On the geometry of higher dimensional anabelian varieties

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1. Anabelian varieties

The étale fundamental group of a geometrically connected variety X over a field k sits in a short exact sequence

$$1 \to \pi_1 X_{\overline{k}} \to \pi_1 X \to \mathrm{Gal}_k \to 1.$$

Grothendieck conceived that for special varieties, this short exact sequence captures the geometry and arithmetic of the underlying (category of) varieties. Such varieties are called anabelian and coincide in dimension 1 with hyperbolic curves, whereas in higher dimensions the notion of anabelian varieties is unclear. The talk focused on the geometric property of being an algebraic $\mathrm{K}(\pi,1)$ space and derived consequences for the geometry of such varieties[1].

2. Algebraic $\mathrm{K}(\pi,1)$ spaces

Let X/k be a connected variety. The finite étale site $X_{\text{fét}}$ is by definition via a choice of a base point isomorphic to the classifying site $\mathscr{B}\pi_1 X$ of $\pi_1 X$. The map

$$\gamma : X_{\text{ét}} \to X_{\text{fét}} \cong \mathscr{B}\pi_1 X$$

induces comparison maps $\mathrm{H}^*(\pi_1 X, A) \to \mathrm{H}^*_{\text{ét}}(X, \mathscr{A})$ for each finite continuous $\pi_1 X$-module A with locally constant system $\mathscr{A} = \gamma^* A$. An **algebraic** $\mathrm{K}(\pi,1)$ **space** is a variety, such that all comparison maps are isomorphisms. As cohomology is always killed locally in the respective topology for which it is computed, the Leray spectral sequence for γ_* shows that being an algebraic $\mathrm{K}(\pi,1)$ is equivalent to the following: all $\mathrm{H}^q(X', \mathbb{Z}/n\mathbb{Z})$ for finite étale covers X'/X, $n \in \mathbb{N}$ and $q > 0$ are killed upon restriction to suitable finite étale covers. Hence, the $\mathrm{K}(\pi,1)$ property forces $\pi_1 X$ to be sufficiently rich. The comparison map is always bijective for H^1 and injective for H^2, as classes in H^1 are étale torsors which kill themselves.

Examples for algebraic $\mathrm{K}(\pi,1)$ spaces are: curves except for $\mathbb{P}^1_k$, abelian varieties, and more generally varieties such that for all finite étale covers the cohomology ring is generated by classes in H^1. The latter implies for projective varieties that the Albanese map is finite. Being an algebraic $\mathrm{K}(\pi,1)$ space goes up and down along finite étale covers and behaves well in fibrations in the following sense. Let $f : X \to S$ be a smooth, projective map with $f_*\mathscr{O}_X = \mathscr{O}_S$ and S is connected of characteristic 0. Then one geometric fibre is $\mathrm{K}(\pi,1)$ if and only if all geometric fibres are $\mathrm{K}(\pi,1)$. With X_s being the geometric fibre over $s \in S$ the fibre sequence

$$1 \to \pi_1 X_s \to \pi_1 X \to \pi_1 S \to 1$$

is exact if the base S is $\mathrm{K}(\pi,1)$ or X/S admits a section.

Then comparison of Hochschild–Serre and Leray spectral sequences yields: if two out of fibre X_s, base S and total space X are $\mathrm{K}(\pi,1)$ then also the third. For

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the exactness of the fibre sequence in case S is a $\mathrm{K}(\pi,1)$ only the injectivity of $\pi_1 X_s \to \pi_1 X$ needs a proof. For this we need to extend any connected G-torsor $\varphi : \pi_1 X_s \twoheadrightarrow G$ of the fibre to a G-torsor $\tilde{\varphi} : \pi_1 X' \to G$, where $X' = X \times_S S'$ with a finite étale cover S'/S. As the nonabelian $\mathrm{R}^1 f_* G$ is locally constant, we may choose, upon substitution of S by a finite étale cover, a global isomorphism class of a G-torsor in $\mathrm{H}^0(S, \mathrm{R}^1 f_* G)$ that restricts to φ in the fibre above s. The obstruction for this class to come from an actual torsor lies in $\mathrm{H}^2(S, Z(G))$ where $Z(G)$ is the center of G and the lien of the corresponding gerbe, because the G-torsor φ is connected. This obstruction vanishes on a finite étale cover of S.

A smooth projective $\mathrm{K}(\pi,1)$ space X of dimension $\dim X \geq 2$ is never a hyperplane section of a smooth projective variety Y. By Lefschetz, the inclusion induces an isomorphism $\pi_1 X \xrightarrow{\sim} \pi_1 Y$ which for an anabelian X would conjecturally lead to an unlikely retraction for the inclusion. In the diagram

$$\begin{array}{ccc}
\mathrm{H}^2(\pi_1 Y, \mathbb{Z}_\ell(1)) & \xrightarrow{\cong} & \mathrm{H}^2(\pi_1 X, \mathbb{Z}_\ell(1)) \\
\big\downarrow & & \big\downarrow \cong \\
\mathrm{H}^2(Y, \mathbb{Z}_\ell(1)) & \hookrightarrow & \mathrm{H}^2(X, \mathbb{Z}_\ell(1))
\end{array}$$

the Weak Lefschetz Theorem and comparison for H^2 of Y imply the injections. Hence all maps are bijective. In particular, the class h of a hyperplane of Y comes from group cohomology, so that $h^{\dim Y}$ can be computed in the cohomology ring $\mathrm{H}^*(\pi_1 X)$ and hence vanishes, a contradiction.

3. Algebraic $\mathrm{K}(\pi,1)$ spaces and the Minimal Model Program

By Zariski–Nagata purity $\pi_1 X$ is a birational invariant of a smooth projective variety. As birational maps have seldom retractions, a smooth projective anabelian variety must be an absolutely minimal variety in its birational class. This is indeed the case already for smooth projective $\mathrm{K}(\pi,1)$ spaces.

Kollàr defines in [Ko93] Def 2.7 the notion of a variety X with **large algebraic fundamental group**: the image of $\pi_1 Z \to \pi_1 X$ is infinite for all nonconstant algebraic maps $f : Z \to X$. Projective algebraic $\mathrm{K}(\pi,1)$ spaces have large algebraic fundamental group. Arguing by contradiction we may restrict to smooth projective curves Z and finite maps f, such that $\pi_1 f$ is trivial. The degree $\deg_Z f^*\mathscr{L}$ for an ample line bundle $\mathscr{L}$ on X must be positive. On the other hand, by the commutativity of the following diagram

$$\begin{array}{ccccc}
\mathrm{Pic}(X) & \xrightarrow{c_1} & \mathrm{H}^2(X, \mathbb{Z}_\ell(1)) & \xleftarrow{\cong} & \mathrm{H}^2(\pi_1 X, \mathbb{Z}_\ell(1)) \\
\big\downarrow f^* & & \big\downarrow \mathrm{H}^2(f) & & \big\downarrow 0 \\
\mathrm{Pic}(Z) & \xrightarrow{c_1} & \mathrm{H}^2(Z, \mathbb{Z}_\ell(1)) & \longleftarrow & \mathrm{H}^2(\pi_1 Z, \mathbb{Z}_\ell(1))
\end{array}$$

and the formula $\deg_Z f^*\mathscr{L} = c_1(f^*\mathscr{L}) \in \mathrm{H}^2(Z, \mathbb{Z}_\ell(1)) = \mathbb{Z}_\ell$, the degree vanishes.

Having large fundamental group implies nonexistence of rational curves, which tremendously restricts geometry. Mori's bend and break technique inhibits nontrivial families of pointed maps $(C, c) \to (X, x)$. Moreover, the canonical bundle ω_X is nef and X is already minimal in the sense of the Minimal Model Program.

4. An abelian fibration

Abelian varieties are not really anabelian. Hence it is desirable to get rid of the abelian part of an algebraic $\mathrm{K}(\pi, 1)$ space. The following is inspired by its birational version [Ko93] Thm 6.3. of Kollàr.

An **almost regular fibration** on X is a projective map $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ defined on a projective birational modification $\sigma : \widetilde{X} \to X$, such that for a dense open $V \subset \widetilde{Y}$ the preimage $\widetilde{f}^{-1}(V)$ is mapped isomorphically by σ onto an open subset $U \subset X$ and the restriction $\widetilde{f}|_U : U \to V$ is smooth projective.

Theorem. *Let X/k be a smooth projective algebraic $\mathrm{K}(\pi, 1)$ space in characteristic 0. Let $\widetilde{f}$ be an almost regular fibration on X such that a general fibre admits a finite étale cover by an abelian variety.*

Then there exists a finite étale cover X'/X and a map $f' : X' \to Y'$ that is birational to the prolongation of $\widetilde{f}$ such that X'/Y' is an abelian scheme and Y' is a smooth projective $\mathrm{K}(\pi, 1)$ space.

Though for algebraic $\mathrm{K}(\pi, 1)$ spaces the proof is easier as in [Ko93] Thm 6.3, the proof follows the strategy of loc. cit. adding a final fourth step. (I) Replacing X by a finite étale cover, we may assume that a given fibre is an abelian variety. Hence all smooth fibres are algebraic $\mathrm{K}(\pi, 1)$ spaces with abelian fundamental group, thus are abelian varieties. (II) From Grothendiecks monodromy description of families of abelian varieties [Gr66], we obtain good reduction of the relative Albanese family over all of $\widetilde{Y}$. Here Kollàr uses a more involved argument via Hodge theory in order to verify Grothendieck's condition. (III) Again replacing X by a finite étale cover, we may assume that $\widetilde{X}/\widetilde{Y}$ is itself a family of abelian varieties. (IV) in the final step, we descend the family of abelian varieties using [Gr66] to the image under σ of the zero section of $\widetilde{X}/\widetilde{Y}$ as an abelian scheme. It turns out, that the necessary contraction of the total space is precisely the map σ, and the abelian fibration on a cover of X is achieved.

Almost abelian fibrations on minimal models are given by: (A) the nef reduction of ω_X as in [Nef01], (B) the Iitaka fibration under a conjecture on Kodaira dimension 0 and numerically trivial ω_X, (C) suitable pluricanonical maps under the abundance conjecture. Conjecturally all three examples agree and lead to a base Y' which is of general type. Following Kawamata Campana and Peternell conclude that the canonical bundle is even ample, see [Ka92] Appendix.

References

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