Birational Galois sections with local conditions for hyperbolic curves

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The section conjecture in anabelian geometry describes rational points of anabelian varieties in terms of profinite groups. We report on progress made in [Sx12].

1. The section conjecture of anabelian geometry

1.1. **The conjecture.** Let k be a number field and $\mathrm{Gal}_k = \mathrm{Gal}(\bar{k}/k)$ its absolute Galois group. A rational point $a \in X(k)$ of a geometrically connected variety X/k yields by functoriality a section

$$s_a : \mathrm{Gal}_k = \pi_1(\mathrm{Spec}(k)) \to \pi_1(X)$$

of the projection map $\mathrm{pr}_* : \pi_1(X) \to \mathrm{Gal}_k$, which, due to neglecting base points, is only well defined up to conjugation by elements of $\pi_1(X_{\bar{k}}) \subset \pi_1(X)$.

A section s is **cuspidal** if X has a smooth completion $X \subset \overline{X}$ and a k-rational point $a \in (\overline{X} \setminus X)(k)$ such that s factors over the corresponding decomposition subgroup $D_a \subset \pi_1(X)$, well defined up to $\pi_1(X_{\bar{k}})$-conjugacy.

Conjecture 1 (Grothendieck [Gr83])**.** *Let k be a number field and X/k a smooth, geometrically connected curve with non-abelian $\pi_1(X_{\bar{k}})$. Then the map $a \mapsto s_a$*

$$X(k) \to \mathscr{S}_{\pi_1(X/k)} = \{\text{sections } s : \mathrm{Gal}_k \to \pi_1(X) \text{ of } \mathrm{pr}_*\} / \pi_1(X_{\bar{k}})\text{-}conjugacy$$

is a bijection onto the complement of the set of cuspidal sections $\mathscr{S}^{\mathrm{cusp}}_{\pi_1(X/k)}$.

2. Local conditions for Galois sections

2.1. **A hierarchy of sections.** We introduce local conditions on sections that are shared by sections s_a coming from rational points. For a place v of the number field k we denote by $k \hookrightarrow k_v$ its completion and consider $\mathrm{Gal}_{k_v} \subset \mathrm{Gal}_k$ as a subgroup by fixing a choice of a prolongation of v to $\bar{k}$.

A **Selmer** section is a section s that locally comes from a point, i.e., such that for all v we have $a_v \in \overline{X}(k_v)$ and

$$s|_{\mathrm{Gal}_{k_v}} = s_{a_v} : \mathrm{Gal}_{k_v} \to \pi_1(X_{k_v}) \subset \pi_1(X).$$

Since the map $a \mapsto s_a$ is injective over number fields as well as over local fields when X is a curve, we obtain a well defined map: the **associated adele**

$$\underline{a} : \mathscr{S}^{\mathrm{Selmer}}_{\pi_1(X/k)} = \{s \in \mathscr{S}_{\pi_1(X/k)} \ ; \ \text{Selmer section}\} \to \overline{X}(\mathbb{A}_k)_{\bullet}$$

where $\mathbb{A}_k$ denotes the ring of k-adeles and $(-)_{\bullet}$ means that we have replaced the archimedian components by their connected components.

Let K be the function field of X/k. Then a **birationally liftable** section is a section s that lifts to a section $\tilde{s} : \mathrm{Gal}_k \to \mathrm{Gal}_K$ along the natural surjection $\mathrm{Gal}_K \twoheadrightarrow \pi_1(X)$. It follows from Koenigsmann's lemma [Ko05] §2.4, see also [Sx12] Prop. 1, that birationally liftable sections are Selmer sections.

An **adelic** section is a Selmer section $s : \mathrm{Gal}_k \to \pi_1(X)$ such that $\underline{a}(s) \in X(\mathbb{A}_k)_{\bullet}$ is even an adelic point of X. Finally, a **birationally adelic** section is a section

s that admits a lift $\tilde{s} : \mathrm{Gal}_k \to \mathrm{Gal}_K$ such that for all open $U \subset X$ the induced section

$$s_U : \mathrm{Gal}_k \xrightarrow{\tilde{s}} \mathrm{Gal}_K \twoheadrightarrow \pi_1(U)$$

yields an adelic or cuspidal section of U. Clearly we obtain a hierarchy of sections:

$$X(k) \amalg \mathscr{S}^{\mathrm{cusp}}_{\pi_1(X/k)} \subseteq \{s \in \mathscr{S}_{\pi_1(X/k)} \ ; \ \text{birationally adelic}\} \subseteq \mathscr{S}^{\mathrm{Selmer}}_{\pi_1(X/k)}.$$

2.2. **The support.** The **support** of a Selmer section $s : \mathrm{Gal}_k \to \pi_1(X)$ with adele $\underline{a}(s) = (a_v(s))_v$ is the Zariski closure

$$Z(s) = \overline{\bigcup_v \mathrm{im}(a_v(s) : \mathrm{Spec}(k_v) \to \overline{X})} \subseteq \overline{X}.$$

We say that a Selmer section s has **finite support** if $Z(s)$ is finite over k. If $Z = Z(s)$ is finite and the genus of $\overline{X}$ is ≥ 1, then by Stoll [St07] Theorem 8.2,

$$Z(k) = \big\{(a_v) \in \overline{X}(\mathbb{A}_k)^{\mathrm{f-desc}}_\bullet \ ; \ a_v \in Z(k_v) \text{ for a set of places } v \text{ of density } 1\big\}$$

where $(-)^{\mathrm{f-desc}}$ means that we require (a_v) to survive all descent obstructions imposed by torsors over $\overline{X}$ under finite groups G/k.

Since for a Selmer section s the adele $\underline{a}(s)$ survives all finite descent obstructions, we conclude by the usual limit argument with the tower of all neighbourhoods of s, that the image of the map

$$X(k) \amalg \mathscr{S}^{\mathrm{cusp}}_{\pi_1(X/k)} \hookrightarrow \mathscr{S}^{\mathrm{Selmer}}_{\pi_1(X/k)}$$

consists precisely of the Selmer sections with finite support.

Theorem 2. *Let k be a totally real number field or an imaginary quadratic number field. Then we have*

$$X(k) \amalg \mathscr{S}^{\mathrm{cusp}}_{\pi_1(X/k)} = \{s \in \mathscr{S}_{\pi_1(X/k)} \ ; \ \textit{birationally adelic}\}$$

for X/k a smooth, geometrically connected curve with non-abelian $\pi_1(X_{\bar{k}})$.

Proof: It suffices to find an open $U \subseteq X$ and a quasi-finite map $f : U \to \mathbb{T}$ to a torus $\mathbb{T}$ such that all adelic sections $\mathrm{Gal}_k \to \pi_1(\mathbb{T})$ come from rational points, because if $\pi_1(f) \circ s = s_t$ for $t \in \mathbb{T}(k)$, then $Z(s) \subseteq f^{-1}(t)$ will be finite. If $k/\mathbb{Q}$ is imaginary quadratic, then $\mathbb{T} = \mathbb{G}_{\mathrm{m}}$ suffices since $\mathfrak{o}_k^*$ is finite. When k is totally real, we can use for $\mathbb{T}$ the norm 1 torus of a totally imaginary quadratic extension of k. For details we refer to [Sx12] §3+4. □

3. Almost compatible systems of ℓ-adic representations

3.1. **The representations.** Let $f : E \to X$ be a family of elliptic curves. Any section $s : \mathrm{Gal}_k \to \pi_1(X)$ leads to a system of ℓ-adic representations

$$\rho_s = (\rho_{s,\ell}) = \Big(\rho_{E/X,s,\ell} : \mathrm{Gal}_k \xrightarrow{s} \pi_1(X, \bar{x}) \xrightarrow{\rho_{E/X,\ell}} \mathrm{GL}_2(\mathbb{Z}_\ell)\Big)_\ell$$

where $\rho_{E/X,\ell}$ is the monodromy representation on the fibre $\mathrm{T}_\ell(E_{\bar{x}}) \cong \mathbb{Z}_\ell^2$ corresponding to $\mathrm{R}^1 f_* \mathbb{Z}_\ell(1)$. If s is a Selmer section, and E/X has bad semistable reduction, then

(i) $\det(\rho_s) = \varepsilon$ is the cyclotomic character,
(ii) for all finite places v of k the local representation $\rho_{s,\ell}|_{\mathrm{Gal}_{k_v}}$ for $v \nmid \ell$ has a semisimplification $\rho_{s,v,\ell}^{\mathrm{ss}}$ with

$$\det(\mathbf{1} - \mathrm{Frob}_v \cdot T | \rho_{s,v,\ell}^{\mathrm{ss}}) = 1 - a_v(\rho)T + N(v)T^2 \in \mathbb{Z}[T]$$

and the trace of Frobenius $a_v(\rho)$ is independent of ℓ,
(iii) moreover, the semisimplification $\rho_{s,v,\ell}^{\mathrm{ss}}$ has weight -1 or weights 0 and -2.

3.2. **Integrality.** Let $G_\ell \subseteq \mathrm{GL}_2(\mathbb{F}_\ell)$ be the image of $\rho_{s,\ell}$ mod ℓ, and let $M_\ell \subseteq G_\ell$ be the subset of elements with one eigenvalue ± 1. Then either $\rho_{s,\ell}$ is reducible for all $\ell \gg 0$ or otherwise $\#M_\ell / \#G_\ell \to 0$ when $\ell \to \infty$. Combining this argument with Chebotarev's density theorem and the description of monodromy of the Legendre family of elliptic curves allows to prove the following, see [Sx12] §5+6.

Theorem 3. *Let $s : \mathrm{Gal}_k \to \pi_1(X)$ be a birationally liftable section of a smooth, geometrically connected curve X/k with non-abelian $\pi_1(X_{\bar{k}})$ over a number field k and with smooth completion $\overline{X}$. Then the associated adele $\underline{a}(s) \in \overline{X}(\mathbb{A}_k)_\bullet$ is either integral for a set of places v of Dirichlet density 1, or the section s is cuspidal.*

Note that the statement on integrality in Theorem 3 is well defined although integrality depends on the chosen model of X over $\mathrm{Spec}(\mathfrak{o}_k)$.

3.3. **Modularity.** The following result, see [Sx12] §7, requires $k = \mathbb{Q}$ since it relies on the arithmetic of the Eisenstein quotient of the modular jacobians $J_0(\ell)$ and on Serre's modularity conjecture proven by Khare and Wintenberger.

Theorem 4. *Let $X/\mathbb{Q}$ be a smooth, geometrically connected curve with non-abelian $\pi_1(X_{\bar{\mathbb{Q}}})$. A section $s : \mathrm{Gal}_\mathbb{Q} \to \pi_1(X)$ comes from a rational point or is cuspidal, if and only if s is birationally liftable (to say $\tilde{s}$) and for every open $U \subseteq X$ and every family E/U of elliptic curves the associated family of ℓ-adic representations $\rho_{E/U,\tilde{s}}$ has one of the following properties:*

(i) ***Finite conductor:*** *There exists a finite set of places S independent of ℓ such that $\rho_{E/U,\tilde{s},\ell}$ is unramified outside ℓ and the places in S.*
(ii) ***Reducible:*** *There is a character $\delta : \mathrm{Gal}_\mathbb{Q} \to \{\pm 1\}$ such that for all ℓ we have an exact sequence $0 \to \delta\varepsilon \to \rho_{E/U,\tilde{s},\ell} \to \delta \to 0$, where ε is the ℓ-adic cyclotomic character.*

The two cases in Theorem 4 reflect the dichotomy of the section s being associated to a rational point $a \in U(k)$ or to being already cuspidal for U/k.

References

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