Nonabelian examples for the section conjecture in anabelian geometry

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The present report continues [Sx09a], that dealt with p-adic local obstructions, by elaborating on the passage from local to global in the section conjecture. More details can be found in [Sx08] and the preprint [Sx09b].

1. The section conjecture of anabelian geometry

1.1. **The conjecture.** Let k be a field, k^{sep} a separable closure, and $\mathrm{Gal}_k = \mathrm{Gal}(k^{\mathrm{sep}}/k)$ its absolute Galois group. The étale fundamental group $\pi_1(X, \bar{x})$ of a geometrically connected variety X/k with a geometric point $\bar{x} \in X$ above k^{sep}/k forms an extension

$$1 \to \pi_1(X \times_k k^{\mathrm{sep}}, \bar{x}) \to \pi_1(X, \bar{x}) \to \mathrm{Gal}_k \to 1, \tag{1}$$

which we abbreviate by $\pi_1(X/k)$ ignoring base points. A k-rational point $a \in X(k)$ yields by functoriality a section s_a of (1), which depends on the choice of an étale path from a to $\bar{x}$. Thus s_a is well defined only up to conjugation by elements from $\pi_1(X \times_k k^{\mathrm{sep}}, \bar{x})$. The section conjecture speculates the following, see Grothendieck [Gr83] for the case of a number field k.

Conjecture 1. *The map $a \mapsto s_a$ is a bijection of the set of rational points $X(k)$ with the set $\mathscr{S}_{\pi_1(X/k)}$ of conjugacy classes of sections of $\pi_1(X/k)$ if k is a number field or a finite extension of $\mathbb{Q}_p$ and X is a smooth, geometrically connected curve of genus g with boundary divisor D in its smooth projective completion, such that*

(i) $2 - 2g - \deg(D)$ *is negative, i.e., X is hyperbolic, and*
(ii) *D has no k-rational point.*

1.2. **Evidence.** It was known already to Grothendieck, that the map $a \mapsto s_a$ of Conjecture 1 is injective by an application of the weak Mordell-Weil theorem.

The real analogue of Conjecture 1 was proven by Mochizuki [Mz03] with alternative proofs later in [Sx08] Appendix A, and by Pal. Koenigsmann was able to prove a birational form over p-adic local fields [Ko05].

The note [Sx08] contains a p-adic local obstruction to the existence of sections and thus k-rational points that leads to a wealth of positive examples where Conjecture 1 holds, yet in the case that there are neither sections nor points. However, it is known that this ostensibly dull case of *empty curves* is crucial, see [Sx08] Appendix C. Shortly afterwards, in [HS08] Harari, Szamuely and Flynn gave examples for the section conjecture with still no points globally over $\mathbb{Q}$ but with local points everywhere.

Further evidence for the section conjecture can be found in the work of Ellenberg, Esnault–Hai, Esnault–Wittenberg, Wickelgren, Saïdi–Tamagawa, and Hoshi–Mochizuki.

Harari and Szamuely work with the abelianized extension $\pi_1^{(\mathrm{ab})}(X/k)$ obtained by pushing with the characteristic quotient $\pi_1 \twoheadrightarrow \pi_1^{\mathrm{ab}}$ of the maximal abelian quotient. The aim of this report is to discuss structural aspects of Conjecture 1 which go beyond the abelianized extension.

2. Adelic sections and Brauer–Manin obstructions

2.1. **Adelic sections.** An extension of algebraically closed base fields does not alter the fundamental group in characteristic 0. Hence for an extension K/k the extension $\pi_1(X \times_k K/K)$ is a pullback of $\pi_1(X/k)$ and we get a natural map

$$\mathscr{S}_{\pi_1(X/k)} \to \mathscr{S}_{\pi_1(X/k)}(K) := \mathscr{S}_{\pi_1(X \times_k K/K)} \quad s \mapsto s \otimes K.$$

Let k be a number field, k_v its completion at a place v and $\mathbb{A}_k \subset \prod_v k_v$ its ring of adels $\mathbb{A}_k \subset \prod_v k_v$. The space of adelic sections $\mathscr{S}_{\pi_1(X/k)}(\mathbb{A}_k) \subset \prod_v \mathscr{S}_{\pi_1(X/k)}(k_v)$ of $\pi_1(X/k)$ is the set of all tuples (s_v) such that for all quotients $\varphi : \pi_1(X) \twoheadrightarrow G$ with finite G all but finitely many of the $\varphi \circ s_v : \mathrm{Gal}_{k_v} \to G$ are unramified.

2.2. **Brauer–Manin obstruction for sections.** A class $\alpha \in \mathrm{H}^2(\pi_1(X), \mu_n)$ describes a function $\langle \alpha, - \rangle \ : \ \mathscr{S}_{\pi_1(X/k)}(\mathbb{A}_k) \to \mathbb{Q}/\mathbb{Z}$ on adelic sections of $\pi_1(X/k)$ by the formula $\langle \alpha, (s_v) \rangle = \sum_v \mathrm{inv}_v(s_v^*(\alpha))$, where the maps inv_v are the local invariant maps $\mathrm{H}^2(k_v, \mu_n) \subset \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$.

Theorem 2. *The function $\langle \alpha, - \rangle$ is well defined because only finitely many summands in $\sum_v \mathrm{inv}_v(s_v^*(\alpha))$ do not vanish. The image of the global sections under the diagonal map $\mathscr{S}_{\pi_1(X/k)} \to \prod_v \mathscr{S}_{\pi_1(X/k)}(k_v)$ lands in the Brauer kernel*

$$\mathscr{S}_{\pi_1(X/k)}(\mathbb{A}_k)^{\mathrm{Br}} := \{(s_v) \in \mathscr{S}_{\pi_1(X/k)}(\mathbb{A}_k) \ ; \ \langle \alpha, (s_v) \rangle = 0 \textit{ for all } \alpha\}.$$

Proof: We only prove the second part which was independently also observed by O. Wittenberg. We compute

$$\langle \alpha, (s \otimes k_v) \rangle = \sum_v \mathrm{inv}_v\big((s \otimes k_v)^*(\alpha)\big) = \sum_v \mathrm{inv}_v\big(s^*(\alpha) \otimes_k k_v\big) = 0$$

by the Hasse–Brauer–Noether local global principle for the Brauer group. □

2.3. **Conditional results.** Because $\bigcup_n \mathrm{H}^2(\pi_1 X, \mu_n) \twoheadrightarrow \mathrm{H}^2(X, \mathbb{G}_m)$ is surjective for hyperbolic curves, the classical Brauer–Manin obstruction for rational points as in [Ma71] is subsumed under the corresponding obstruction for sections. We can therefore prove the following conditional result.

Theorem 3. *Let k be a number field such that Conjecture 1 holds for each completion k_v. If the Brauer–Manin obstruction against rational points is the only one for curves over k, then the section conjecture holds for hyperbolic curves over k.*

3. Beyond abelian sections

3.1. **The Reichardt–Lind curve.** We present an affine curve over $\mathbb{Q}$ that by an application of Section 2 can be shown not to admit sections. The corresponding empty example for the section conjecture over $\mathbb{Q}$ has adelic points but none that satisfies the Brauer–Manin obstructions, and moreover is not accounted for by the explicit examples of [HS08].

The affine Reichardt–Lind curve $U/\mathbb{Q}$ is defined by $2Y^2 = Z^4 - 17$ with $Y \neq 0$. Let $X/\mathbb{Q}$ be its smooth completion. The class $\alpha_U = \chi_Y \cup \chi_{17} \in \mathrm{H}^2(\pi_1(U), \mu_2)$, the cup product of the two characters defined via Kummer theory by Y and 17, lifts to $\alpha \in \mathrm{H}^2(X, \mu_2)$. The corresponding function $\langle \alpha, - \rangle$ takes the constant value $\frac{1}{2}$ on adelic sections subject to an extra condition.

Theorem 4. *The fundamental group extension $\pi_1(U/\mathbb{Q})$ for the affine Reichardt–Lind curve $U/\mathbb{Q}$ does not split. In particular, the section conjecture holds trivially for $U/\mathbb{Q}$ as there are neither rational points nor sections.*

More precisely, the maximal geometrically pro-2 quotient $\pi_1^{(2)}(X/\mathbb{Q})$ of $\pi_1(X/\mathbb{Q})$ for the projective Reichardt–Lind curve $X/\mathbb{Q}$ does not admit a section s that allows a lifting $\tilde{s}_p$

$$\begin{tikzcd}
 & \mathrm{Gal}_{\mathbb{Q}_p} \arrow[dl, dotted, "\tilde{s}_p"'] \arrow[d, "s\otimes\mathbb{Q}_p"] \\
\pi_1^{(2)}(U) \arrow[r, two heads] & \pi_1^{(2)}(X)
\end{tikzcd}$$

locally at $p = 2$ and $p = 17$.

3.2. **Genus 2 curves.** Potentially, the Brauer–Manin obstruction against sections occurs only on a finer quotient than $\pi_1^{(\mathrm{ab})}(U/k)$, because it depends on H^2. This hope turns out to be illusory for the Reichardt–Lind curve. In order to have an explicit example X, where $\pi_1^{(\mathrm{ab})}(X/k)$ splits and yet there is no section, we resort to an argument of [Sx08] with some more care to prove the following.

Theorem 5. *Let $k/\mathbb{Q}_p$ be a finite extension for $p > 2$, and let X/k be a smooth projective curve of genus 2.*

1. *If X has period 1, then $\pi_1^{\mathrm{ab}}(X/k)$ admits a section.*
2. *If X has index 2, then the maximal geometrically metabelian quotient $\pi_1^{\mathrm{metab}}(X/k)$ does not split.*

Explicit examples for the setup of Theorem 5, even of curves over number fields that satisfy the conditions locally at some place, can be found in abundance.

References

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