Anabelian geometry with étale homotopy types II

Jakob Stix

(joint work with Alexander Schmidt)

We report on the second of two consecutive talks. Therefore we keep the notation of [Sch18]. In particular X_{et} denotes the étale homotopy type of X and $\mathrm{Ho(pro\text{-}spc)} \downarrow k_{et}$ denotes the over-category of the homotopy category of pro-spaces over $\mathrm{Spec}(k)_{\mathrm{et}}$. The first result is a weakly anabelian statement:

Theorem 1 (Schmidt-Stix, [SS16])**.** *Let $k/\mathbb{Q}$ be a finitely generated field. Let X and Y be smooth geometrically connected varieties over k such that for both X and Y we have a locally closed embedding into a product of hyperbolic curves over k. Then there is a map*

$$r : \mathrm{Isom}(X_{\mathrm{et}}, Y_{\mathrm{et}}) \longrightarrow \mathrm{Isom}_k(X, Y)$$

such that

(i) *r is a retraction: $r(f_{\mathrm{et}}) = f$ for all isomorphisms $f : X \cong Y$ over k,*

(ii) *r is functorial: if Z is a further such variety and $\gamma : X_{\mathrm{et}} \cong Y_{\mathrm{et}}$ and $\delta : Y_{\mathrm{et}} \cong Z_{\mathrm{et}}$ are isomorphisms in the homotopy category over k_{et}, then $r(\delta\gamma) = r(\delta)r(\gamma)$,*

(iii) *for all dominant maps $f : Y \to C$ with C a hyperbolic curve over k, we have for all $\gamma : X_{\mathrm{et}} \cong Y_{\mathrm{et}}$*

$$f_{\mathrm{et}}\gamma = f_{\mathrm{et}} r(\gamma)_{\mathrm{et}}.$$

Note that property (iii) determines the retraction r uniquely, and that (i) and (ii) follow at once from (iii).

Corollary 2. *Let X and Y be as in Theorem 1. Then X and Y are isomorphic as varieties over k if and only if $X_{\mathrm{et}} \cong Y_{\mathrm{et}}$ as pro-spaces over k_{et} up to homotopy.*

The proof of Theorem 1 proceeds in several steps. Let $\gamma : X_{\mathrm{et}} \cong Y_{\mathrm{et}}$ be an isomorphism. We first embed $Y \hookrightarrow W := \prod C_i$ in a product of hyperbolic curves such that each projection $\mathrm{pr}_i : Y \to C_i$ is dominant. Applying Mochizuki's anabelian theorem to $\mathrm{pr}_i\gamma$ in the form of Theorem 5 of [Sch18] we obtain a map $f : X \to W$. Spreading out over a scheme S of finite type over $\mathrm{Spec}(\mathbb{Z})$ as $\mathcal{Y} \hookrightarrow \mathcal{W}$ and $f : \mathcal{X} \to \mathcal{W}$ we must show that $\mathcal{Y} \times_{\mathcal{W}} \mathcal{X} \to \mathcal{X}$ is surjective on $\mathbb{F}_q$-points for sufficiently many q. This follows by applying a technique pioneered by Tamagawa in [Ta97]: auxilliary étale covers of $\mathcal{W}$ separate rational points, and the existence of rational points can be computed by the Lefschetz trace formula (a special *harpoon in the body of the whale of algebraic geometry* as mentioned several times during the workshop). Here we use crucially that the étale homotopy type of the special fibre of $\mathcal{X} \to S$ determines the étale cohomology also of finite étale covers as a Galois representation, hence the isomorphism γ induces an isomorphism with the corresponding Galois representation for $\mathcal{Y} \to S$. The factorization of f yields the desired map $r(\gamma) : X \to Y$.

A refined version based on Chebotarev's theorem yields some modest control about the extent to which the retraction r might actually be an inverse.

Theorem 3 (Schmidt-Stix, [SS16])**.** *Let X be as in Theorem 1, and let γ be an automorphism of X_{et} as pro-spaces over k_{et} up to homotopy with $r(\gamma) = \mathrm{id}_X$. Then (upon choosing base points and lifting to a pointed homotopy, see* [Sch18]*)*

$$\varphi = \pi_1(\gamma)$$

is a class-preserving automorphism of $\pi_1(X)$, i.e., $\varphi(\sigma)$ is conjugate to σ for all $\sigma \in \pi_1(X)$.

Recall that a *good Artin neighbourhood* is a smooth variety X that admits the structure of an iterated fibration

$$X = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = \mathrm{Spec}(k),$$

such that for all i the fibration $X_i \to X_{i-1}$ is an elementary fibration in hyperbolic curves. In characteristic 0, good Artin neighbourhoods are $K(\pi, 1)$-spaces. We define a *strongly hyperbolic Artin neighbourhood* to be a good Artin neighbourhood such that furthermore all X_i admit locally closed embeddings into a product of hyperbolic curves. Refining the classical argument, we obtain that for a variety over an infinite field any smooth point admits a Zariski open neighbourhood that is also a strongly hyperbolic Artin neighbourhood. Therefore the following strong anabelian statement establishes a positive answer to Conjecture 2 mentioned in [Sch18].

Theorem 4 (Schmidt-Stix, [SS16])**.** *For strongly hyperbolic Artin neighbourhoods X and Y the retraction r of Theorem 1 is a bijection.*

Note that Theorem 4 was obtained by Hoshi [Ho14] by different means and restricted to $\dim(X) \leq 4$.

The proof proceeds by induction on the dimension. The compatibility with a choice of a fibration structure follows from Theorem 3 above. The induction hypothesis simplifies the situation of the last fibration $X_n \to X_{n-1}$ to the extent that we have an induced map on homotopy types for the generic fibres. Here we are back in the case of hyperbolic curves and Mochizuki's theorem applies again in the form of Theorem 5 of [Sch18]. This shows that all homotopy equivalences are geometric and a posteriori that the retraction r is an inverse.

References

[Ho14] Y. Hoshi, *The Grothendieck conjecture for hyperbolic polycurves of lower dimension.* J. Math. Sci. Univ. Tokyo **21** (2014), no. 2, 153–219.

[Sch18] A. Schmidt, *Anabelian geometry with étale homotopy types I*, extended abstract of a talk at Oberwolfach April 18, 2018, this volume.

[SS16] A. Schmidt, J. Stix, *Anabelian geometry with étale homotopy types.* Annals of Math. **184** (2016), Issue 3, 817–868

[Ta97] A. Tamagawa, *The Grothendieck conjecture for affine curves.* Compos. Math. **109** (1997), no. 2, 135–194.