On Galois sections for hyperbolic p-adic curves

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(joint work with Florian Pop)

This note advocates a valuation theoretic point of view on Grothendieck's section conjecture in general, and for hyperbolic curves over p-adic fields in particular.

1. Valuative point of view towards the section conjecture

1.1. **Packets of sections.** Let X/k be a normal, geometrically irreducible variety with function field K. Let Gal_K be the absolute Galois group of K, and view the étale fundamental group $\pi_1(X)$ as its maximal quotient unramified over X:

$$\mathrm{Gal}_K \twoheadrightarrow \mathrm{Gal}(\tilde{K}/K) = \pi_1(X).$$

Let w be a Krull k-valuation of K with residue field $\kappa(w) = k$. The decomposition group $D_{\tilde{w}|w} \subseteq \pi_1(X)$ determined by a prolongation $\tilde{w} \mid w$ to $\tilde{K}$ admits a natural projection $D_{\tilde{w}|w} \twoheadrightarrow \mathrm{Gal}_{\kappa(w)}$ that always has a splitting $\sigma : \mathrm{Gal}_{\kappa(w)} \to D_{\tilde{w}|w}$. We obtain a **Galois section**, i.e., a section of $\pi_1(X) \to \mathrm{Gal}_k$, as follows:

$$s_w : \mathrm{Gal}_k = \mathrm{Gal}_{\kappa(w)} \xrightarrow{\sigma} D_{\tilde{w}|w} \to \pi_1(X).$$

The section s_w depends on the choice of splitting σ and on the choice of $\tilde{w}$. The collection of all such s_w associated to w is the **packet** of sections at w.

1.2. **The section conjecture.** Recall that a **hyperbolic** curve is a smooth geometrically connected curve with non-abelian geometric étale fundamental group.

Conjecture 1 (Grothendieck's section conjecture [G83])**.** *Let k be a number field and X/k a hyperbolic curve. Then every Galois section $s : \mathrm{Gal}_k \to \pi_1(X)$ is of the form s_w for a suitable choice of k-valuation w on the function field of X.*

Remark 2. (1) Since the injectivity of the section map for hyperbolic curves

$$X(k) \to \{s : \mathrm{Gal}_k \to \pi_1(X) \ ; \ \text{Galois section}\}, \quad a \mapsto s_a$$

is well known, Conjecture 1 is equivalent to the original version from [G83].

(2) In fact, the valuation theoretic formulation of Conjecture 1 takes care of the necessary correction of the original statement, see already in [G83], due to cuspidal sections coming from rational points from the boundary of the compactification.

(3) With $\mathrm{Gal}_K \to \mathrm{Gal}_k$ instead of $\pi_1(X) \to \mathrm{Gal}_k$ we obtain a birational version of the section conjecture. This is in fact a theorem for the variant where k is a finite extension of $\mathbb{Q}_p$ due to Koenigsmann [K03].

2. Valuations on p-adic fields

2.1. **The main theorem.** We are now concerned with the p-adic version of Conjecture 1. From now on, let $k/\mathbb{Q}_p$ be a finite extension with p-adic valuation v, ring of integers $\mathfrak{o}_k$, and residue field $\mathbb{F}$. The variety X/k will be a hyperbolic curve. We define

$$\mathrm{Val}_v(K) = \{w \ ; \ \text{Krull valuation on } K \text{ extending } v \text{ on } k\}$$

and similarly $\mathrm{Val}_v(\tilde{K})$. Then the main result of [PS09] is the following.

Theorem 3. *Let $k/\mathbb{Q}_p$ be a finite extension and X/k a hyperbolic curve with function field K. Then for every Galois section $s : \mathrm{Gal}_k \to \pi_1(X) = \mathrm{Gal}(\tilde{K}/K)$ there is a valuation $\tilde{w} \in \mathrm{Val}_v(\tilde{K})$ such that with $w = \tilde{w}|_K$*

$$s(\mathrm{Gal}_k) \subseteq D_{\tilde{w}|w} \subseteq \pi_1(X).$$

Remark 4. (1) Theorem 3 confirms a p-adic version of Conjecture 1: every Galois section is of the form s_w for a suitable valuation. Only the class of valuations has to take into account also the more "arithmetic" compactification by flat projective $\mathfrak{o}_k$-models of X, see below for the description of $\mathrm{Val}_v(\tilde{K})$. For an assertion towards the uniqueness of the valuation w in Theorem 3 we refer to [PS09].

(2) We set v_a for the k-valuation of K corresponding to the k-rational point $a \in X(k)$. The composition of valuations $w_a = v \circ v_a$ yields a map

$$X(k) \to \mathrm{Val}_v(K), \quad a \mapsto w_a$$

such that $D_{w_a} = s_a(\mathrm{Gal}_k)$ up to conjugation. The p-adic section conjecture follows from Theorem 3 if only valuations of the form w_a admit sections of $D_{\tilde{w}|w} \to \mathrm{Gal}_k$.

(3) If the p-adic section conjecture turns out to be wrong, then Theorem 3 yields the analogous correction with sections coming from valuations centered at infinity as in the case for affine curves with Grothendieck's original conjecture in [G83].

(4) There are conditional results due to Saïdi to lift Galois sections at least partially towards birational Galois sections, namely to the cuspidally abelian quotient of Gal_K relative X, with the idea in mind to reduce the p-adic section conjecture to Koenigsmann's Theorem recalled above. Further weaker but unconditional lifting results are obtained by Borne/Emsalem together with the author.

(5) Hoshi has shown that the geometrically pro-p version of the section conjecture fails in explicit examples where non-geometric sections exist.

(6) Mochizuki deals with an analogue regarding Galois sections for the tempered fundamental group of André, a group which is pro-discrete rather than pro-finite.

2.2. **An application.** Theorem 3 has the following consequence for Galois sections (trivial for Galois sections coming from k-rational points).

Theorem 5. *Let $k/\mathbb{Q}_p$ be a finite extension and X/k a proper hyperbolic curve with proper flat model $\mathscr{X} \to \mathrm{Spec}(\mathfrak{o}_k)$. Let $Y = \mathscr{X}_{\mathbb{F}}$ be the special fibre.*

1. *If there is a Galois section $s : \mathrm{Gal}_k \to \pi_1(X)$, then the geometric specialisation map $\overline{\mathrm{sp}} : \pi_1(X \otimes k^{\mathrm{alg}}) \twoheadrightarrow \pi_1(Y \otimes \mathbb{F}^{\mathrm{alg}})$ is surjective.*
2. *Every Galois section $s : \mathrm{Gal}_k \to \pi_1(X)$ specialises to a unique Galois section $t : \mathrm{Gal}_{\mathbb{F}} \to \pi_1(Y)$, i.e., there is a commutative diagram*

$$\begin{array}{ccc} \pi_1(X) & \xrightarrow{\mathrm{sp}} & \pi_1(Y) \\ {\scriptstyle s}\uparrow\downarrow & & \downarrow\uparrow{\scriptstyle t} \\ \mathrm{Gal}_k & \twoheadrightarrow & \mathrm{Gal}_{\mathbb{F}}\,. \end{array}$$

2.3. **The Riemann–Zariski space.** The space of valuations $\mathrm{Val}_v(\tilde{K})$ can be more geometrically understood as the Riemann–Zariski pro-space of (the closed fibres of) all models. Let $X_H \to X$ be the finite étale cover corresponding to an open subgroup $H \subseteq \pi_1(X)$, and let $\mathscr{X}_H$ be a proper flat $\mathfrak{o}_k$-model of X_H. Any $\tilde{w} \in \mathrm{Val}_v(\tilde{K})$ has a unique center in the special fibre $\mathscr{X}_{H,\mathbb{F}}$ by the valuative criterion of properness, i.e., a point $z_{\tilde{w}}$ such that the valuation ring of $\tilde{w}$ dominates the local ring $\mathcal{O}_{\mathscr{X},z_{\tilde{w}}}$. In fact, the map assigning the compatible system of centers

$$(\star) \qquad \mathrm{Val}_v(\tilde{K}) \xrightarrow{\sim} \varprojlim_{H,\mathscr{X}_H} \mathscr{X}_{H,\mathbb{F}}, \quad \tilde{w} \mapsto z_{\tilde{w}}$$

is a homeomorphism of pro-finite spaces (for the patch topology on the left and the constructible topology on the right).

2.4. **Fixed points.** The map $(\star)$ is equivariant under $\pi_1(X) = \mathrm{Gal}(\tilde{K}/K)$ and $D_{\tilde{w}|w}$ is precisely the stabilizer of $\tilde{w}$. By the usual compactness argument with projective limits it suffices for Theorem 3 to show that $\Sigma = s(\mathrm{Gal}_k) \subset \pi_1(X)$ has a fixed point (generic or closed)

$$(\mathscr{X}_{H,\mathbb{F}})^{\Sigma} \neq \emptyset$$

for a cofinal set of open normal subgroups $H \lhd \pi_1(X)$ and equivariant models $\mathscr{X}_H$ on which Σ acts via a finite subgroup of $\pi_1(X)/H$. Thus we first may assume $\mathscr{X}_H$ is a regular semistable model. The fibres of the projection to the stable model

$$\mathscr{X}_H \to \mathscr{X}_{H,\mathrm{stable}}$$

are trees of projective lines. Since a tree is a CAT(0)-space, any action by a finite group on a tree has fixed points. It follows that the fibre over a Σ-fixed point of $(\mathscr{X}_{H,\mathrm{stable}})_{\mathbb{F}}$ again has a Σ-fixed point. We may therefore restrict to stable models.

3. The ℓ-adic Brauer group method

3.1. **The locus of a Brauer class.** Although it is counterintuitive that ℓ-adic methods actually are able to detect the arithmetic in a Galois section, we next fix a prime $\ell \neq p$. The Brauer group method going back to Neukirch in the study of absolute Galois groups of number fields is here based on the following.

The relative Brauer group $\ker(\mathrm{Br}(k) \to \mathrm{Br}(X))$ is cyclic of order the index of X due to Roquette and Lichtenbaum. By [S10] the presence of a section implies that the index is in fact a power of p, so that the map on ℓ-torsion

$$\mathrm{Br}(k)[\ell] \hookrightarrow \mathrm{Br}(X)[\ell] \subseteq \mathrm{Br}(K)[\ell]$$

is injective. In the limit over all neighbourhoods of s, i.e., for the fixed field $M = \tilde{K}^{\Sigma}$, the map $\mathrm{Br}(k)[\ell] \hookrightarrow \mathrm{Br}(M)[\ell]$ remains injective. We now need a fine local–global principle for the Brauer group due to Pop:

Theorem 6 ([P88] Thm 4.5)**.** *Let $k/\mathbb{Q}_p$ be a finite extension and M/k a function field of transcendence degree 1 over k. Then the restriction map*

$$\mathrm{Br}(M) \hookrightarrow \prod_{w \in \mathrm{Val}_v(M)} \mathrm{Br}(M_w^{\mathrm{h}})$$

is injective. Here M^{h}_w denotes the henselisation of M in the valuation w.

It follows that there is a valuation $w_M \in \mathrm{Val}_v(M)$ such that $\mathrm{Br}(k)[\ell]$ survives in $\mathrm{Br}(M^{\mathrm{h}}_{w_M})$. Let $\tilde{w}$ be an extension of w_M to $\tilde{K}$. Since $\mathrm{Gal}(\tilde{K}/M) = \Sigma \simeq \mathrm{Gal}_k$, all intermediate fields are composita with extensions k'/k of the same degree. It follows that $[(\tilde{K} \cap M^{\mathrm{h}}_{w_M}) : M]$ is prime to ℓ since otherwise $\mathrm{Br}(k)[\ell]$ would not survive. Therefore a suitable choice of ℓ-Sylow subgroup $\Sigma_\ell \subset \Sigma$ is contained in

$$(\star\star) \qquad \Sigma_\ell \subseteq \mathrm{Gal}(\tilde{K}/\tilde{K} \cap M^{\mathrm{h}}_{w_M}) = D_{\tilde{w}|w_M} \subseteq D_{\tilde{w}|w}.$$

3.2. **Inertia.** Let $\Theta \subseteq \Sigma$ be the image under s of the inertia group $I_k \subseteq \mathrm{Gal}_k$ and let $I_{\tilde{w}|w} \subseteq D_{\tilde{w}|w}$ denote the inertia group of $\tilde{w}$. Based on $(\star\star)$ with considerable more work for valuations $\tilde{w}$ associated to generic points of components of the special fibre one may show the following.

Proposition 7. *It is possible to choose $\tilde{w}$ such that $\Theta_\ell \subseteq I_{\tilde{w}|w}$, where Θ_ℓ is a choice of ℓ-Sylow group of Θ.*

4. Independence of ℓ-adic ramification

4.1. **The kernel of specialisation.** Let $H \lhd \pi_1(X)$ be an open normal subgroup such that X_H has a stable model $\mathscr{X}_{H,\mathrm{stable}}$. We write $Y = \bigcup_\alpha Y_\alpha$ for the union of irreducible components of its reduced special fibre and may further assume that all Y_α are smooth and have genus ≥ 1. We consider the kernel of specialisation

$$N_H := \ker\big(H = \pi_1(X_H) \twoheadrightarrow \pi_1(\mathscr{X}_H)\big)$$

which contains $I_{\tilde{w}|w} \cap H$ for every valuation $\tilde{w} \in \mathrm{Val}_v(\tilde{K})$. We further set

$$V_H = N_H^{\mathrm{ab}} \widehat{\otimes} \mathbb{Q}_\ell$$

and for each $\tilde{w} \in \mathrm{Val}_v(\tilde{K})$ we define a set of cardinality 1 or 2

$$A_{\tilde{w}} = \{\alpha \ ; \ Y_\alpha \text{ contains the center of } \tilde{w} \text{ on } \mathscr{X}_{H,\mathrm{stable}}\}.$$

By ℓ-adic étale cohomology computations and logarithmic geometry we show the following statement on independence of ℓ-adic inertia. For simplicity of notation we denote the discrete rank 1 valuation of $\tilde{K}^H$ associated to Y_α by α.

Proposition 8. (1) *For any choice of prolongation $\tilde{\alpha} \in \mathrm{Val}_v(\tilde{K})$ of each α, the natural map*

$$\bigoplus_\alpha I^{\mathrm{ab}}_{\tilde{\alpha}|\alpha} \otimes \mathbb{Q}_\ell \hookrightarrow V_H$$

is injective.

(2) *For every $\tilde{w} \in \mathrm{Val}_v(\tilde{K})$ the map $I_{\tilde{w}|w} \cap H \to N_H \to V_H$ factors as*

$$I_{\tilde{w}|w} \cap H \to \bigoplus_{\alpha \in A_{\tilde{w}}} I^{\mathrm{ab}}_{\tilde{\alpha}|\alpha} \otimes \mathbb{Q}_\ell \hookrightarrow V_H.$$

4.2. **Sketch of proof for the existence of fixed points.** Let $\sigma \in \Sigma = s(\mathrm{Gal}_k)$ be arbitrary. Since Θ is a normal subgroup in Σ we obtain a commutative diagram

$$\begin{array}{ccccc} \Theta_\ell \cap H & \subseteq & I_{\tilde{w}|w} \cap H & & \\ & \searrow & & \searrow & \\ & & \Theta \cap N_H & \longrightarrow & V_H \\ & \nearrow & & \nearrow & \\ \sigma\Theta_\ell\sigma^{-1} \cap H & \subseteq & I_{\sigma(\tilde{w})|w} \cap H & & \end{array}$$

Because s is a Galois section, the composition

$$\mathbb{Z}_\ell(1) \simeq \Theta_\ell \cap H \to V_H \to I_k^{\mathrm{ab}} \otimes \mathbb{Q}_\ell \simeq \mathbb{Q}_\ell(1)$$

is non-trivial. On the other hand, the image of $\Theta \cap N_H$ in V_H spans at most a 1-dimensional subspace, since any closed subgroup of I_k has pro-ℓ completion of rank at most 1. It follows from Proposition 8 that $\Theta \cap N_H$ maps to the subspace

$$\bigcup_{A_{\tilde{w}} \cap A_{\sigma(\tilde{w})}} I_{\tilde{\alpha}|\alpha}^{\mathrm{ab}} \otimes \mathbb{Q}_\ell \hookrightarrow V_H$$

whence $A_{\tilde{w}} \cap A_{\sigma(\tilde{w})} \neq \emptyset$. A combinatorial argument relying again on Proposition 8 shows that either an $\alpha \in A_{\tilde{w}}$ is fixed by Σ, or $A_{\tilde{w}}$ is fixed by Σ as a set and consists of two elements corresponding to components meeting in a unique node. In this way we have found a fixed point under Σ on $\mathscr{X}_{H,\mathrm{stable}}$ and the sketch of the proof of Theorem 3 is complete.

References

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