Hierarchical Random Walks

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Abstract. This is a survey of recent work on hierarchical random walks with emphasis on transience-recurrence phenomena and in particular on the notion of order of transience. A family of hierarchical random fields is also described. Applications of hierarchical random walks and fields to the behavior near critical dimensions in statistical physics and branching processes are discussed.

1. Introduction

Random walks and random fields on the d-dimensional lattice $\mathbb{Z}^d$ are basic probabilistic objects that are not only of intrinsic mathematical interest but provide stochastic models of phenomena in the physical and biological sciences. An important aspect of both subjects is the role of dimension. In a seminal paper in 1921, Polya [**Pol**] proved that random walks have qualitatively different long-time behavior in low dimensions ($d = 1, 2$) and in high dimensions ($d \geq 3$). The class of Gibbs random fields plays an important role in statistical physics, non-negative infinitely divisible random fields arise in the study of superprocesses and Gaussian random fields arise in the study of fluctuations in both statistical physics and superprocesses. Dimension also plays a key role in the study of fluctuations of random fields in statistical physics and in particular the dimension $d = 4$ arises as a "critical dimension" in this case. It turns out that many questions involving the phenomena of critical dimensions are difficult to resolve and this has led to the introduction of the class of *hierarchical models* that, among other things, serve as caricatures of random walks and random fields in the neighbourhood of a critical dimension. The purpose of this paper is to review some recent developments in the study of hierarchical random walks and hierarchical random fields and their application to the study of large scale behavior of some classes of interacting systems.

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For a class of Markov processes on locally compact Abelian groups, the definition of *order of transience* is introduced in [**DGW2**]. The related notions of k*-strong transience* and k*-weak transience* for each integer $k \geq 1$, which generalize the definitions of weak and strong transience of [**PS**] and provide a classification of transient processes, were given in [**DGW1**]. For example, d-dimensional Brownian motion is k-strongly transient for $d > 2(k+1)$ and k-weakly transient for $2k < d \leq 2(k+1)$. The case $k = 1$ corresponds to the usual strong and weak transience. These concepts arose in the context of occupation time fluctuations of infinite particle systems with multilevel branching, where the particles migrate independently according to the given process and k is the number of branching levels. These definitions were originally stated in terms of powers of the Green potential operator of the process [**DGW1**] but for processes on metric groups the definitions can be recast in terms of moments of last exit times from closed balls.

In particular, we consider the countable Abelian group Ω_N consisting of sequences of numbers in $\{0, 1, \ldots, N-1\}$ only a finite number of which are different from zero with componentwise addition modulo N. Random walks on Ω_N, called *hierarchical random walks*, were introduced by Spitzer [**Sp**](p. 93) and used by Sawyer and Felsenstein [**SF**] to study genetic relatedness in a spatially structured population and are closely related to the hierarchical random field models in statistical physics first introduced by Dyson [**Dy**]. In addition to the study of random walks on Ω_N for a fixed N, we also consider sequences of random walks on Ω_N indexed by N and chosen so that they have the same order of transience. The limiting behavior of this sequence as $N \to \infty$ (the *hierarchical mean field limit*) involves another level of simplification but this limit can provide useful information. In particular, under the appropriate parameterization we can often construct a sequence of caricatures and by taking the limit as $N \to \infty$ obtain a probabilistic object that provides information on the critical behavior.

In Section 5 we indicate the role of hierarchical random walks and fields and the related notions of critical dimension and order of transience in the study of interacting systems. Examples from statistical physics and branching systems are given.

2. Hierarchical Structures and Ultrametrics

Hierarchical structures arise in the physical, biological and social sciences due to the multiscale organization of many natural objects. A simple example is a phylogenetic tree in which each node at one level has a number of descendant nodes at the next level and the *hierarchical distance* between nodes is the depth of their common ancestor. Other examples arise in statistical physics such as quasi-fractal energy landscapes with random energy barriers separating local minima [**RTV**]. In this case the hierarchical distance between two local minima in the landscape is the minimum (over all possible paths) of the height climbed in going from one to the other.

To begin the mathematical formulation we consider a *finite hierarchy* of height $K \in \mathbb{N}$.

$$\Omega^K_{\{N_i\}_{i=1}^K} = \{\omega = \{\omega_i\}_{i=1,\ldots,K}\}$$
$$\omega_i \in \{0, \ldots, N_i\}$$

where $N_i \in \mathbb{N}\backslash\{1\}$. In the case $N_i = N \; \forall \; i$ we denote this by Ω_N^K.

Next for each $N \in \mathbb{N}$ we define the *countable hierarchy*

$$\begin{aligned} \Omega_N = \{\omega = &\{\omega_i\}_{i=1,,2\ldots}\} \\ &\omega_i \in \{0, \ldots, N-1\}, \; \omega_i = 0 \text{ for a.a. } i \end{aligned}$$

The notion of hierarchical distance can then be defined by

$$d(\omega, \omega') := \max_j \{\omega_j \neq \omega_j'\}$$

The distance measure $d(\omega, \omega')$ is an *ultrametric*, that is, it satisfies the strong (or non-archimedean) triangle inequality

$$d(\omega, \omega'') \leq \max(d(\omega, \omega'), d(\omega'', \omega'))$$

In some ways ultrametric spaces are very different from Euclidean spaces; e.g., two balls are either disjoint or one contains the other and all points in the interior of a closed ball of radius k are at distance k from the boundary (see e.g. [**Sch**]). These spaces grew out of the study of p-adic numbers at the end the 19th century. Let p be a prime number. Every rational $x \neq 0$ can be written in the form $x = \frac{s}{t} p^n$ for some integer n, where s and t are integers not divisible by p. Let $|x|_p = p^{-n}$ (and $|0|_p = 0$). Then $d(x, y) := |x - y|_p$ satisfies the strong triangle inequality. (This leads to the p-adic numbers by an extension procedure).

Random fields and random walks on ultrametric spaces have become a useful tool in statistical physics, biology, optimization theory and other fields [**RTV**]. Some references on applications in physics emphasizing ultrametricity are [**BAHM**], [**FT**], [**OS**], [**Sc**]. In the case of p-adics with p prime, random walks have been studied in detail in [**AKZ**], [**AZ**] (and references therein). We will describe in more detail some applications in Section 5.

We now return to the space Ω_N and note that it can be given the structure of a countable Abelian group

$$\Omega_N = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}_N,$$

that is, the direct sum over $\mathbb{N}$ of copies of the cyclic group $\mathbb{Z}_N = \{0, \ldots, N-1\}$. In other words, Ω_N is a countable Abelian group with componentwise addition modulo N. Note that the balls $\Omega_N^k = \{\xi : d(0, \xi) \leq k\}$ are subgroups of Ω_N. In addition, the ultrametric d is invariant under translations so that we can write

$$d(\omega, \omega') = |\omega - \omega'|.$$

3. Random Walks and Lévy Processes

3.1. Random Walks and Lévy Processes on Abelian Groups. In this section we give a brief review of the basic notions of random walks and Lévy processes on groups.

Let S be a locally compact (additive) Abelian group with countable base and with Haar measure ρ. A discrete time random walk, $\{W_n\}_{n \in \mathbb{Z}_+}$, is prescribed by a transition function

$$P(x, dy) := P(W_{n+1} \in dy | W_n = x) = p(d(y - x))$$

where p is a probability measure on S. The corresponding k-step transition function is

$$P^k(x,dy) := P(W_{n+k} \in dy | W_n = x).$$

A continuous time random walk $\{W_t : t \geq 0\}$ with jump rate 1 is then defined by the transition function

$$P_t(x,dy) := P_x(W_t \in dy), \quad t \geq 0,$$

$$P_t(x,dy) = \sum_{k=0}^{\infty} \frac{e^{-t}t^k}{k!} P^k(x,dy).$$

A natural generalization of continuous time random walks is the notion of Lévy process.

Definition 3.1. A S-valued process $\{X_t : t \geq 0\}\}$ is a *Lévy process* if it is stochastically continuous and has stationary and independent increments.

We associate to a Lévy process a semigroup $\{T_t : t \geq 0\}$ on $\mathcal{B}_c(S)$, the space of bounded measurable functions on S with compact support, as follows:

$$T_t\varphi(x) = E_x(\varphi(X_t)),$$

The λ-potential of X is the operator

$$G^{(\lambda)}\varphi = \int_0^\infty e^{-\lambda} T_t \varphi dt, \quad \varphi \in \mathcal{B}_c(S), \;\; 0 \leq \lambda < \infty.$$

The classical Green operator is given by $G = G^{(0)} = \int_0^\infty T_t dt$. The *fractional operator powers* of G are given by

$$G^\zeta \varphi = \frac{1}{\Gamma(\zeta)} \int_0^\infty t^{\zeta - 1} T_t \varphi dt, \;\; \zeta \geq 0 \quad \varphi \in \mathcal{B}_c(S).$$

3.2. Transience-Recurrence Properties. In order to introduce the basic notion of transience and recurrence, we consider the *last exit time*, L_A, of W from a non-empty set A defined by

$$L_A := \begin{cases} \sup\{t \geq 0 : X_t \; \in \; A\} & \text{if } \{t \geq 0 : X_t \; \in \; A\} \neq \emptyset \\ \infty & \text{otherwise.} \end{cases}$$

Definition 3.2. The Lévy process X_t on S is *transient* if for any compact set K

$$P(L_K < \infty) = 1.$$

and *recurrent* if it is not transient.

The following result of Sato and Watanabe is the basis for a finer classification of the transience properties of random walks in terms of the moments of last exit times.

Proposition 3.3. (Sato, Watanabe [**SW1**],[**SW2**]) Assume that X_t is transient, for any compact set $K \subset S$

$$\sup_{x \in K} G1_K(x) < \infty$$

and for any compact set $C \subset K^\circ$

$$\inf_{x \in C} G^{(\lambda)} 1_K(x) > 0, \;\; \lambda > 0.$$

Then there exist positive constants c_1 and c_2 such that for all $\zeta > 0$ and $x \in S$

$$c_1 G^{\zeta+1} 1_C(x) \le E_x L_C^{\zeta} \le c_2 G^{\zeta+1} 1_K(x).$$

Definition 3.4. The *order of transience*, γ, of the random walk W is defined by

$$\gamma := \sup\{\zeta \ge 0 : E_0 L_K^{\zeta} < \infty \quad \text{for all compact } K\},$$

or equivalently

$$\gamma := \sup\{\zeta \ge 0 : G^{\zeta+1}\varphi < \infty \text{ for } \varphi \in C_c^+(S)\}$$

where $C_c^+(S)$ denotes the space of nonnegative continuous functions on S with compact support.

Remark 3.5. Following Sato and Watanabe let

$$\mathcal{T} := \begin{cases} \{\zeta \ge 0 : E_0 L_K^{\zeta} < \infty & \text{for all compact } K\} \text{ if the process is transient,} \\ \emptyset & \text{if the process is recurrent.} \end{cases} \tag{3.1}$$

In the terminology of [**DGW1**], given $k \in \mathbb{Z}_+$, the process is

$$\begin{aligned} k - \text{strongly transient} \qquad & \text{if } \; k \in \mathcal{T}, \quad \text{and} \\ k - \text{weakly-transient} \quad & \text{if } \; k-1 \in \mathcal{T} \text{ and } k \notin \mathcal{T}. \end{aligned}$$

Remark 3.6. The order of transience can be viewed as a generalization of the notion of "critical dimension". Note that $G^{\zeta+1}\varphi$ at $\zeta = \gamma$ can be either finite or infinite - we will give examples of both possibilities below.

3.3. The classical processes in $\mathbb{R}^d$. In this section we briefly review the classical results on random walks and Lévy processes in $\mathbb{Z}^d$ and $\mathbb{R}^d$.

First recall that symmetric nearest neighbour random walks in $\mathbb{Z}^d$ are recurrent in dimensions $d = 1, 2$ and transient in dimensions $d \ge 3$. Moreover, since the rate of decay of the transition probabilities for simple symmetric d-dimensional random walk is $p_t(0,0) \sim \text{const}.t^{-d/2}$, its order of transience is $\gamma = d/2 - 1$.

We next recall the classical characterization of Lévy processes in $\mathbb{R}^d$.

Theorem 3.7. *(Lévy-Khintchine representation) A Lévy process in $\mathbb{R}^d$ has the characteristic function (i.e. Fourier transform)*

$$\begin{aligned} &E[e^{i(z,X_t)}] \\ &= exp\left[t\left(-\frac{1}{2}(z,Az) + \int_{\mathbb{R}^d}(e^{i(z,x)} - 1 - i(z,x)1_{\{|x|\le 1\}}(x))\nu(dx) + i(m,z)\right)\right] \end{aligned}$$

where A is a symmetric nonnegative definite $d\times d$ matrix, ν is a measure on $\mathbb{R}^d\backslash\{0\}$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty$, and $m \in \mathbb{R}^d$.

For the proof see [**S**].

The case $A = Id,\ \nu = 0,\ m = 0$ is the standard Brownian motion and the case $A = 0,\ m = 0$ and $\nu(dx) = |x|^{-\alpha-d}dx$, is the symmetric α-stable process.

Proposition 3.8. For the α-stable process on $\mathbb{R}^d$ the order of transience is

$$\gamma = \frac{d}{\alpha} - 1 \tag{3.2}$$

and in this case

$$\int_0^t s^{\gamma} T_s \varphi ds \sim \text{const} \cdot \log t \to \infty$$

as $t \to \infty$.

The distribution of jumps of the α-stable process has "long tails". Another example of a random walk with large jumps is the following.

EXAMPLE 3.9. Fractal random walk in $\mathbb{R}$. Hughes, Montroll and Shlesinger [**HMS**],[**HSM**] studied the multiscale structure of the paths of an interesting class of random walks. In particular they studied the random walk in $\mathbb{R}^1$ with jumps of size b^ℓ at rates $a^{-\ell}$, $\ell \in \mathbb{N}$, $1 < a < b^2$. These random walks are transient iff $\mu := \frac{\ln a}{\ln b} < 1$ and the results of [**HSM**] suggest that the order of transience is $\frac{1-\mu}{\mu}$. In the transient case the *range*, that is, the infinite but locally finite random set visited by the walk, has a natural multiscale structure. In particular the range can be divided into clusters of order k, $k \in \mathbb{N}$, that is, a set of points visited in a ball of radius b^k. The observation of [**HSM**] is that this random set is self-similar in the sense that a cluster of a given order can be divided into a set of $O(a)$ clusters one order lower in the hierarchy and a cluster of one order when expanded by a factor of b is statistically equivalent to a cluster of the next order. The class of hierarchical random walks described below can be viewed as an idealization of this intrinsically multiscale character of natural classes of random walks.

3.4. Hierarchical Random Walks.

3.4.1. *Introduction.* In this section we consider random walks on the countable Abelian groups Ω_N (a basic reference for background on random walks on countable Abelian groups is [**KS**]). In some ways random walks on ultrametric spaces such as Ω_N have some qualitatively different behaviors from those on d-dimensional spaces; e.g., a random walk can leave a closed ball of radius R only by making a single jump of size greater than R, and not by a sequence of small jumps. But in terms of long time behavior there are classes of random walks on Ω_N that have important properties in common with d-dimensional random walks. This means that they can be constructed to mimic certain aspects of the $d-$dimensional behavior, in particular regarding k-strong/weak transience, and serve as analogues for walks with non-integer dimension d. In particular our goal is to study the behavior of particle systems in dimensions infinitesimally close to values of d which are critical in some sense. For example, we shall see that by means of hierarchical random walks it is possible to approach the borderline between k-weak transience and k-strong transience arbitrarily closely from the strong side.

3.4.2. *Random Walks on Ω_N.* We begin with a discrete time random walk on Ω_N defined by

$$\xi_n = \sum_{i=1}^{n} \eta_i, n = 1, 2, \ldots$$

where $\{\eta_i\}$ are i.i.d. on the additive group Ω_N with distribution

$$P[\eta = y] = \frac{r_{|y|}}{N^{|y|-1}(N-1)},\ y \neq 0,$$
$$P[\eta = 0] = 0$$

where $(r_j)_{j=1,2,\ldots}$ is a probability vector. In other words $\{\xi_n\}$ jumps by first choosing distance k with probability r_k and then choosing y with uniform probability on the sphere of radius k. The term in the denominator $N^{|y|-1}(N-1)$ is equal to the "volume" of the sphere of radius $|y|$.

Using Fourier transforms (see [**SF**],[**FG**],[**K**]) the n-step transition probability $P^n_{x,y}$ of $\{\xi_n\}$ can be computed to be

$$P^n_{0,y} = (\delta_{0,|y|} - 1)\frac{f^n_{|y|}}{N^{|y|}} + (N-1)\sum_{k=|y|+1}^{\infty} \frac{f^n_k}{N^k}$$

where

$$f_k = \sum_{j=1}^{k-1} r_j - \frac{r_k}{N-1}$$

The corresponding continuous time random walk on Ω_N has transition function

$$P_{0,y}(t) = e^{-t}\sum_{n=0}^{\infty}\frac{t^n}{n!}P^n_{0,y},\ t \geq 0.$$

This becomes

$$P_{0,y}(t) = (\delta_{0,|y|} - 1)\frac{e^{-h_{|y|}t}}{N^{|y|}} + (N-1)\sum_{j=|y|+1}^{\infty}\frac{e^{-h_j t}}{N^j}$$

where

$$h_j = 1 - f_j = r_j \frac{N}{N-1} + \sum_{i=j+1}^{\infty} r_i,\ j = 1, 2, \ldots$$

The random walk is also characterized by its generator that we refer to as a (discrete) *Laplacian* operator $\Delta^N_{\{r_j\}}$ defined by

$$\Delta^N_{\{r_j\}}\psi(x) = \sum_{y\in\Omega_N\setminus\{0\}} \frac{r_{|y|}}{N^{|y|-1}(N-1)}\left(\psi(x+y) - \psi(x)\right)$$

Finally, in this case the Green's kernel is given by

$$G_N(x) := \int_0^{\infty} P_{0,x}(t)dt.$$

3.4.3. *The Class of $c(N)$ random walks.* We first consider the following class of random walks.

DEFINITION 3.10. Given N and $c > 0$ the c-random walk on Ω_N has jump rates

$$r_j = \left(1 - \frac{c}{N}\right)\left(\frac{c}{N}\right)^{j-1}, \quad j \geq 1, \quad \text{where } 0 < c < N.$$

By [**DGW1**], the order of transience for the c-random walk is

$$\gamma = \frac{\log c}{\log N/c}.$$

The special case of the c-random walk on Ω_N with

$$c = c(N) = \eta N^{\frac{\zeta}{1+\zeta}}, \quad \text{with} \quad \zeta, \eta > 0$$

leads to jump rates

$$r_j = D\frac{\eta^{j-1}}{N^{\frac{j-1}{\zeta+1}}}, \quad j \geq 1$$

and is called the $(\zeta + 1, (\eta^j), N)$ random walk.

In the case $\eta = 1$

$$p_t(0,0) \sim \kappa t^{-(\zeta+1)}h_t$$

where κ is a positive constant and h_t is a slowly oscillating function (cf. [**DGW1**]). Hence the order of transience is $\gamma = \zeta$ and $E_0(L^{\gamma}_{\{0\}}) = \infty$. In the case $\eta \neq 1$ $E_0(L^{\gamma}_{\{0\}}) < \infty$ iff $\eta > 1$.

Let $G_{N,\gamma,j}(\cdot)$ denote the kernel of the jth operator power of the Green operator of the $(\gamma+1,1,N)$-random walk. Consider the special case $\gamma = \frac{d}{2}-1$. If $d > 2j$, that is, $\gamma > j-1$, we obtain ([**DGW1**], (4.2.2))

$$\begin{aligned} G_{N,\gamma,j}(x) &= \text{const} \cdot N^{-|x|(1-\frac{j}{\gamma+1})} \\ &= \text{const} \cdot \rho(x)^{-(d-2j)} \end{aligned} \tag{3.3}$$

where we define the "Euclidean radial distance" of x from 0 as

$$\rho(x) = N^{|x|/d}$$

(chosen so that the volume of a ball of radius ρ grows as ρ^d). The Green's function decay (3.3) agrees with the asymptotic decay of the Green's function of simple symmetric random walk in $\mathbb{Z}^d$. It is in this sense of same order of transience and Green's function behavior that this hierarchical random walk can be viewed as a caricature of the latter.

The fact that the order of transience and asymptotic decay of the Green functions are independent of N in this parameterization is the key to the hierarchical mean field method which involves looking at the limit as $N \to \infty$ of such a sequence.

The corresponding hierarchical Laplacian Δ^N_γ is given by

$$\Delta^N_\gamma \psi(x) = \sum_{\ell=1}^{\infty} \frac{c(N)^\ell}{N^\ell} \sum_{\{y:|y-x|=\ell\}} (\psi(y)-\psi(x))$$

The spectral properties of Δ^N_γ on the subgroup corresponding to the ball of hierarchical radius R is obtained by [**BAHM**] and applied to study the critical behavior of the so-called spherical model of statistical physics (see section 5).

3.4.4. *The class of* $(k+1,(c_j),N)$ *random walks.* In this section we consider a class of random walks that have integer order of transience $k = \gamma$ and that are closer to the critical regime than the $(\gamma+1,(\eta^j),N)$ with $\eta \neq 1$ random walks.

DEFINITION 3.11. The $(k+1,(c_j),N)$ random walk on Ω_N has jump rates

$$r_j = D\frac{c_{j-1}}{N^{\frac{j-1}{k+1}}}, \quad j \geq 1$$

PROPOSITION 3.12. ([**DGW2**]) (a) For integer $k \geq 0$, consider a $(k+1,(c_j),N)$-random walk such that c_j satisfies

$$\lim_{j\to\infty} \frac{c_{j+1}}{c_j} = 1. \tag{3.4}$$

Then the order of transience of the random walk is $\gamma = k$.
(b) In addition, $E((L_K)^k) < \infty$ iff

$$\sum_{j=1}^{\infty} \frac{1}{c_j^{k+1}} < \infty.$$

EXAMPLE 3.13. The j^β-random walks.
Consider the $(k+1,(c_j),N)$-random walks with

$$c_j = (j+1)^\beta, \ \beta \geq 0.$$

This class of random walks has been has been studied in [**DGW2**]. The j^β-random walk is k-strongly transient iff $\beta > \frac{1}{k+1}$, k-weakly transient iff $\frac{1}{k} < \beta \le \frac{1}{k+1}$.

3.4.5. *The Distance Markov chain.* Let $\{\xi_n\}$ be the random walk on Ω_N determined by the probabilities $\{r_k\}$, and let

$$Z_n = |\xi_n|.$$

$\{Z_n\}$ is a Markov chain on $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ called the *distance Markov chain.* This is the analogue of the Bessel process when the underlying motion is d-dimensional Brownian motion.

Denote the transition probabilities of $\{Z_n\}$ by

$$p_{ij} = P[Z_{n+1} = j | Z_n = i].$$

Next, starting from i the expected distance from 0 of the distance chain after one step is given by

$$D_i := \sum_{j=1}^{\infty} j p_{ij}$$

Proposition 3.14. ([**DGW2**]) Let $\{Z_n\}$ denote the distance process of a c-random walk on Ω_N. Then

$$\begin{aligned} D_0 &= \frac{N}{N-c}, \\ D_i &= i + \left(\frac{c}{N}\right)^{i-1} \left[\frac{c}{N-c} - \frac{(N-c)(N^i-1)}{N^i(N-1)^2}\right], \quad i > 0. \end{aligned}$$

In the borderline between transience and recurrence ($c = 1$),

$$D_i = i + \frac{1}{N^{2i-1}(N-1)}, \quad i > 0.$$

Corollary 3.15. a) For $c \ge 1$, $D_i \ge i$ for all i, hence $\{Z_n\}$ is a submartingale. (b) For $c < 1, D_i < i$ iff $i > L_N(c) := \frac{1}{\log N}(-\log(1 - c\left(\frac{N-1}{N-c}\right)^2))$.

Remark 3.16. Note that for $c < 1$, $\{Z_n\}$ behaves like a submartingale for $i \le L_N(c)$, and when it goes beyond $L_N(c)$ it stops behaving that way and tends to go towards the origin. Also, $L_N(c) \to \infty$ as $c \nearrow 1$. Note that this does not occur for the Bessel process $\mathbb{R}^d$ with $d \ge 1$.

We next introduce some related objects. The *maximum process* is defined by

$$Z_n^* := \max_{k \le n} Z_k.$$

Note that Z_n^* is also a Markov chain as a consequence of the ultrametric structure of Ω_N.

Theorem 3.17. *([**DGW2**]) For the c-random walk, and $k \ge 0$, (a)*

$$\lim_{N\to\infty} P_i \left[Z^*_{\lfloor N^{j/(k+1)} \rfloor} \le j\right] = \begin{cases} 0, & c > N^{\frac{k}{k+1}} \\ 1/e, & c = N^{\frac{k}{k+1}} \\ 1, & c < N^{\frac{k}{k+1}} \end{cases}$$

(b) Let $c = c(N) = \eta N^{k/(k+1)}$ (i.e. the $(k+1, (\eta^j), N)$ random walk). For $i < j$,

$$\lim_{N\to\infty} P_i\left[Z^*_{\lfloor N^{j/(k+1)}\rfloor} = \ell\right] = \begin{cases} e^{-\eta^j}, & \ell = j, \\ 1 - e^{-\eta^j}, & \ell = j+1. \end{cases}$$

Note that $c = N^{k/(k+1)}$ is the borderline between k-strong and k-weak transience for $k \geq 1$, (between transience and recurrence for $k = 0$).

Remark 3.18. These results indicate that the right time scale to observe phenomena related to k–strong/weak transience of the c-random walk in a closed ball of radius j is $N^{j/(k+1)}$. For large N only the closed ball of radius j and the surrounding closed ball of radius $j+1$ matter.

Remark 3.19. Let $\tau_k = \inf\{n : Z_n \geq k\}$, and $0 \leq i < k$. Then for the c-random walk:

$$P_i[\tau_k = m] = \left(1 - \left(\frac{c}{N}\right)^{k-1}\right)^{m-1} \left(\frac{c}{N}\right)^{k-1},$$
$$E_0(\tau_k) = \left(\frac{N}{c}\right)^{k-1}.$$

For $c < 1$ (recurrence), let

$$T_N(c) = \lfloor L_N(c) \rfloor + 1,$$

i.e., the first time of jump over the threshold where the $\{Z_n\}$ stops behaving like a submartingale. Then

$$E_0(\tau_{T_N(c)}) \sim \frac{1}{1-c} \quad \text{as} \quad N \to \infty,\ c < 1.$$

3.5. Lévy process limits of c-random walks. The countable group Ω_N is useful for the study of large scale properties of hierarchical random walks. For the small scale properties it is necessary to pass to a continuum hierarchical group (consisting of semi-infinite sequences), namely,

$$\Omega_N^{*,\infty} = \left\{(\omega_\ell)_{\ell\in\mathbb{Z}} \,|\omega_\ell \in \{0,1,\ldots,N-1\},\ \exists \ell_0 : \omega_\ell = 0 \quad \forall \ell \geq \ell_0\right\}.$$

The space $\Omega_N^{*,\infty}$ with componentwise addition modulo N is a totally disconnected locally compact Abelian group.

By analogy with the construction of Lévy processes in $\mathbb{R}^d$, Lévy processes on $\Omega_N^{*,\infty}$ can be constructed as a limit of a rescaled class of c-random walks on the hierarchical lattices

$$\Omega_N^{*,j} = \left\{(\omega_\ell)_{\ell\in\mathbb{Z},\ell\geq -j} \,|\omega_\ell \in \{0,1,\ldots,N-1\},\ \exists \ell_0 : \omega_\ell = 0 \quad \forall \ell \geq \ell_0\right\}.$$

The limit yields a class of Lévy processes on $\Omega_N^{*,\infty}$ introduced by Evans [**E**]. These have also been investigated by [**M**] and applied in [**EF**] to study cluster formation in a stepping-stone model and in [**CDG**] to study mutually catalytic branching.

4. Random Fields on Hierarchical Groups

4.1. H-exchangeable random fields on Ω_N. In this section we briefly introduce a class of real-valued random fields indexed by the hierarchical group Ω_N.

The law of the random field is given by a probability measure, P, on $\mathbb{R}^{\Omega_N}$. We introduce two groups of transformations on $\mathbb{R}^{\Omega_N}$. The first is the group $\mathcal{G}_{\mathcal{T}}$ generated by translations

$$\{x_\xi : \xi \in \Omega_N\} \to \{x_{\xi+\xi'} : \xi \in \Omega_N\}$$

and the second group $\mathcal{G}_{\mathcal{E}}$,

$$\{x_\xi : \xi \in \Omega_N\} \to \{x_{\Phi_{\pi,k}\xi} : \xi \in \Omega_N\}$$

is generated by the mappings $\Phi_{\pi,k} : \Omega_N \to \Omega_N$, where $k \in \mathbb{N}$, π is a permutation of $\{0, 1, \dots, N-1\}$ and

$$\Phi_{\pi,k}(\omega)(i) := \begin{cases} \omega(i) & \text{if } i \neq k, \\ \pi\omega(i) & \text{if } \ i = k. \end{cases}$$

.

The random field with probability law P is said to be:

- *stationary*: if the law is invariant under the group $\mathcal{G}_{\mathcal{T}}$,
- *H-exchangeable*: if the law is invariant under the group $\mathcal{G}_{\mathcal{E}}$.

Given a stationary and H-exchangeable random field we introduce the sequence of *block averages*:

$$x^N_{\xi,k} := \frac{1}{N^k} \sum_{\xi' \in B_{\xi,k}} x_{\xi'} \tag{4.1}$$

where the ball (or "block") of hierarchical radius k and center ξ is given by

$$B_{\xi,k} := \{\xi' \in \Omega_N : |\xi - \xi'| \leq k\}.$$

Also, note that $B_{0,k}$ is a subgroup of Ω_N.

4.2. Hierarchical Gibbs Random Field. Gibbs random fields are fundamental objects in statistical physics - see Georgii [**Ge**] for a systematic introduction to this subject.

To define a hierarchical Gibbs random field in our setting we begin with a *pair potential* $J : \Omega_N \times \Omega_N \to \mathbb{R}$, $J_{\xi',\xi''} = J(|\xi' - \xi''|)$ and single site self-interaction potential $f : \mathbb{R} \to \mathbb{R}_+$. For $B_{0,M} \subset \Omega_N$, we then define the finite system *pair Hamiltonian function* on $\mathbb{R}^{\Omega_N}$:

$$\mathcal{H}_M(x) := \sum_{\xi' \in B_{0,M}, \xi'' \in \Omega_N} J_{\xi',\xi''} x_{\xi'} x_{\xi''}. \tag{4.2}$$

Given $\mathcal{H}_M$, assume that the function f grows sufficiently fast at $\pm\infty$ so that for all $M \in \mathbb{N}$,

$$Z_M := \int \exp\left(-\beta \mathcal{H}_M(\tilde{x})\right) \prod_{\xi' \in B_{0,M}} e^{-\beta f(x_{\xi'})} dx_{\xi'} < \infty$$

where for $x \in B_{0,M}$

$$\tilde{x}(\eta) = \begin{cases} x(\eta) & \text{if } \eta \in B_{0,M} \\ \theta & \quad \text{otherwise.} \end{cases} \tag{4.3}$$

Then the corresponding *Gibbs random field* on $B_{0,M}$ with inverse temperature $\beta \geq 0$, and frozen external field $x_\eta \equiv \theta \ \forall \eta \in B^c_{0,M}$, is a probability measure, $P_{M,\theta}$

on $\mathbb{R}^{B_{0,M}}$ with

$$(4.4)\quad P_{M,\theta}(dx|x_\eta=\theta\;\forall\eta\in B^c_{0,M})=\frac{1}{Z_M}\exp\left(-\beta\mathcal{H}_M(\tilde{x})\right)\prod_{\xi'\in B_{0,M}}e^{-\beta f(x_{\xi'})}dx_{\xi'}.$$

Given $P_{M,\theta}$ one can associate a probability measure $\tilde{P}_{M,\Theta}=P_{M,\theta}\times\delta_{\mathbf{\Theta}}$ on $\mathbb{R}^{\Omega_N}$ where Θ is the configuration outside $B_{0,M}$ identically equal to θ. A *thermodynamic limit*, if it exists, is given by the weak limit of $\tilde{P}_{M,\Theta}$ as $M\to\infty$. The number of distinct thermodynamic limits is called the number of *phases*. A value of β at which the number of phases changes is called a *critical point*.

REMARK 4.1. Taking $J(\xi)=p(\xi)$ where $p(\cdot)$ is the one step displacement distribution of a hierarchical random walk we obtain a large class of Gibbs random fields. This relationship between Gibbs random fields and random walks has been been exploited to study critical phenomena such as spontaneous magnetization (see e.g. [**Ge**], Chapt. 20).

4.3. Gaussian random fields. Consider a stationary zero-mean Gaussian random field $\{x_\xi:\xi\in\Omega_N\}$. Let $\Gamma(\xi,\xi'):=Cov(x_\xi,x_{\xi'})$. Note that the random field is H-exchangeable if and only if

$$\Gamma(\xi,\xi')=\gamma(d(\xi,\xi'))$$

and in this case it suffices to specify the function $\{\gamma(\ell):\ell\in\mathbb{Z}_+\}$. Then if $\lim_{\ell\to\infty}\gamma(\ell)=0$, the L^2 ergodic theorem holds and in particular

$$\lim_{k\to\infty}E\left((x^N_{\xi,k}-E(x_0))^2\right)=0$$

where $x^N_{\xi,k}$ is defined as in (4.1).

If $G^k_N(x)$ is the kth operator power of the Green's function of a transient random walk on Ω_N with $k\in\mathcal{T}$, then it is a positive definite function and therefore is a covariance function. Gaussian random fields with covariance functions involving the powers $G^k_N(\cdot)$ arise in the study of limits of rescaled occupation time fluctuations of branching systems (see for example Section 5.2.5).

Given a stationary and H-exchangeable Gaussian random field it is natural to begin by freezing the field outside the ball $B_{0,M}$ and then taking the *thermodynamic limit* as $M\to\infty$. To do this, let $\mathcal{P}_Mx_\xi$ denote the L^2 projection of x_ξ on the orthogonal complement of $L^2((B_{0,M})^c)$ and let $\sigma_M(k):=Cov(\mathcal{P}_Mx_\xi,\mathcal{P}_Mx_{\xi'})$ where $|\xi-\xi'|=k$, $k=0,\dots,M$. Then the resulting covariance structure is given by

$$\sigma_M(0),\sigma_M(1),\dots,\sigma_M(M).$$

On the other hand, if the limit

$$0<\sigma(0)=\lim_{M\to\infty}\sigma_M(0)<\infty,$$

and

$$\rho(\ell)=\lim_{M\to\infty}\frac{\sigma_M(\ell)}{\sigma_M(0)}<1,\quad\text{if}\quad\ell\neq0$$

of a consistent family exists, then there exists a Gaussian random field on Ω_N with covariance function $\sigma(0)\rho(\cdot)$.

4.4. Hierarchical mean field limit. The notion of mean-field approximation (or *caricature*) has proved to be extremely useful in obtaining information about spatially distributed systems in statistical physics, population genetics, communications networks, etc (see e.g. [**CH**], Chapt. 15, and [**D**]). The idea is to replace a spatially structured system by a related *exchangeable* system of size N and then to study its behavior as $N \to \infty$. In many cases this approximation correctly predicts phase transition behavior. However it fails to predict behavior near critical points, particularly in low dimensions. The hierarchical mean field limit introduced in [**DG1**] provides a caricature which is intermediate in complexity between the true spatially distributed system and the mean-field caricature. The hierarchical mean field approximation is based on two simplifications. First, the spatial structure is modelled in a simplified way by replacing the random walk or discrete Laplacian that determines the Hamiltonian (or branching dynamics) with a hierarchical random walk or hierarchical Laplacian having the same order of transience. The second simplification is to consider the limit of the sequence $\{x^N_{0,k} : k \in \mathbb{Z}_+\}$ as $N \to \infty$. If the weak limit, $\{x^\infty_{0,k} : k \in \mathbb{Z}_+\}$ of $\{x^N_{0,k} : k \in \mathbb{Z}_+\}$ as $N \to \infty$ exists, then it is called the *hierarchical mean field limit.* In some important cases $\{x^\infty_{0,k} : k \in \mathbb{Z}_+\}$ is a reverse time Markov chain whose transition kernel can be identified.

5. Applications to Interacting Systems

Interacting systems models in statistical physics and population processes on the Euclidean lattice $\mathbb{Z}^d$ exhibit dimension-dependent behavior in which there is a critical dimension d_c at which the large space-time scale behavior changes qualitatively. The purpose of this section is to give a brief introduction to the application of the hierarchical group caricatures described above to explore the behavior of such systems in the vicinity of the critical dimension. Examples will be given in which the critical dimension is $d_c = 2$ or $d_c = 4$.

5.1. Applications in Statistical Physics. A hierarchical random field model of ferromagnetic behavior was introduced by Dyson using Ω_N with $N = 2$, and has been used by Bleher and Sinai [**BS**],[**BS2**] as a framework in which to carry out a rigorous renormalization group analysis following the ideas of Wilson [**W**]. Renormalization group methods are used in statistical physics to explain scaling laws near the critical point of a phase transition. In the case of ferromagnetism, $d_c = 4$ is the critical dimension and the hierarchical group has been used to study the large scale fluctuations near the critical point in 4–ε dimensions. Renormalization group methods are also used in quantum field theory, where 4 is also a critical dimension. More detailed expositions on these subjects are given in [**JL**],[**CE**],[**Si**],[**W**].

5.1.1. *Dyson's hierarchical ferromagnetic model.* Dyson's hierarchical ferromagnetic model is a continuous spin system described by a Gibbs random field. In Dyson's original formulation the model is viewed as a random field on $\mathbb{Z}^1$ with long range potential. In particular, the finite system with 2^M sites, $x = \{x_j : j = 1, 2, \dots, 2^M\}$, has as Hamiltonian $\mathcal{H}_M$ the function on $\mathbb{R}^{2^M}$ defined by

$$\mathcal{H}_M(x) \;=\; -\frac{1}{2}\sum_{k=1}^{M} c^k \sum_{j=0}^{2^{M-k}-1}\left(\frac{1}{2^k}\sum_{i=1}^{2^k} x_{j2^k+i}\right)^2 .$$

However from our perspective it is more natural to consider the corresponding finite system as a random field $\{x_\xi,\ \xi \in B_{0,M}\}$, on the ball $B_{0,M}$ of hierarchical radius M on Ω_2. In this setting the Hamiltonian (cf. (4.2)) is given by

$$\mathcal{H}_M(x) = \sum_{\xi,\xi' \in B_{0,M}} J(|\xi - \xi'|) x_\xi x_{\xi'},$$

where

$$J(k) = -\frac{1}{2} \sum_{j=k}^{M} \frac{c^j}{2^j} \frac{1}{2^j}$$

and $1 < c < 2$.

REMARK 5.1. Note also that this corresponds to a modification of the $(2, (\eta^j), 2)$-random walks where $\eta = \frac{c}{2^{1/2}}$. In particular the sites to which a jump is made are chosen with uniform distribution on a ball rather than on a sphere. However this does not change the order of transience and the choice of $c = 2^{\frac{1}{2}(1-\varepsilon)}$ corresponds to order of transience $\gamma = \frac{1-\varepsilon}{1+\varepsilon}$, or dimension $d = \frac{4}{1+\varepsilon} \approx 4 - \varepsilon$.

Then (cf. (4.4)) the Gibbs measure with pair Hamiltonian $\mathcal{H}_M$, inverse temperature β and frozen external field $x_\xi \equiv 0\ \forall \xi \in (B_{0,M})^c$, is a probability measure, $P_{M,0}$ on $\mathbb{R}^{B_{0,M}}$ with

$$P_{M,0}(dx) = \frac{1}{Z_M} \exp\left(-\beta \mathcal{H}_M(x)\right) \prod_{\xi \in B_{0,M}} e^{-\beta f(x_\xi)} dx_\xi \tag{5.1}$$

The thermodynamic limit, $M \to \infty$, exists if $1 < c < 2$ but does not exist if $c > 2$ (cf. [**Si**], Chapt. 4, Section 2). When it exists, the thermodynamic limit is an H-exchangeable random field on Ω_2. This system exhibits spontaneous magnetization (i.e. multiple phases) at low temperatures, namely, when $\beta > \beta_{\mathrm{crit}}(c)$ where $\beta_{\mathrm{crit}}(c)$ is the inverse critical temperature.

The behavior at the critical point $\beta_{\mathrm{crit}}(c)$ is of special interest. A program has developed in statistical physics to show that renormalized fluctuations at the critical temperature are Gaussian (as in the spatial central limit theorem for random fields satisfying an appropriate mixing condition) in high dimensions but non-Gaussian in low dimensions. For many systems including ferromagnetic systems dimension $d = 4$ is the critical dimension for this dichotomy. The conjectured high dimensional behavior has been proved for a number of models. However the low-dimensional behavior is much harder to obtain rigorously. Dyson's hierarchical model has played an important role in this development; in particular rigorous results on the non-Gaussian fluctuations have been obtained by Bleher and Sinai [**BS**],[**BS2**] using renormalization group ideas and working in a neighbourhood of the critical point, namely $4 - \varepsilon$ for small positive ε.

To explain their result we consider the block sum $X_M := \sum_{\xi \in B_{0,M}} x_\xi$. If f is symmetric, then X_M has mean zero and we consider the normalized distribution of X_M for large M. For high temperatures, $\beta < \beta_{\mathrm{crit}}(c)$, X_M satisfies a spatial central limit theorem, that is, the distribution of $\frac{1}{2^{M/2}} X_M$ converges to a Gaussian distribution as $M \to \infty$.

At the critical temperature, $\beta = \beta_{\mathrm{crit}}$, X_M scales as $(2/c^{1/2})^M$ and we let $F^{\beta}_{2^M,f}(\cdot)$ denote the probability density function of $\frac{1}{(2/c^{1/2})^M} X_M$. Then as a result of the hierarchical structure it is possible to derive the recursive relations ([**CE**],[**Si**])

$$F^{\beta}_{2^{M+1},f}(u) = F^{\beta}_{2^M,\mathcal{N}^{(\beta)}f}(u) \tag{5.2}$$

where $\mathcal{N}^{(\beta)}f$ is defined via the equation

$$e^{-\beta\mathcal{N}^{(\beta)}f(u)} = e^{\beta u^2/2}(\frac{2}{c^{1/2}})\int e^{-\beta f(uc^{-\frac{1}{2}}+v)-\beta f(uc^{-\frac{1}{2}}-v)}dv$$

Therefore limiting distributions should correspond to non-negative solutions of the fixed point equation

$$\mathcal{N}^{(\beta)}f(\cdot) = f(\cdot) \tag{5.3}$$

Proposition 5.2. ([**CE**],[**Si**]). (1) For all $1 < c < 2$, there is a solution, f^* of the fixed point equation (5.3) corresponding to a Gaussian distribution $F(x) = \frac{1}{Z}e^{-\beta_{\text{crit}}f^*(x)}$ and for $c > 2^{\frac{1}{2}}$, $(\mathcal{N}^{(\beta_{\text{crit}})})^M f \to f^*$ as $M \to \infty$.
(2) For sufficiently small $\varepsilon > 0$ and

$$c = 2^{\frac{1}{2}(1-\varepsilon)}$$

there exists a solution of the fixed point equation (5.3) corresponding to a distribution that is not infinitely divisible and therefore not Gaussian (in addition to the Gaussian fixed point).

Moreover the analysis developed to establish these results also provides information on the "critical indices" which describe singularites that occur at the critical temperature.

5.1.2. *Related applications in statistical physics.* Hierarchical models have been used to investigate a number of phenomena in statistical physics. The following are three illustrative cases.

1. Results were obtained by [**BAHM**] for a ferromagnetic model called the hierarchical spherical model. In this model one considers

$$\mathbb{S}^M := \{x \in \mathbb{R}^{\Omega^M_N} : \sum_{\xi\in\Omega^M_N} x^2_\xi = N^M\}$$

and takes the Hamiltonian

$$\mathcal{H}_M(x) := -(\Delta^{(M)}x, x), \quad x \in \mathbb{S}^M$$

where $\Delta^{(M)}$ is the hierarchical Laplacian associated to a random walk on Ω^M_N. In this model they establish that the rescaled fluctuations at the critical temperature are asymptotically Gaussian in dimensions $d \geq 4$ and non-Gaussian in dimensions $2 < d < 4$.

2. Brydges, Evans and Imbrie [**BEI**] have studied the self avoiding random walk in a hierarchical lattice in four dimensions. More precisely, they first consider the $(2, 1, N)$ random walk which is a caricature of 4-dimensional simple random walk. Then the Green's function (recall (3.3)) is

$$G(x) = \begin{cases} \frac{1-\frac{1}{N}}{1-\frac{1}{N^{1/2}}} & \text{if } x = y, \\ \frac{1}{\rho(x-y)^2} & \text{if } x \neq y. \end{cases} \tag{5.4}$$

where the $\rho(\cdot)$ is the "Euclidean radial distance" of x from 0:

$$\rho(x) = 0 \text{ if } x = 0, \quad \rho(x) = N^{|x|/4} \text{ if } x \neq 0.$$

They then introduce a self-repulsion mechanism as a approximation to true self-avoidance and investigate the Green's function of this interactive process.
3. Molchanov [**Mol**] has studied the question of localization for the random hierarchical Anderson model associated with the random operator

$$H\psi(x) = \Delta\psi(x) + \sigma V(x)\psi(x)$$

where $\sigma > 0$, the $V(x)$ are i.i.d. and Δ is the hierarchical Laplacian associated to a hierarchical random walk on Ω_N.

5.2. Applications to Branching Systems.

5.2.1. *Equilibrium Branching Random Walks on* Ω_N. We now introduce a system of branching random walks. This is given by a system of particles undergoing random walks on Ω_N together with critical branching. Branching and migration are assumed independent. Let us specify the branching mechanism as simply as possible: after a unit exponential time, an individual, independently of all the others, either is removed or splits into two, each with probability 1/2.

A basic result on branching symmetric random walks ([**LMW**], [**GHW**]) is that if the random walks are transient then, for each $\theta > 0$, there exists a unique equilibrium random field with a mean number of θ individuals per site. (Recall that the random walks we consider on Ω_N are all symmetric.) This equilibrium random field is H-exchangeable and is of Poisson type, i.e. the equilibrium population Φ on Ω_N is the superposition $\Phi = \sum_i \Phi_i$ of a Poisson system $\sum_i \delta_{\Phi_i}$ of families (each family consists of a collection of individuals which in a historical picture can be traced back to a common ancestor). In contrast, if the random walks are recurrent, local extinction occurs.

5.2.2. *Equilibrium Two-level Branching Random Walks on* Ω_N.

Multilevel branching systems were introduced by Dawson and Hochberg [**DH**] and they have been studied by several authors [**DGW1**], [**DGW2**], [**DHV**], [**Eth**], [**G1**],[**GH**], [**GHW**], [**Ho**], [**HW**], [**Wu**]. In addition to the individual particle branching there is an independent branching of families of related particles (2-level branching), and this idea can be extended to higher levels of branching. The main difficulty in dealing with these models is that the independence of behavior of individual particles no longer holds due to the higher-level branchings.

Assume that the random walk is strongly transient on Ω_N, and let $\theta > 0$. Start with the family system $\Psi_0 = \sum_i \delta_{\Phi_i}$ where each Φ_i is a family, that is, a collection of individuals and introduce, in addition to the individual branching and migration, a *family branching*: independently of everything else, after a unit exponential time each family, Φ_i either vanishes or reproduces resulting in two identical copies, Φ_i',Φ_i'' with probability 1/2 each. After a reproduction event Φ_i',Φ_i'' evolve as independent one level branching systems. This creates the basic *two-level branching* system Ψ_t. We can represent this process as a Markov process with space the space of measures on $M_c(M_c(\Omega_N))$ (where $M_c(\Omega_N)$ denotes the space of locally finite counting measures on Ω_N and $M_c(M_c(\Omega))$ denotes the space of counting measures, Ψ on $M_c(\Omega_N)$ such that $\int_{M_c(\Omega_N)} \mu(A)\Psi(d\mu) < \infty$ for every finite subset A of Ω_N). The following establishes the existence of a non-trivial equilibrium in the strongly transient case.

PROPOSITION 5.3. ([**GHW**]) Assume that
(i) the random walk on Ω_N is strongly transient and

(ii) $\Psi_0 = \sum_i \delta_{\Phi_i}$ where $\{\Phi_i\}$ correspond to the family decomposition of an equilibrium state for the one level branching random walk with mean number θ of individuals per site.
Then as $t \to \infty$, the two-level branching system Ψ_t converges to an $M_c(M_c(\Omega_N))$-valued equilibrium, Ψ_{equ}, with a mean number θ of individuals per site.

5.2.3. *Hierarchical Mean-field limit of Branching Systems.* In this section we consider the application of the hierarchical mean field limit to the study of the structure of the equilibrium random fields of sections 5.2.1, 5.2.2. Recall that $\gamma = 0$, i.e. $d = 2$ is critical for one level branching and $\gamma = 1$, i.e. $d = 4$ is critical for two level branching.

Proposition 5.4. ([**DG2**]) Let $\{x^N_{0,k}, k \in \mathbb{Z}_+\}$ be the block average process arising from the equilibrium random measure on Ω_N for the one-level branching $(1, (c_j), N)$-random walk such that the random walk is transient. Let

$$Z^N_k := x^N_{0,-k}, \quad k \in \mathbb{Z}_-.$$

Then as $N \to \infty$, $\{Z^N_k, k \in \mathbb{Z}_-\}$ converges to an inhomogeneous Markov chain, $\{Z_k\}_{k\in\mathbb{Z}_-}$, on $[0, \infty)$ with transition function

$$P(Z_k \in dy|Z_{k-1} = x) = \Gamma_k(x, dy)$$

where $\Gamma_k(x, dy)$ is given by the equilibrium distribution of a subcritical Feller branching diffusion process with immigration. Moreover

$$\lim_{k\to-\infty} Z_k = \theta.$$

Remark 5.5. A similar result is established in [**DGW3**] for two-level branching $(2, (c_j), N)$ random walk such that the random walk is strongly transient.

The point of these results is that for branching systems having critical dimensions $d = 2$ or $d = 4$, the hierarchical mean field limit leads to objects that can be explicitly described and provide information on the equilibrium behavior in Ω_N, at least for large N.

5.2.4. *HMF Analysis of Related Models.* The hierarchical mean field approximation has been used to study not only equilibrium behavior but also the multiple space-time scale behavior of a number of other spatially structured population models. These include interacting Fleming-Viot processes ([**DGV**]), and mutually catalytic branching systems ([**CDG**]).

5.2.5. *Occupation time fluctuations of branching systems.* Another dimension-dependent phenomenon arises in the analysis of the large time fluctuations of occupation time of spatially homogeneous systems of independent random walks or branching random walks. For example in the analysis of large time occupation time fluctuations of a particle systems on a locally compact Abelian group S, a basic problem consists in finding an appropriate norming a_t such that the occupation time fluctuation

$$\frac{1}{a_t}\int_0^t (X_s - E(X_s))ds \tag{5.5}$$

has a non-trivial limit in distribution as $t \to \infty$, where X_s is the empirical measure on S given by the particle system at time s. In the case in which the branching laws have finite variance the limits are represented by Gaussian random fields on S.

On the other hand, in the case of some infinite variance branching laws the limits are given by stable random fields [**DGW1**].

There are three regimes to consider: low dimensions, critical dimension and high dimensions. In the case of k-level branching systems, the high dimensional case ($\gamma > k$) behavior involves the scaling $a_t = t^{\frac{1}{2}}$ and the limiting Gaussian random fields have covariance functions which are expressed in terms of the powers of the Green potential operator $G, G^2, \ldots, G^{k+1}$ (see [**DGW1**], Theorems 2.2.1, 2.2.2, 2.2.3).

In the critical and low dimensions the limiting random field is a Gaussian random multiple of Haar measure (i.e. the limiting correlation function is identically equal to 1). In the case of systems of independent recurrent random walks motion a key role is played by the growth of the operator $G_t = \int_0^t T_s ds$ as $t \to \infty$, where $\{T_t\}$ is the semigroup of the motion. In the case of k-level branching systems (with k=1 corresponding to ordinary one level branching) and k-weakly transient motion this role is played by the growth of the power G_t^k [**DGW1**]. It is shown in [**DGW1**] that for each k-level branching system, if the growth of G_t, G_t^2, etc., is given by an increasing function f_t, then the norming a_t for the occupation time fluctuation (5.3) is

$$a_t = \left(\int_0^t f_s ds \right)^{1/2}. \tag{5.6}$$

For the α-stable process or Brownian motion ($\alpha = 2$) on $\mathbb{R}^d$ (with no branching),

$$a_t = \begin{cases} t^{1-d/2\alpha} & \text{for} \quad \alpha > d, \\ (t \log t)^{1/2} & \text{for} \quad \alpha = d, \\ t^{1/2} & \text{for} \quad \alpha < d. \end{cases}$$

Note that $t^{1-d/2\alpha} \to t^{1/2}$ as $\alpha \searrow d$, so there is a discontinuity in the order of the growth at $\alpha = d$, and for this value of α the "critical" fluctuations of the occupation time are bigger than $t^{1/2}$. The critical case corresponds to $\gamma = 0$, which represents the borderline between transience and recurrence. Similarly, the values $\gamma = k, k = 1, 2, \ldots$, which represent the borderline between k-strong and k-weak transience, are critical cases for which occupation time fluctuations are bigger. The same pattern of the non-branching system is repeated for the 1-level branching system (individual particle branching) 2 dimensions higher [**CG**], and for the 2-level branching system (individual branching and family branching) 4 dimensions higher [**DGW1**].

The occupation time fluctuations of branching systems of c-random walks on Ω_N was investigated in detail in [**DGW1**] for multilevel branching. For k-level branching (with $k = 0$ corresponding to random walks with no branching) we obtain

$$a_t = \begin{cases} (t^{k+2-\gamma} h_t^{(k)})^{1/2} & \text{for} \quad k - 1 < \gamma < k, \\ (t \log t)^{1/2} & \text{for} \quad \gamma = k, \\ t^{1/2} & \text{for} \quad \gamma > k. \end{cases}$$

where $h^{(k)}$ is a class of differentiable functions that are bounded and bounded away from zero and are periodic in logarithmic scale.

We next consider the class of critically recurrent random walks of Example 3.13. Then from [**DGW2**] we obtain the following results for $k = 0$ (0-level system), $k = 1$

(1-level system) and $k = 2$ (2-level system):

$$a_t = \begin{cases} t^{1/2}(\log t)^{(1-(k+1)\beta)/2} & \text{for} \quad \beta < 1/(k+1), \\ (t \log\log t)^{1/2} & \text{for} \quad \beta = 1/(k+1), \\ t^{1/2} & \text{for} \quad \beta > 1/(k+1). \end{cases}$$

5.3. Universality. The idea of universality, originally introduced in statistical physics, is that interacting systems can be classified into equivalence classes called *universality classes* that share the same large scale behavior. For example, a class of Gibbs random fields in which the pair potentials are associated to random walks having the same order of transience and ferromagnetic-type local interaction could be a candidate for a universality class. The renormalization group method was introduced (see e.g. [**KW**],[**Wi**]) to study the behavior of universality classes near a common critical point. The hierarchical models provide a setting for a rigorous study of the universality class in which the fluctuations and critical indices have the same behavior for a class of ferromagnetic models (e.g. different choices of f in (5.1)) in dimensions $d = 4 - \varepsilon$ (see [**CE**],[**Si**]). In the setting of population models the identification of universality classes has been carried out using the hierarchical mean field limit for spatially structured branching models [**DG1**], [**BCGH2**] and interacting Fisher-Wright models [**DGV**], [**BCGH**].

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