Anabelian properties of the moduli spaces of smooth projective curves

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The talk delivered at the meeting and the report below contain a survey of the results obtained by the author in [3].

1. Rational curves

In [2] Oort constructs non-constant maps $\mathbb{P}^1(k) \to \mathrm{M}_g(k)$ for large, suitable g by exploiting 'Parshin's trick'. Here k is an algebraically closed field and M_g is the coarse moduli variety of smooth, projective, geometrically connected curves of genus g. In contrast, the fine moduli stack $\mathscr{M}_g$ parametrising families of smooth, projective, geometrically connected curves of genus $g \geq 2$ does not contain rational curves. In characteristic 0 this follows for example from the uniformisation of $(\mathscr{M}_g)^{\mathrm{an}}$ by Teichmüller space which is a ball and the simply connectedness of $\mathbb{P}^1$. Indeed, any map $f : T \to (\mathscr{M}_g)^{\mathrm{an}}$ from a simply connected complex variety T must lift to Teichmüller space and hence is forced to be constant if T is proper or by Liouville for T the complex plane. The Brødy hyperbolicity of $(\mathscr{M}_g)^{\mathrm{an}}$ follows.

The moduli space of principally polarised abelian varieties $\mathscr{A}_{g,1}$ behaves similar at first sight. It is uniformised by the Siegel upper half plane. But, again in [2], Oort constructs rational families of abelian varieties in positive characteristic, essentially by using rational families of maps from α_p to a given abelian variety with p-rank exceeding 1. This raises the question of the existence of simply connected subvarieties of $\mathscr{M}_g$ in positive characteristic and in particular of an algebraic reasoning.

2. Anabelian Geometry

Anabelian geometry deals with the arithmetical/geometrical content of the profinite étale fundamental group of a variety. In homotopy theory, for X an Eilenberg-MacLane $\mathrm{K}(\pi, 1)$ space, the fundamental group $\pi_1 X = \pi$ determines all maps up to homotopy from CW-spaces with target X. One consequence is that group cohomology of π with coefficients in F computes the singular cohomology of X with coefficients in the associated locally constant sheaf $\mathscr{F}$.

We define an **algebraic** $\mathrm{K}(\pi, 1)$ **space** to be a variety over an algebraically closed field, such that the canonical map

$$\gamma^* : \mathrm{H}^*(\pi_1^{\text{ét}} X, F) \to \mathrm{H}^*(X_{\text{ét}}, \mathscr{F})$$

is an isomorphism for all finite F and associated $\mathscr{F}$. In general, we call the cohomology classes in the image of γ^* **group theoretic** cohomology classes.

One difference between $\mathscr{M}_g$ and $\mathscr{A}_{g,1}$ consists in the pro-finite properties of their respective analytic fundamental groups: the Mapping class group is conjectured to be good in the sense of Serre, whereas $\mathrm{Sp}_{2g}(\mathbb{Z})$ is definitely not good. This might explain why the 'analytic $\mathrm{K}(\pi, 1)$ behaviour' of $\mathscr{A}_{g,1}$ does not carry over to the algebraic and moreover positive characteristic setting.

3. Constant maps

Theorem 1. *Let k be an algebraically closed field and X/k a quasi-projective, connected variety such that for some prime ℓ different from the characteristic and for all $n \in \mathbb{N}$ all classes in $H^2(X, \mathbb{Z}/\ell^n\mathbb{Z})$ are group theoretic. Let $f : T \to X$ be a regular map from a proper, reduced and connected variety T/k such that $\pi_1 f$ is the trivial map. Then f must be constant.*

Proof: One just notices that f is constant if and only if it contracts all proper curves in T, which is a numerical condition. The pullback of an ample numerical class can be computed via group cohomology, hence vanishes, and we are done.

In order to apply Theorem 1 to the moduli space of smooth, projective curves we either need to prove that the mapping class group is good in cohomological degree 2 (announced by Boggi) or find different means. The alternative proof deals only with some quotient of the fundamental group of $\mathscr{M}_g$. Thus it yields a much stronger theorem also strengthening the characteristic 0 case, which for the full fundamental group follows from the analytic argument given above as the mapping class group is residually finite.

Let X/S be a family of smooth, projective, geometrically connected curves of genus $g \geq 2$ with S connected. Let $\mathbb{L}$ be a set of primes invertible on S. Then one can proof, see [3] and the comments in loc. cit., that the homotopy sequence

$$\pi_1(\text{fibre}) \to \pi_1 X \to \pi_1 S \to 1$$

yields an outer monodromy representation $\rho : \pi_1 S \to \operatorname{Out}\left(\pi^{\mathbb{L}}\right)$ where $\pi^{\mathbb{L}}$ is the pro-$\mathbb{L}$ completion of the fundamental group of a geometric fibre of X/S. The construction is compatible with pullback, hence comes by composition with the characteristic map from the universal outer representation

$$\rho^{\text{univ}} : \pi_1\big(\mathscr{M}_g \otimes \mathbb{Z}[\frac{1}{\mathbb{L}}]\big) \to \operatorname{Out}\left(\pi^{\mathbb{L}}\right).$$

Theorem 1^{bis}. *Let T be a reduced, connected variety over an algebraically closed field k. Let $f : T \to \mathscr{M}_g$, $g \geq 2$, be a map such that for the associated curve the outer pro-$\mathbb{L}$ representation $\rho : \pi_1 T \to \operatorname{Out}\left(\pi^{\mathbb{L}}\right)$ is the trivial homomorphism for one of the following collections of sets of prime numbers $\mathbb{L}$ and additional conditions on k:*

(A) *$\mathbb{L} = \{\ell\}$ for some prime number ℓ and k is of characteristic* 0*, or*
(B) *$\mathbb{L} = \{\ell_1, \ell_2\}$ for all pairs of sufficiently large prime numbers ℓ_1, ℓ_2 invertible in k and k is of positive characteristic.*

Then f is constant in the sense that the corresponding T-curve $X/T \in \mathscr{M}_g(T)$ comes by base extension from a curve in $\mathscr{M}_g\big(\operatorname{Spec}(k)\big)$.

The basic idea of the proof is to look at the corresponding family of Jacobians which has constant $\mathbb{L}$-primary torsion by the condition of the theorem. Then use the Torelli theorem. This explains the easier condition in characteristic 0.

But this approach fails in positive characteristic. We only deduce that our Jacobians all have the same p-rank. A construction of auxiliary covers following

Tamagawa (after Raynaud) that has already been exploited by Saïdi allows nevertheless to deduce the theorem. It is in the construction of the auxiliary covers that condition (B) on the sets $\mathbb{L}$ comes up.

4. Monodromy controls good reduction

There are several known criteria for good reduction that ask for unramified Galois action. Galois action is nothing but monodromy action in the arithmetic case. The question of good reduction turns out to be the question of extendibility for the representing map to the respective moduli space. Put together, these ideas indicate that extending a map $f : U \to \mathscr{M}_g$ from some open dense subscheme $U \subset S$ in the normal scheme S to a map $\tilde{f} : S \to \mathscr{M}_g$ is controlled by monodromy.

Theorem 2 (Moret-Bailly)**.** *The above f always extends uniquely for S regular and $S \setminus U$ of codimension at least* 2*.*

This is the purity result for smooth, projective curves of Moret-Bailly in [1]. If we combine Zariski–Nagata's purity of the branch locus, Moret-Bailly's theorem above and the following criterion for good reduction from [4], Thm 5.3,

Theorem 3 (Oda–Tamagawa)**.** *Let S be the spectrum of a discrete valuation ring, and U be the generic point. Then a curve of genus $g \geq 2$ over U extends uniquely over S iff the associated outer pro-ℓ monodromy representation is unramified, i.e., factors over* $\mathrm{Gal}_K = \pi_1 \operatorname{Spec}(K) \to \pi_1 S$*, for some ℓ invertible in K.*

we obtain the case of regular bases S of the following monodromy criterion of good reduction of smooth, projective curves.

Theorem 4. *Let U be a dense open subscheme of a normal, connected, excellent scheme S. Let X/U be a U-curve in $\mathscr{M}_g(U)$ for some $g \geq 2$. Then X/U extends to an S-curve in $\mathscr{M}_g(S)$ if and only if the pro-$\mathbb{L}$ monodromy representation*

$$\rho : \pi_1\big(U \otimes \mathbb{Z}[\frac{1}{\mathbb{L}}]\big) \to \mathrm{Out}\left(\pi^{\mathbb{L}}\right)$$

factors over $\pi_1\big(U \otimes \mathbb{Z}[\frac{1}{\mathbb{L}}]\big) \to \pi_1\big(S \otimes \mathbb{Z}[\frac{1}{\mathbb{L}}]\big)$ for one of the following collections of sets of prime numbers $\mathbb{L}$ and additional conditions on S:

- (A) *$\mathbb{L} = \{\ell\}$ for some prime number ℓ and S is of characteristic* 0*, or*
- (B) *$\mathbb{L} = \{\ell_1, \ell_2\}$ for all pairs of sufficiently large prime numbers ℓ_1, ℓ_2 and no additional conditions on S.*

For the proof and details on the matter of this report see [3].

References

[1] L. Moret-Bailly, *Un théorème de pureté pour les familles de courbes lisses*, C. R. Acad. Sci. Paris Sér. I Math. **300** (1985), no. 14, 489–492.

[2] F. Oort, *Subvarieties of moduli spaces*, Inventiones Math. **24** (1974), 95–119.

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[4] A. Tamagawa, *The Grothendieck conjecture for affine curves*, Compositio Math. **109** (1997), 135–194.