (3.3) that $u_1 = (1,0,0)^T$, $u_2 = (0, \sqrt{2}, \sqrt{2}/2)^T$, $u_3 = (0, \sqrt{6}/2x, \sqrt{6}/2)^T$, $u_4 = (0, \sqrt{2}/2x, 0)^T$ provides the desired algebraic certificate $-1 \in M_A$, where the u_i are as in (3.2). We remark that u_4 can be omitted due to $u_4^T A(x) u_4 = 0$.

Origin in the interior. We shortly point out a fine point which explains a technical assumption in later statements. Clearly, if the constant matrix A_0 of a pencil $A(x)$ is positive semidefinite then the origin is contained in the spectrahedron S_A. However, in general it is *not* true that A_0 is positive definite if and only if the origin is contained in the interior of S_A. Fortunately, by [33, Corollary 5], if a spectrahedron S_A is full-dimensional, then there exists a so-called reduced linear pencil that is positive definite exactly on the interior of S_A. Hence, in the case of a reduced pencil we have $0 \in \operatorname{int} S_A$ if and only $A_0 \succ 0$. Moreover, for arbitrary dimension of S_A, we have $0 \in \operatorname{int} S_A$ if and only if there is a linear pencil $A'(x)$ with the same positivity domain such that $A'_0 = I_k$ (see [15]). Such a pencil is called *monic*.

3.2. **Boundedness.** In order to certify that a given spectrahedron is bounded, the quadratic module (3.1) is applied as well. Recall that a quadratic module $M \subseteq \mathbb{R}[x]$ is called *archimedean* if it contains a polynomial of the form $N - \sum_{i=1}^n x_i^2$ for some $N > 0$.

Theorem 3.6 (Klep, Schweighofer [23]). *Given $A(x) \in \mathcal{S}_k[x]$, the spectrahedron S_A is bounded if and only if M_A is archimedean.*

Example 3.7. In order to show that the spectrahedron S_A of

$$A(x) = \begin{pmatrix} x & 1 & 0 \\ 1 & x & 0 \\ 0 & 0 & -x+2 \end{pmatrix}$$

is bounded, we ask for $u \in \mathbb{R}[x]^3$ and sos-polynomials s_0, s_1 with

$$N - \sum_{i=1}^{3} x_i^2 \;=\; u^T A u + s_1^2(-x+2) + s_0$$

for some $N > 0$. The choice $u = (x - \frac{1}{2}, -x+1, 0)^T$, $s_1 = 2x^2 + \frac{17}{4}$, $s_0 = 0$ and $N = \frac{17}{2}$ gives an algebraic certificate for the boundedness of S_A.

There exist spectrahedra whose elements have coordinates of double-exponential size in the number of variables and whose distance to the origin grows double-exponentially in the number of variables (see [1, 33]). Hence, in general one cannot expect to have a certificate of polynomial size for the boundedness of the spectrahedron.

4. Containment problems

As a next step in the class of algorithmic problems on spectrahedra, we consider containment problems: Given two linear pencils $A(x) \in \mathcal{S}_k[x]$ and $B(x) \in \mathcal{S}_l[x]$, is $S_A \subseteq S_B$?

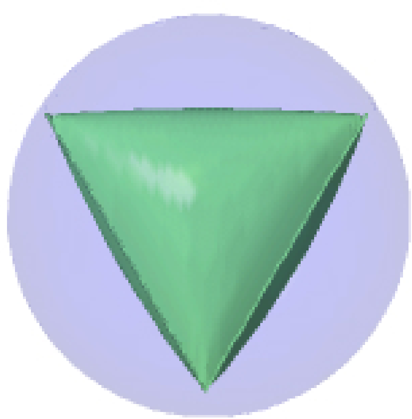

Figure 2. Visualization of an elliptope in a ball.

Containment problems of convex sets are a classical topic in convex geometry (see, e.g., Gritzmann and Klee for the containment of polytopes and a number of computational aspects [10]). In the context of spectrahedra, the study of algorithmic approaches and relaxations has been initiated by Ben-Tal and Nemirovski [2] who investigated the case where S_A is a cube and S_B is an arbitrary spectrahedron ("matrix cube problem"). Figure 2 visualizes an elliptope in a ball.

4.1. **Complexity of containment problems for spectrahedra.** It is useful to start from the case of polytopes. Here, it is well-known that the computational complexity of deciding containment of a given polytope P in a given polytope Q strongly depends on the type of input representations. We assume that all input data is given in terms of rational numbers, and the dimension is part of the input.

Proposition 4.1. [7] *The following problems can be decided in polynomial time.*

(1) *Given $\mathcal{H}$-polytopes P and Q, is $P \subseteq Q$?*
(2) *Given $\mathcal{V}$-polytopes P and Q, is $P \subseteq Q$?*
(3) *Given a $\mathcal{V}$-polytope P and an $\mathcal{H}$-polytope Q, is $P \subseteq Q$?*

In contrast to this, deciding whether an $\mathcal{H}$-polytope is contained in a $\mathcal{V}$-polytope is co-NP-complete.

In [21], this classification has been extended to containment problems involving polytopes and spectrahedra, where the spectrahedra are given by a linear pencil with rational entries. The main hardness results are given by the subsequent Theorems 4.2 and Proposition 4.3.

Theorem 4.2. [21] *The following problems are co-NP-hard:*

(1) *Given a spectrahedron S_A and a $\mathcal{V}$-polytope Q, is $S_A \subseteq Q$?*
(2) *Given an $\mathcal{H}$-polytope P and a spectrahedron S_B, is $P \subseteq S_B$?*

The latter hardness statement persists if the $\mathcal{H}$-polytope is a standard cube or if the outer spectrahedron is a ball.

Since deciding whether a given rational matrix is positive semidefinite can be done in polynomial time, it can be decided in polynomial time whether a $\mathcal{V}$-polytope is contained in a spectrahedron. As mentioned earlier, the question “Can semidefinite feasibility problems SDFP be solved in polynomial time?” is an open complexity question. Consequently, the following statement on containment of a spectrahedron in an $\mathcal{H}$-polytope does not give a complete answer concerning polynomial solvability of these containment questions in the Turing machine model. If the additional inequalities were non-strict, then we had to decide a finite set of problems from the complement of the class SDFP.

Proposition 4.3. [21] *The problem of deciding whether a spectrahedron is contained in an $\mathcal{H}$-polytope can be formulated by the complement of semidefinite feasibility problems (involving also strict inequalities), whose sizes are polynomial in the input data.*

Since Theorem 4.2 also implies that deciding containment of a spectrahedron in a spectrahedron is co-NP-hard, all the relevant cases are covered. See Table 1 for a condensed presentation, where $\mathcal{H}$, $\mathcal{V}$ and $\mathcal{S}$ stand for $\mathcal{H}$-polytope, $\mathcal{V}$-polytope and spectrahedron, respectively.

For the computational question of deciding whether a spectrahedron is a polyhedron see Bhardwaj, Rostalski and Sanyal [3], and for sos-based approaches to the NP-hard containment problem of deciding whether an $\mathcal{H}$-polytope is contained in a $\mathcal{V}$-polytope see Kellner and Theobald [20].

	$\mathcal{H}$	$\mathcal{V}$	$\mathcal{S}$
$\mathcal{H}$	P	co-NP-complete	co-NP-hard
$\mathcal{V}$	P	P	P
$\mathcal{S}$	"SDFP"	co-NP-hard	co-NP-hard

Table 1. Computational complexity of containment problems, where the rows refer to the inner set and the columns to the outer set. "SDFP" refers to the formulations through semidefinite feasibility problems as described in Proposition 4.3.

4.2. **From Farkas-type characterizations for polytopes to relaxations for spectrahedra.** In this section, we present some recent results on semidefinite relaxations which provide a sufficient criterion for the containment problem of spectrahedra. Here, relaxation means that some conditions are omitted from the original problem in order to obtain a more tractable, semidefinite formulation.

It is helpful to start from the containment problem for pairs of $\mathcal{H}$-polytopes, which by Proposition 4.1 can be decided in polynomial time. Indeed, as a consequence of the affine form of Farkas' Lemma, this can be achieved by solving a linear program, as stated by the following necessary and sufficient criterion (see, e.g., [21]). Recall that a real matrix with non-negative entries is called right stochastic if each row sums to one.

Proposition 4.4. *Let $P_A = \{x \in \mathbb{R}^n : \mathbb{1}_k + Ax \geq 0\}$ and $P_B = \{x \in \mathbb{R}^n : \mathbb{1}_l + Bx \geq 0\}$ be polytopes. Then $P_A \subseteq P_B$ if and only if there exists a right stochastic matrix C with $B = CA$.*

For the treatment of containment of spectrahedra, a good starting point is the sufficient criterion given by Helton, Klep and McCullough [14]. As earlier, let $A(x) \in \mathcal{S}_k[x]$ and $B(x) \in \mathcal{S}_l[x]$ be linear pencils. In the subsequent statement, the indeterminate matrix $C = (C_{ij})_{i,j=1}^k$ is a symmetric $kl \times kl$-matrix where the C_{ij} are $l \times l$-blocks.

Theorem 4.5. *(*[14, Theorem 4.3]*, see also* [21, Theorems 4.3 and 4.4]*) Let $A(x) \in \mathcal{S}_k[x]$ and $B(x) \in \mathcal{S}_l[x]$ be linear pencils. If one of the systems*

$$C = (C_{ij})_{i,j=1}^k \succeq 0, \quad \forall p = 0, \ldots, n: \; B_p = \sum_{i,j=1}^k a_{ij}^p C_{ij} \tag{4.1}$$

or

$$C = (C_{ij})_{i,j=1}^k \succeq 0, \quad B_0 - \sum_{i,j=1}^k a_{ij}^0 C_{ij} \succeq 0, \quad \forall p = 1, \ldots, n: \; B_p = \sum_{i,j=1}^k a_{ij}^p C_{ij} \tag{4.2}$$

is feasible, then $S_A \subseteq S_B$. Here, a_{ij}^p denotes the (i,j)-entry of A_p.

Note that whenever (4.1) is satisfied, condition (4.2) is satisfied as well. However, (4.2) contains an additional sos-condition. An elementary proof of Theorem 4.5 was given in [21] – here, we provide a slight variant of that proof.

Proof of Theorem 4.5. For $x \in S_A$, the last two conditions in (4.2) imply

$$B(x) = B_0 + \sum_{p=1}^{n} x_p B_p \ \succeq \ \sum_{i,j=1}^{k} a_{ij}^0 \, C_{ij} + \sum_{p=1}^{n} \sum_{i,j=1}^{k} x_p \, a_{ij}^p \, C_{ij} \ = \ \sum_{i,j=1}^{k} (A(x))_{ij} \, C_{ij} \, . \tag{4.3}$$

For any block matrices $S = (S_{ij})_{ij}$ and $T = (T_{ij})_{ij}$, consisting of $k \times k$ blocks of size $p \times p$ and $q \times q$, the *Khatri-Rao product* of S and T is defined as the block-wise Kronecker product of S and T, i.e.,

$$S * T = (S_{ij} \otimes T_{ij})_{ij} \ \in \ \mathcal{S}_{kpq} \, .$$

If both S and T are positive semidefinite, then the Khatri-Rao product $S * T$ is positive semidefinite as well, see [27, Theorem 5].

In our situation, we have $p = 1$ and $q = l$, and the Khatri-Rao product

$$A(x) * C \ = \ ((A(x))_{ij} \otimes C_{ij})_{i,j=1}^{k} \ = \ ((A(x))_{ij} C_{ij})_{i,j=1}^{k}$$

is positive semidefinite. And since $B(x)$ is given in (4.3) as a sum of submatrices of $A(x) * C$, we obtain that $B(x)$ is positive semidefinite, i.e., $x \in S_B$.

When starting from system (4.1), the inequality chain in (4.3) becomes an equality, and the remaining part of the proof remains valid. □

For both systems (4.1) and (4.2) the feasibility depends on the linear pencil representation of the sets involved. If S_B is contained in the positive orthant, a stronger version can be given.

Corollary 4.6. *Let $A(x) \in \mathcal{S}_k[x]$ and $B(x) \in \mathcal{S}_l[x]$ be linear pencils and let S_A be contained in the non-negative orthant. If the system*

$$C = (C_{ij})_{i,j=1}^{k} \succeq 0, \quad B_0 - \sum_{i,j=1}^{k} a_{ij}^0 C_{ij} \succeq 0, \quad \forall p = 1, \ldots, n : \ B_p - \sum_{i,j=1}^{k} a_{ij}^p C_{ij} \succeq 0 \tag{4.4}$$

is feasible, then $S_A \subseteq S_B$.

Proof. Since S_A is contained in the non-negative orthant, any $x \in S_A$ has non-negative coordinates, and hence,

$$B(x) = B_0 + \sum_{p=1}^{n} x_p B_p \ \succeq \ \sum_{i,j=1}^{k} a_{ij}^0 \, C_{ij} + \sum_{p=1}^{n} \sum_{i,j=1}^{k} x_p \, a_{ij}^p \, C_{ij} \ = \ \sum_{i,j=1}^{k} (A(x))_{ij} \, C_{ij} \, .$$

□

The version (4.4) is strictly stronger than system (4.1). There are cases, where a solution to the condition (4.4) exists, even though the original system (4.1) is infeasible.

4.3. **Exact cases of the relaxation.** It turns out that the sufficient semidefinite criteria (4.1) and (4.2) even provide exact containment characterizations in several important cases.

Recall the normal form for polyhedral spectrahedra introduced in Section 2, and let us also introduce a normal form for the class of centered and aligned ellipsoids. Here, an ellipsoid is called *centered* if it is centrally symmetric, and it is called *aligned* if its axes are aligned to the directions of the coordinate axes. A centered and aligned ellipsoid with semi-axes of lengths $a_1, \ldots, a_n$ can be written as the spectrahedron S_A of the monic linear pencil

$$A(x) \;=\; I_{n+1} + \sum_{p=1}^{n} \frac{x_p}{a_p}(E_{p,n+1} + E_{n+1,p}), \tag{4.5}$$

where E_{ij} denotes the matrix with a one in position (i,j) and zeros elsewhere. This representation is called the the *normal form of the ellipsoid*. If $a_1 = \cdots = a_n$, this gives the *normal form of a ball*. The exact characterizations also use the following *extended form* $S_{\widehat{A}}$ of a spectrahedron S_A. Given a linear pencil $A(x) \in \mathcal{S}_k[x]$, we call the linear pencil with an additional 1 on the diagonal

$$\widehat{A}(x) \;=\; \begin{pmatrix} 1 & 0 \\ 0 & A(x) \end{pmatrix} \in \mathcal{S}_{k+1}[x] \tag{4.6}$$

the *extended linear pencil* of $S_A = S_{\widehat{A}}$. Note that the spectrahedra $S_A = S_{\widehat{A}}$ coincide. The entries of $\widehat{A}_p$ in the pencil $\widehat{A}(x) = \widehat{A}_0 + \sum_{p=1}^{n} x_p \widehat{A}_p$ are denoted by $\widehat{a}_{ij}^{p}$ for $i,j = 0, \ldots, k$.

Theorem 4.7. [21] *Let $A(x) \in \mathcal{S}_k[x]$ and $B(x) \in \mathcal{S}_l[x]$ be monic linear pencils. In the following cases the criteria* (4.1) *as well as* (4.2) *are necessary and sufficient for the inclusion $S_A \subseteq S_B$:*

(1) *if $A(x)$ and $B(x)$ are normal forms of centered and aligned ellipsoids,*
(2) *if $A(x)$ and $B(x)$ are normal forms of a ball and an $\mathcal{H}$-polyhedron, respectively,*
(3) *if $B(x)$ is the normal form of a polytope,*
(4) *if $\widehat{A}(x)$ is the extended form of a spectrahedron and $B(x)$ is the normal form of a polyhedron.*

Recently, Fritz, Netzer and Thom have shown the following exactness result which distinguishes the simplex situation within the situation that S_A is a polytope.

Theorem 4.8. [8, Cor. 5.3] *For a fixed polytope S_A, the criterion* (4.1) *is exact for any spectrahedron S_B if and only if S_A is a simplex, and this statement is independent of the representing pencil of the polytope S_A.*

Note that all the exactness statements presented in this section refer to exact characterizations of the containment problem in terms of a formulation as semidefinite program. Similar to the case of the infeasibility certificates in Section 3, when it comes to actually solving the semidefinite programs, in case of employing numerical solvers this involves additional numerical aspects.

5. Sufficient semidefinite hierarchies for containment of spectrahedra

In this section, we present two hierarchical approaches for the containment problem in terms of polynomial matrix inequalities (PMI). The underlying PMI hierarchy was developed by Kojima [24], Hol and Scherer [17], as well as Henrion and Lasserre [16], and it generalizes the Lasserre hierarchy for polynomial optimization [25]. We then discuss the relation of the two approaches for containment to each other as well as the connection to positive maps.

5.1. **From the sufficient criterion to a moment hierarchy of sufficient criteria.** As before, let $A(x) \in \mathcal{S}_k[x]$ and $B(x) \in \mathcal{S}_l[x]$, and assume that $S_A \neq \emptyset$. By definition of a positive semidefinite matrix, we have $S_A \subseteq S_B$ if and only if the infimum μ of the polynomial optimization problem

$$\begin{aligned} \mu \;=\; \inf\;\; & z^T B(x) z \\ \text{s.t.}\;\; & A(x) \succeq 0 \\ & g(z) := z^T z - 1 = 0 \end{aligned} \tag{5.1}$$

in the variables $(x, z) = (x_1, \ldots, x_n, z_1, \ldots, z_l)$ is non-negative (cf. [22] for improved numerical stability). Setting $G_A(x, z)$ to be the matrix with blocks $A(x)$ as well as the two 1×1-blocks $g(z)$ and $-g(z)$, the constraints can be written as $G_A(x, z) \succeq 0$.

The general framework of moment relaxations for PMIs translates the optimization problem into a semidefinite hierarchy as a relaxation to problem (5.1). Assuming, for ease of notation, that we are working over the variables $x = (x_1, \ldots, x_n)$, let $y = (y_\alpha)$ be a real sequence indexed by the monomials in x. Let $M(y)$ be the infinite moment matrix defined by $(M(y))_{\alpha,\beta} = L_y(([x][x]^T)_{\alpha,\beta}) = y_{\alpha+\beta}$, where $[x]$ is the infinite vector of monomials in $x_1, \ldots, x_n$ and L_y is the linearization operator that maps a monomial x^α to the associated moment variable y_α. $M_t(y)$ denotes the truncated moment matrix that contains only entries $(M(y))_{\alpha,\beta}$ with $|\alpha|, |\beta| \leq t$.

The positive semidefiniteness constraint on a matrix polynomial $G(x) \in \mathcal{S}_k[x]$ is captured by the *localizing matrices*. The truncated localizing matrix $M_t(Gy)$ is defined as $M_t(Gy) = L_y([x]_t [x]_t^T \otimes G(x))$, where application of the linearization operator L_y is component-wise. If d_G denotes the highest degree of a polynomial appearing in $G(x)$, then only linearization variables coming from monomials of degree at most $2t + d_G$ appear in $M_t(Gy)$.

For $t \geq 2$, the t-th relaxation of the polynomial optimization problem (5.1) becomes

$$\begin{aligned} \mu_{\text{mom}}(t) \;=\; \inf\;\; & L_y(z^T B(x) z) \\ \text{s.t.}\;\; & M_t(y) \succeq 0 \\ & M_{t-1}(G_A y) \succeq 0\,. \end{aligned} \tag{5.2}$$

Note that $t = 2$ is the initial relaxation order. The sequence $\mu_{\text{mom}}(t)$ for $t \geq 2$ is monotone non-decreasing. If for some t^* the condition $\mu_{\text{mom}}(t^*) \geq 0$ is satisfied, then $S_A \subseteq S_B$.

The following connection will be further refined and extended in Theorems 5.5 and 5.7.

Theorem 5.1. [22] *Let $S_A \neq \emptyset$. Then $\mu_{\text{mom}}(2) \geq 0$ (and thus $\mu_{\text{mom}}(t) \geq 0$ for all $t \geq 2$) if and only if the SDFP* (4.1) *has a solution $C \succeq 0$, that is, if and only if the sufficient containment criterion in Theorem 4.5 is satisfied.*

5.2. **The Hol-Scherer hierarchy.** The background of the second hierarchical approach is provided by Hol-Scherer's Positivstellensatz. In order to characterize matrix polynomials which are positive semidefinite on a spectrahedron, we consider a generalization of the quadratic module (3.1) for a matrix polynomial $G \in \mathcal{S}_k[x]$. For any $l \geq 0$, let

$$M_{G,l} \;=\; \{S_0 + \langle S, G\rangle_l \,:\, S_0 \in \mathbb{R}[x]^{l \times l} \text{ sos-matrix},\ S \in \mathbb{R}[x]^{kl \times kl} \text{ sos-matrix}\}\,,$$

where for matrices $U = (U_{ij})_{i,j=1}^{l} \in \mathcal{S}_{kl}$ and $V \in \mathcal{S}_k$ the l^{th} *scalar product* is defined by

$$\langle U, V\rangle_l \;=\; (\langle U_{ij}, V\rangle)_{i,j=1}^{l} \;\in\; \mathcal{S}_l\,.$$

Proposition 5.2 (Hol, Scherer [17]). *Let $G(x)$ be a matrix polynomial in $\mathcal{S}_k[x]$. Further assume that there exists a polynomial $p(x) = s(x) + \langle S(x), G(x)\rangle$ for some sos-polynomial $s(x) \in \mathbb{R}[x]$ and some sos-matrix $S(x) \in \mathcal{S}_k[x]$, such that the level set $\{x \in \mathbb{R}^n \,:\, p(x) \ge 0\}$ is compact. Then every matrix polynomial $F \in \mathcal{S}_l[x]$ which is positive semidefinite on $\{x \in \mathbb{R}^n \,:\, G(x) \succeq 0\}$ is contained in the quadratic module $M_{G,l}$.*

As before, let $A(x) \in \mathcal{S}_k[x]$ and $B(x) \in \mathcal{S}_l[x]$ be linear pencils, and consider for $t \ge 0$ the truncated quadratic module

$$M_{A,l}^{(t)} \;=\; \{S_0 + \langle S, A\rangle_l \,:\, S \in \mathbb{R}[x]_{2t}^{l\times l} \text{ sos-matrix}, S \in \mathbb{R}[x]_{2t}^{kl\times kl} \text{ sos-matrix}\}\,. \tag{5.3}$$

Proposition 5.3. [19, 22] *Let $A(x) \in \mathcal{S}_k[x]$, $B[x] \in \mathcal{S}_l[x]$ be linear pencils.*

1. *If $B(x) \in M_{A,l}^{(t)}$ for some $t \ge 0$, then $S_A \subseteq S_B$.*
2. *Let S_A be bounded and $B(x)$ be a reduced pencil. If S_A is contained in the interior of S_B then there exists some $t \ge 0$ such that $B(x) \in M_{A,l}^{(t)}$.*

For computational purposes and to relate the hierarchy to the moment approach in Section 5.1, it is useful to pass over to a robust optimization version. First note that $S_A \subseteq S_B$ if and only there exists some $\lambda \ge 0$ with

$$B(x) - \lambda I_l \succeq 0 \quad \text{ for all } x \in S_A\,.$$

Now we consider the hierarchy of optimization problems

$$\begin{aligned} \lambda_{\text{sos}}(t) = \sup \;\; & \lambda \\ \text{s.t.} \;\; & B(x) - \lambda I_l - (\langle S_{i,j}(x), A(x)\rangle)_{i,j=1}^{l} \text{ sos-matrix} \\ & S(x) = (S_{i,j}(x))_{i,j=1}^{l} \in \mathcal{S}_{kl}[x] \text{ sos-matrix,} \end{aligned} \tag{5.4}$$

where $S(x)$ has $l \times l$ blocks of size $k \times k$ with entries of degree at most $2t \ge 0$. Given some $t \ge 0$, we observe that $\lambda_{\text{sos}}(t) \ge 0$ implies that $S_A \subseteq S_B$.

Theorem 5.4. [22] *Let $A(x) \in \mathcal{S}_k[x]$ be a linear pencil such that the spectrahedron S_A is bounded. Then the optimal values of the moment relaxation* (5.2) *and of the sos-relaxation* (5.4) *converge from below to the optimal value of the polynomial optimization problem* (5.1)*, i.e., $\mu_{\text{mom}}(t) \uparrow \mu$ and $\lambda_{sos}(t) \uparrow \mu$ as $t \to \infty$.*

The following theorem shows that that the sufficient criteria coming from the hierarchies of relaxations are at least as strong as the criterion (4.1) by showing that feasibility of the criterion (4.1) implies $\mu(t) \ge 0$ and $\lambda_{\text{sos}}(t) \ge 0$ in the initial relaxation steps of the semidefinite hierarchies (5.2) and (5.4). From this relation, we get that in some cases already the initial relaxation step of the hierarchies gives an exact answer to the containment problem; see Section 5.3.

Theorem 5.5. [22] *Let $S_A \neq \emptyset$. Then for the properties*

1. *the SDFP* (4.1) *has a solution $C \succeq 0$,*

(2) $\lambda_{\text{sos}}(0) \geq 0$,
(3) $\mu_{\text{mom}}(2) \geq 0$,
(4) $S_A \subseteq S_B$,

we have the implications $1 \iff 2 \Longrightarrow 3 \Longrightarrow 4$.

For further aspects on the Hol-Scherer hierarchy for containment see also Kellner's dissertation [19].

5.3. **(Completely) positive maps.** We briefly discuss the connection of the hierarchies to the theory of positive maps and completely positive maps. For background on positive and completely positive maps see, e.g., [31].

Definition 5.6. Given two linear subspaces $\mathcal{A} \subseteq \mathbb{R}^{k\times k}$ and $\mathcal{B} \subseteq \mathbb{R}^{l\times l}$, a linear map $\Phi : \mathcal{A} \to \mathcal{B}$ is called *positive* if $\Phi(A) \succeq 0$ for any $A \in \mathcal{A}$ with $A \succeq 0$.

The map Φ is called *d-positive* if the map $\Phi_d : \mathbb{R}^{d\times d}\otimes\mathcal{A} \to \mathbb{R}^{d\times d}\otimes\mathcal{B}$, $M\otimes A \mapsto M\otimes\Phi(A)$ is positive, i.e., if $M \otimes \Phi(A) \succeq 0$ whenever $M \otimes A \succeq 0$. And Φ is called *completely positive* if Φ_d is positive for all $d \geq 1$.

As explained in the following, checking positivity of a map on a subspace is equivalent to checking containment for spectrahedra. This does not only provide a structural connection, but also allows to apply the hierarchy for the containment question to positivity questions of maps on subspaces, such as the ones in [13]. Note that for the special case of detecting positivity of a map on the whole space, Nie has recently shown that this can be done by solving a finite number of semidefinite relaxations [29].

For simplicity, we restrict to the situation that $A_0, \ldots, A_n$ are linearly independent and that S_A is bounded. Let the linear map $\Phi_{AB} : \mathcal{A} \to \mathcal{B}$ be defined through

$$\Phi_{AB}(A_p) \ = \ B_p \quad \text{for } 0 \leq p \leq n\,.$$

Then the following extension of Theorem 5.5 states the connection of the semidefinite hierarchies with positive and completely positive maps.

Theorem 5.7. [22] *Let $A_0, \ldots, A_n$ be linearly independent and S_A be non-empty and bounded. Then for the properties*

(1) *Φ_{AB} is completely positive,*
(2) *the SDFP (4.1) has a solution $C \succeq 0$,*
(3) $\lambda_{\text{sos}}(0) \geq 0$,
(4) $\mu_{\text{mom}}(2) \geq 0$,
(5) $S_A \subseteq S_B$,
(6) *Φ_{AB} is positive,*

we have the implications $1 \Longleftarrow 2 \iff 3 \Longrightarrow 4 \Longrightarrow 5 \iff 6$. *If $\mathcal{A}$ contains a positive definite matrix, then the implication $1 \Longleftarrow 2$ is an equivalence.*

Note that Theorem 5.4 implies a partial converse of the implication $3 \Longrightarrow 4$. Namely, if $\emptyset \neq S_A \subseteq S_B$ and S_A is bounded, then $\mu_{\text{mom}}(t) \uparrow \mu \geq 0$ for $t \to \infty$.

Theorem 5.7 allows to extend the exactness results from Theorem 4.7 to the initial step of the hierarchy (5.2).

Remark 5.8. It is well-known that the map Φ_{AB} connects to the characterization of biquadratic forms in Proposition 2.6 (see [5]). A positive linear map $\Phi : \mathcal{S}_k \to \mathcal{S}_l$ is completely positive if and only if Φ can be written as $\Phi(A) = \sum_s V_s^T A V_s$ for some matrices $V_s \in \mathbb{R}^{k \times l}$ if and only if the corresponding biquadratic form $F(x, y)$ is a sum of squares of bilinear forms, $F(x, y) = \sum_s (x^T V_s y)^2$.

6. Final remarks

We have reviewed some recent developments on fundamental algorithmic problems in spectrahedral computation. While containment questions for spectrahedra are co-NP-hard in general, the hierarchical relaxation techniques give a practical way of certifying containment. For detailed experiments of the two approaches (5.2) and (5.4), see [21] and [22]. In practice, the sufficient criteria perform well already for small relaxation orders.

While in many situations the running times of the two hierarchical approaches for containment are comparable, the number of linearization variables in the moment approach (5.2) does not depend on the size k of the pencil $A(x)$. Therefore, for problems with relatively large k, this approach to the containment problem seems to be superior to the approach based on Hol-Scherer's hierarchy.

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