On the period–index problem in light of the section conjecture

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1. The section conjecture

Let k be a field, $k^{\rm sep}$ a fixed separable closure and $\mathrm{Gal}_k = \mathrm{Gal}(k^{\rm sep}/k)$ its absolute Galois group. The étale fundamental group $\pi_1(X,\bar{x})$ of a geometrically connected variety X/k with a geometric point $\bar{x} \in X$ above $k^{\rm sep}/k$ sits naturally in a short exact sequence

$$1 \to \pi_1(X \times_k k^{\rm sep}, \bar{x}) \to \pi_1(X,\bar{x}) \to \mathrm{Gal}_k \to 1, \tag{1}$$

which we abbreviate by $\pi_1(X/k)$. A k-rational point $a \in X(k)$ yields by functoriality a section s_a of (1), which depends on the choice of an étale path from a to $\bar{x}$ and thus is well defined only up to conjugation by elements from $\pi_1(X \times_k k^{\rm sep})$. The section conjecture speculates the following.

Conjecture 1 (Grothendieck [Gr83])**.** *The map $a \mapsto s_a$ is a bijection of the set of rational points $X(k)$ with the set of conjugacy classes of sections of $\pi_1(X/k)$ if k is a number field and X is a smooth, projective curve of genus at least* 2*.*

It was known to Grothendieck, that $a \mapsto s_a$ is injective by an application of the weak Mordell-Weil theorem.

2. Evidence so far

Only bits of evidence for the section conjecture have emerged over the years. The most convincing piece consists perhaps in J. Koenigsmann's proof in [Ko05] of a birational analogue for function fields in one variable over a local p-adic field.

A **neighbourhood** of a section $s : \mathrm{Gal}_k \to \pi_1(X,\bar{x})$ consists of a connected finite étale cover $X' \to X$ together with a lift $t : \mathrm{Gal}_k \to \pi_1(X',\bar{x}')$ of the section. The geometric covers contained in neighbourhoods of a given section form a cofinal system due to $\pi_1(X \times_k k^{\rm sep})$ being finitely generated and the use of characteristic subgroups.

The technique of neighbourhoods was pioneered in the work of Nakamura and Tamagawa and leeds to the equivalence of the section conjecture with the weak section conjecture, see [Ko05].

Conjecture 2 (weak section conjecture)**.** *Let k be a number field. A smooth projective curve X/k of genus at least* 2 *has a rational point if and only if its fundamental group extension $\pi_1(X/k)$ splits.*

Indeed, if a section s exists that differs from all the finitely many sections associated to rational points, then it has a neighbourhood $X' \to X$ whose fundamental group extension still allows a section, namely the lift t, but which does not contain rational points. It is most unfortunate that this foundational argument relies on the theorem of Faltings-Mordell, which during the infancy of the conjecture was believed to follow from the section conjecture itself. In case of a local p-adic field, we can replace Faltings-Mordell by a compactness argument.

3. Local obstructions to sections

The aim of the talk was to present new evidence for the section conjecture as provided in the authors note [Sx08]. As the new evidence is purely local we take the courage to repond now to a question asked after the talk by S. Wewers and conjecture the following.

Conjecture 3. *Conjecture 1 holds also for smooth, projective curves over a local p-adic field.*

The index of X/k is the gcd of the degrees of all k-rational divisors on X, the period of X/k is the gcd of all k-rational divisor classes of X/k. For curves of genus g, period divides the index which divides $2g-2$. The index furthermore annihilates the kernel $\mathrm{Br}(X/k)$ of $\mathrm{Br}(k) \to \mathrm{Br}(X)$. A theorem of Roquette asserts that for $k/\mathbb{Q}_p$ finite, the relative Brauer group $\mathrm{Br}(X/k)$ is cyclic of order the index.

Of course, if we have a rational point on X, then period and index equals 1 and $\mathrm{Br}(X/k)$ is trivial. In light of the section conjecture, the same should follow from merely the assumption of having a section of $\pi_1(X/k)$. Using results of Lichtenbaum on the period/index problem for curves over p-adic local fields we manage to prove at least the following theorem, see [Sx08] Thm 16. An alternative proof using the cycle class of a section was later given by Esnault and Wittenberg in [EW08].

Theorem 4. *Let k be a finite extension of $\mathbb{Q}_p$ and let X/k be a smooth, projective curve of positive genus, such that the fundamental group extension $\pi_1(X/k)$ admits a section.*

(1) For p odd, period of X equals the index of X and both are powers of p.

(2) For $p = 2$, we have period of X and index of X are powers of 2. If we moreover assume that we have an even degree finite étale cover $X \to X_o$ with X_0 of positive genus, then we have also that period equals the index.

So having a section locally at a p-adic place constaints the numerical data of period and index for the curve base changed to the local field at that place.

The analogue for a real place was known before and admits many proofs (Wojtkowiak, Huisman, Mochizuki, Pal, ...), one of which relies on a theorem of Witt from 1934 and runs parallel to the proof of Theorem 4, see for example [Sx08] Thm 26.

Theorem 5 (Real section conjecture)**.** *Let $X/\mathbb{R}$ be a smooth, projective curve of genus ≥ 1. Then the map*

$$\pi_0\big(X(\mathbb{R})\big) \to \big\{\textit{conjugacy classes of sections of } \pi_1(X/\mathbb{R})\big\},$$

that maps a connected component of the real locus $X(\mathbb{R})$ to the corresponding conjugacy class of sections is a bijection of finite sets.

4. Constructing examples

Examples of curves X over number fields k such that at a place $v|p$ with completion k_v the relative Brauer group of $X \times_k k_v$ over k_v contains nontrivial torsion prime to p can be achieved in several ways. A geometric method using Brauer-Severi varieties is outlined in [Sx08] §7. These lead to the first known examples of curves over number fields where Conjecture 1 holds, albeit for trivial reasons of having neither sections nor rational points. An explicit example is given for $p \equiv 3$ modulo 4 and $n \geq 2$ by the curve in $\mathbb{P}^2_{\mathbb{Q}}$ given as

$$\{X^{2n} + Y^{2n} = pZ^{2n}\}$$

which has neither points nor sections over $\mathbb{Q}$ for local reasons over $\mathbb{Q}_p$. All examples which we can construct are empty in the sense of having neither points nor sections and are obstructed locally. But these examples exist in abundance. For equally empty examples for curves over number fields which are even counter-examples to the Hasse principle (but less abundant) see the talk of T. Szamuely at the same conference and [HS08] for details.

References

[EW08] Esnault, H., Wittenberg, O., *Remarks on the pronilpotent completion of the fundamental group*, preprint, arXiv:0807.2963v3[math.AG], July/December 2008.

[Gr83] Grothendieck, A., *Brief an Faltings (27/06/1983)*, in: Geometric Galois Action 1 (ed. L. Schneps, P. Lochak), LMS Lecture Notes **242** (1997), 49–58.

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[Ko05] Koenigsmann, J., *On the 'section conjecture' in anabelian geometry*, J. Reine Angew. Math. **588** (2005), 221–235.

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