# A logarithmic view towards semistable reduction


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**Abstract**

A smooth, proper family of curves creates a monodromy action of the fundamental group of the base on the $\mathrm{H}^1$ of a fibre. The geometric condition of T. Saito for the action of the wild inertia of a boundary point to be trivial is transformed to the condition of logarithmic smooth reduction. The proof emphasizes methods and results from logarithmic geometry. It applies to quasi–projective smooth curves with étale boundary divisor.


## 1 Introduction

The potential geometry of the degenerate fibres of a proper, generically smooth curve $f : X \to S$ is very much controlled by the monodromy action on $\ell$-adic cohomology. More precisely, let $U \subset S$ be the smooth locus of $f$ and $\overline{u} \in U$ a geometric point. The sheaf $\mathrm{R}^1 f_{*,\text{ét}}\mathbb{Q}_\ell$ is an étale local system above $U$, hence corresponds to an action of $\pi_1(U, \overline{u})$ on the cohomology $\mathrm{H}^1\big(X_{\overline{u}}, \mathbb{Q}_\ell\big)$ of the fibre $X_{\overline{u}}$. Let $s$ be a point in the degeneration locus $S - U$ that is a normal point of codimension 1 in $S$, such that $X|_U$ does not extend as a proper smooth curve into $s \in S$. Then the monodromy action of the respective inertia subgroup $I_s \subset \pi_1(U, \overline{u})$, defined up to conjugation, is nontrivial but quasi-unipotent. It is unipotent if and only if $X|_U$ admits semistable reduction in $s$.

Nevertheless, trivial action of inertia corresponds to good reduction, i.e., smoothness. That raises the question about consequences for the geometry of the reduction if only the wild inertia subgroup acts trivially.

Let us introduce some notation that remain valid throughout the paper. We define a **trait** $S$ to be the spectrum of an excellent, strict henselian, discrete valuation ring $R$ with perfect residue field $k$, uniformizer $\pi$ and field of fractions $K$. We will denote the generic point by $\eta$ and the closed/special point by $s$. We fix geometric points $\overline{\eta}, \overline{s}$ above $\eta$ and $s$, such that — with $\eta^t$ being the maximal tamely ramified subextension of $\overline{\eta}/\eta$ — the normalization $S^t$ of $S$ in $\eta^t$ has $\overline{s}$ as its closed point. These data fix an **inertia subgroup** $I_s < G_K = \mathrm{Gal}(\overline{K}/K)$ in the absolute Galois group of $K$. In case the residue characteristic $\mathrm{char}(k) = p$ is positive, there is also the **wild inertia subgroup** $P \subset I_s$ which is the $p$-Sylow subgroup of $I_s$. When $\mathrm{char}(k)$ equals 0 we set $P = 1$.


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A **proper curve** $X/S$ is a flat, proper map $X \to S$ of finite presentation and pure relative dimension 1 with geometrically connected fibres.

On the above question the following answer exists.

**Theorem 1.1 (Saito, [Sai87, Theorem 3]).** *Let $S$ be a trait with algebraically closed residue field and residue characteristic $p > 0$. Let $X/S$ be a proper curve of genus $\geq 2$ with smooth generic fibre $X_\eta$ that is minimal with respect to $X$ being regular and the reduced special fibre $X_{s,\mathrm{red}}$ being a normal crossing divisor on $X$. Then the following are equivalent:*

*(a) $P$ acts trivially on $\mathrm{H}^1_{\text{ét}}\big(X_{\overline{\eta}}, \mathbb{Q}_\ell\big)$ for some prime number $\ell \neq p$.*

*(b) Any component of the special fibre $X_s$ with multiplicity divisible by $p$ is isomorphic to $\mathbb{P}^1_k$ and intersects the rest of the special fibre in exactly two points lying on components of multiplicity prime to $p$.*

The first impetus for writing this paper was to stress that Saito's theorem is actually about good reduction — but in logarithmic geometry. For an introduction to logarithmic geometry see [Il02, Ka89].

The trait $S$ admits a **canonical** fs log structure: $M_S = \mathcal{O}_S - \{0\} \subset \mathcal{O}_S$. Logarithmic smoothness over $S$ generalizes semistable maps, i.e., étale locally isomorphic to $\operatorname{Spec} R[t_1, \ldots, t_n]/t_1 \cdot \ldots \cdot t_r - \pi$ over $S$.

In parallel to the classical situation, trivial action of wild inertia corresponds to an action of the fundamental group of the base $\pi_1^{\log}(S, \overline{\eta}) \cong G_K/P$. We propose to prove the following theorem.

**Theorem 1.2.** *Let $S$ be a trait with canonical log structure and residue characteristic $p \geq 0$. Let $X_\eta$ be a proper, log smooth curve over the generic point $\eta \in S$ with negative Euler characteristic $\chi = \sum (-1)^q \dim \mathrm{H}^q_{\text{két}}\big(X_{\overline{\eta}}, \mathbb{F}_\ell\big)$. Then the following are equivalent:*

*(c) $P$ acts trivially on $\mathrm{H}^1_{\text{két}}\big(X_{\overline{\eta}}, \mathbb{F}_\ell\big)$ for some prime number $\ell \neq p$.*

*(d) $X_\eta$ has good reduction over $S$ as a log scheme, i.e., there is a proper, log smooth $X/S$ such that the generic fibre is isomorphic to $X_\eta$.*

*Addendum: (1) Moreover, if the above conditions are satisfied, then the minimal proper, regular model $X/S$ of $X_\eta$ with respect to the reduced special fibre plus the locus of non-trivial log structure in the generic fibre being a normal crossing divisor can be endowed uniquely with a fs log structure such that $X/S$ is log smooth.*

*(2) Finally, if there is log smooth reduction then there is a log smooth model (not necessarily regular) such that no component of the special fibre has multiplicity divisible by $p$. If $p = 0$ the divisibility condition is empty.*

We replaced the coefficients $\mathbb{Q}_\ell$ by $\mathbb{F}_\ell$ because the statement is ostensibly stronger. But actually both variants are equivalent due to group theory alone, see Corollary 3.2(1).

One should notice, however, that though we might have log smooth reduction we still can have nontrivial action of the wild inertia group on $\mathrm{H}^1\big(X_{\overline{\eta}}, \mathbb{Q}_p\big)$ for $p = \ell$. For example, look at an elliptic curve over a $p$-adic field $K$ with semistable reduction and Tate-model $\mathbb{C}_p^*/q^{\mathbb{Z}}$ such that $q$ is no $p$th power in $K$.

### A guide through the paper

When the generic fibre has trivial log structure, the fastest way to Theorem 1.2 is via §5. There one finds a quick proof relying on the theorem of semistable reduction for curves. However, §§2–6 supply a selfcontained proof based on logarithmic geometry, thus reproving the semistable reduction theorem. In §3 and §8 the connection between Theorem 1.1 and Theorem 1.2 is dealt with.

One purpose of writing these notes was to separate cohomology (§2), group theory (§3), logarithmic geometry (§4, §5, §8) and the combinatorial argument (§6). In particular, due to the use of logarithmic geometry, the author considers the combinatorial treatment as easier than in [Sai87] or [Ab00], though it is along the same lines.

### Thanks

I am grateful to Ivan Fesenko for inquiring about a logarithmic treatment of Saito's theorem. Thanks go to Moshe Jarden for an invitation to Tel Aviv and the excellent working conditions there, and to Isabelle Vidal for pointing out an inaccuracy. Special thanks I owe to Tina & Ken and of course Sabine for usage of their computers during my stay in KL.

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**Remark.** After the author finished this manuscript he was notified by Takeshi Saito of his related work [Sai04].

## Contents



## 2 Computation of cohomology

The basics of logarithmic geometry can be found in [Ka89], whereas the fundamentals of Kummer étale cohomology are contained in [Il02]. Of this section, only Theorem 2.3 is used lateron.

We basically encounter fs log structures of the following type. Let $D$ be a divisor on the normal noetherian scheme $X$ with complement $j : U \to X$. We denote by $M(\log D)$ the fs log structure $j_* \mathbb{G}_{m,U} \cap \mathcal{O}_X \hookrightarrow \mathcal{O}_X$ and call it 'induced by $D$'. Sometimes we need to distinguish between the log scheme $X$ and its underlying scheme $\overset{\circ}{X}$ and exploit the 'forget log' map $\varepsilon : X \to \overset{\circ}{X}$.

## 2.1 Log smooth curves

We first describe log smooth curves $X/K$ over a field $K$ with trivial log structure on $\operatorname{Spec} K$, i.e., $\overset{\circ}{X}$ is geometrically connected of pure dimension 1 and $X \to \operatorname{Spec} K$ is log smooth.

**Lemma 2.1.** *Let $X/K$ be a log smooth curve. Then the underlying scheme $\overset{\circ}{X}/K$ is a (classically) smooth curve and the log structure is induced by a divisor $D$ that is relatively étale over $K$.*

*Proof:* Let $x \in X$ be a point. By [Ka89, 3.13] the map $\Omega^1_{X/K} \otimes \kappa(x) \to \overline{M}^{\mathrm{gp}}_{X,x} \otimes \kappa(x)$ is surjective. Hence $\overline{M}^{\mathrm{gp}}_{X,x}$ is of rank $\leq 1$.

At points of the open set where the log structure is trivial the assertion follows from [Ka89, Prop 3.8]. On the other hand, if $\overline{M}^{\mathrm{gp}}_{X,x}$ is of rank 1, we may choose a generator $t$ of the log structure, such that $\Omega^1_{X,x} = \mathcal{O}_{X,x} \cdot d\log t$. By [Ka89, Prop 3.12] and [Ka89, Prop 3.8] the corresponding map $X \to \operatorname{Spec} K[\mathbb{N}]$ sending $1 \in \mathbb{N}$ to $t$ is étale near $x$ in the classical sense. That proves the lemma with $D = \{x \in X \mid \mathrm{rk}_{\mathbb{Z}} \overline{M}^{\mathrm{gp}}_{X,x} = 1\}$. □

## 2.2 Kummer étale cohomology

Let $X/K$ be a proper, log smooth curve with $M_X = M(\log D)$. Kummer étale covers of $X$ correspond to covers of $\overset{\circ}{X}$ that are at most tamely ramified at most over $D$, hence an isomorphism $\pi_1^{\log}(X) \cong \pi_1^{\mathrm{tame}}(X, D)$. The usual description of $\mathbb{Z}/\ell$-torsors shows

$$\mathrm{H}^1_{\mathrm{két}}\big(X, \mathbb{Z}/\ell\big) = \mathrm{Hom}\big(\pi_1^{\log}(X), \mathbb{Z}/\ell\big) \ . \tag{2.1}$$

Let now $K$ be algebraically closed, and let $\Lambda$ be a finite ring with $\#\Lambda$ invertible in $K$. For the computation of the Kummer étale cohomology of $X$ we exploit the spectral sequence associated to $\varepsilon : X_{\mathrm{két}} \to \overset{\circ}{X}_{\mathrm{ét}}$. Indeed, via the logarithmic Kummer sequence we deduce $\big(\Lambda^q \overline{M}^{\mathrm{gp}}_X\big) \otimes \Lambda(-q) \cong \mathrm{R}^q\varepsilon_*\Lambda$, see [Il02, 5.2], and thus the following 5-term exact sequence (all coefficients are constant $\Lambda$):

$$0 \to \mathrm{H}^1_{\mathrm{ét}}\big(X\big) \to \mathrm{H}^1_{\mathrm{két}}\big(X\big) \to \bigoplus_{x \in D} \overline{M}^{\mathrm{gp}}_{X,x} \otimes \Lambda(-1) \xrightarrow{\sum} \mathrm{H}^2_{\mathrm{ét}}\big(X\big) \to \mathrm{H}^2_{\mathrm{két}}\big(X\big) \to 0 \ . \tag{2.2}$$

We summarize the computation of $\mathrm{H}^*_{\mathrm{két}}\big(X, \Lambda\big)$ by the following:

**Proposition 2.2.** *All cohomology groups $\mathrm{H}^q_{\mathrm{két}}(X, \Lambda)$ are free $\Lambda$-modules of finite rank, the logarithmic Euler characteristic being*

$$\chi(X_{\mathrm{két}}, \Lambda) = \sum_q (-1)^q \, \mathrm{rk}_\Lambda \, \mathrm{H}^q_{\mathrm{két}}\big(X, \Lambda\big) = 2 - 2g - \deg D$$

*where $g$ is the genus of $\overset{\circ}{X}/K$ and $M_X = M(\log D)$.* □

## 2.3 Logarithmic Euler characteristic

We will need a combinatorial expression for the Euler characteristic of a degenerate fibre of a relative curve in terms of the intersection configuration. Saito obtained such an expression via inspection of the vanishing cycles sheaf. We will use Kummer étale cohomology instead.

For intersection theory in this context see for example [Li68]. In particular, we use intersection numbers $\big(C \bullet D\big)$ on a regular surface between arbitrary divisors $D$ via their associated line bundles or more generally line bundles itself and divisors $C$ whose support is proper over some fixed field.

Let $S$ be a trait with its canonical log structure $M(\log s)$. Let $X/S$ be a proper curve such that the generic fibre $X_\eta$ is log smooth, the underlying scheme $\overset{\circ}{X}$ is regular and the log structure $M_X$ is induced by a normal crossing divisor $Y_0 + H \subset X$ where the horizontal part $H$ is finite and generically étale over $S$, and the vertical part $Y_0 = \sum C$ is the divisor associated to the reduced special fibre $X_{s,\mathrm{red}}$. Here a divisor $D \subset X$ is called **normal crossing** if étale locally $D = \{t_1 \cdot \ldots \cdot t_r = 0\}$ and $t_1, \ldots, t_r \in \mathcal{O}_{X,x}$ is part of a regular parameter system, i.e., a regular system with $t_i$ linear independent in $\mathfrak{m}_x/\mathfrak{m}_x^2$.

Let $Y_1 = \sum m_C C$ be the divisor on $X$ where the sum is taken over all irreducible components of the special fibre $X_s$ and $m_C = \#\mathbb{Z}[\frac{1}{p}]/(r_C)$ is the prime-to-$p$ part of the multiplicity $r_C$ of $C$ in $X_s$ which as a divisor is $Y = \sum r_C C$.

**Theorem 2.3 (compare [Sai87, Prop 1.1]).** *Let $\Lambda$ be a finite ring such that $\#\Lambda$ is invertible in $\mathcal{O}_S$. The respective logarithmic Euler characteristic of the fibres is as follows.*

*(g)* $\chi\big((X_{\overline{\eta}})_{\mathrm{k\acute{e}t}}, \Lambda\big) = \sum (-1)^q \operatorname{rk}_\Lambda \mathrm{H}^q_{\mathrm{k\acute{e}t}}\big(X_{\overline{\eta}}, \Lambda\big) = -\big(Y \bullet Y_0 + H + \omega\big).$

*(s)* $\chi\big((X_{\tilde{s}})_{\mathrm{k\acute{e}t}}, \Lambda\big) = \sum (-1)^q \operatorname{rk}_\Lambda \mathrm{H}^q_{\mathrm{k\acute{e}t}}\big(X_{\tilde{s}}, \Lambda\big) = -\big(Y_1 \bullet Y_0 + H + \omega\big).$

*Here $\omega$ is the relative dualizing sheaf for $\overset{\circ}{X}/\overset{\circ}{S}$, and $X_{\tilde{s}}$ is the log geometric fibre in a log geometric point $\tilde{s}$ with center in the closed point $s$.*

*Proof:* (g) We apply Grothendieck duality relative $S$ to the coherent sheaf $\mathcal{O}_{X_s}$ on $X$. As $\omega$ is a line bundle we get the usual *Adjunction Formula*

$$2\chi\big((X_s)_{\mathrm{ZAR}}, \mathcal{O}_{X_s}\big) = -\big(Y \bullet Y + \omega\big)\ .$$

Moreover, by coherent Euler characteristic being constant in proper, flat families we have

$$2 - 2g = 2\chi\big((X_\eta)_{\mathrm{ZAR}}, \mathcal{O}_{X_\eta}\big) = 2\chi\big((X_s)_{\mathrm{ZAR}}, \mathcal{O}_{X_s}\big)\ ,$$

hence

$$2 - 2g = -\big(Y \bullet Y + \omega\big) = -\big(Y \bullet Y_0 + \omega\big)$$

as the divisor $Y$ is contained in the kernel of the intersection pairing on the special fibre. Furthermore, $\big(Y \bullet H\big) = \deg H/S$ and (g) is reduced to Proposition 2.2.

(s) We may assume that the residue field $k$ of the trait $S$ is algebraically closed. We have to describe the log geometric special fibre. Let $\tilde{s}$ be the log geometric point $\tilde{s} = \varinjlim_{p \nmid n} s_n$ where $s_n$ is $\operatorname{Spec} k$ with log structure $\frac{1}{n}\mathbb{N} \to k$ sending $\frac{1}{n}$ to 0 and the transition maps in the limit are the natural ones. Let $X_n = X_s \times_s^{\mathrm{fs}} s_n$, such that the log geometric fibre is $X_{\tilde{s}} = \varprojlim_{p \nmid n} X_n$.

The set of **double points** of $X_s$ is $\mathrm{DP}(X_s) = \{x \mid \operatorname{rk} \overline{M}^{\mathrm{gp}}_{X_s,x} = 2\}$. It consists of the singular locus of the normal crossing divisor $Y_0 + H$ and contains $H \cap Y_0$ (**tails**) and intersections of components of $Y_0$ (**actual double points**). We define for $n \in \mathbb{N}$ the set of double points of $X_n$ as $\mathrm{DP}(X_n) = \{x \mid \operatorname{rk} \overline{M}^{\mathrm{gp}}_{X_n,x} = 2\}$.

**Lemma 2.4.** *Let $N = \operatorname{lcm}\{m_C\}$ be the prime-to-p part of the least common multiple of the multiplicities of components of the special fibre.*

*(1) The natural maps $\mathring{X}_{\tilde{s}} \to \mathring{X}_{nN} \to \mathring{X}_N$ are bijective closed immersions.*

*(2) The natural map $\mathrm{pr} : \mathring{X}_{N,\mathrm{red}} \to \mathring{X}_{s,\mathrm{red}}$ is a finite ramified cover at most tamely ramified over the double points. For an irreducible component $C$ of $X_{s,\mathrm{red}}$ we have $\sum_{C' \mapsto C} \deg C'/C = m_C$ where the sum is taken over components of $\mathring{X}_{N,\mathrm{red}}$ mapping to $C$.*

*(3) At $x \in C \cap D$ for irreducible components $C, D$ of $X_s$ the ramification index for $x' \in C' \mapsto x \in C$ is $m_C/\gcd(m_C, m_D)$.*

*Proof:* For (1) see [Vi02, Section I.3]. Up to taking the strict reduced subscheme, the map $s_N \to s$ is a Kummer étale cover. The same holds for its fs-base change. Thus the ramification of $\mathring{X}_{N,\mathrm{red}} \to \mathring{X}_{s,\mathrm{red}}$ is at most tame over the double points. At a regular point $x$ of a component $C$ the degree (as map of schemes) may be calculated as the cardinality of prime-to-$p$ torsion in

$$\overline{M}^{\mathrm{gp}}_{X_s,x} \oplus_{\overline{M}^{\mathrm{gp}}_s} \overline{M}^{\mathrm{gp}}_{s_N} \cong \frac{1}{r_C}\mathbb{Z} \oplus_{\mathbb{Z}} \frac{1}{N}\mathbb{Z} \cong \mathbb{Z}/m_C \oplus \frac{1}{\operatorname{lcm}\{r_C, N\}}\mathbb{Z}\,.$$

This proves (2). A more careful étale local analysis at the double points shows (3) which we won't use in the sequel. □

To compute the Euler characteristic of $\mathrm{H}^*_{\text{két}}(X_{\tilde{s}}, \Lambda) = \varinjlim_{p\nmid n} \mathrm{H}^*_{\text{két}}(X_n, \Lambda)$ we use the Leray spectral sequences for the 'forget log' maps $\varepsilon_n : X_n \to \mathring{X}_n$

$$\mathrm{H}^p_{\text{ét}}(\mathring{X}_n, \mathrm{R}^q\varepsilon_{n,*}\Lambda) \Longrightarrow \mathrm{H}^{p+q}_{\text{két}}(X_n, \Lambda) \tag{2.3}$$

which is compatible with the direct limit in $n$. For simplicity we assume that all $n$ are divisible by $N$ (as in the Lemma 2.4) and therefore all higher direct images live on the same étale site $(\mathring{X}_N)_{\text{ét}}$, provided the obvious identifications are made.

**Lemma 2.5.** *The limits of the higher direct images are as follows:*

*(1)* $\varinjlim_{p\nmid n} \varepsilon_{nN,*}\Lambda = \Lambda$,

*(2)* $\varinjlim_{p\nmid n} \mathrm{R}^1\varepsilon_{nN,*}\Lambda = \varinjlim_{p\nmid n} \overline{M}^{\mathrm{gp}}_{X_{nN}} \otimes \Lambda(-1) \cong \bigoplus_{x \in \mathrm{DP}(X_N)} i_{x,*}\Lambda(-1)$,

*(3)* $\varinjlim_{p\nmid n} \mathrm{R}^2\varepsilon_{nN,*}\Lambda = \varinjlim_{p\nmid n} \Lambda^2\overline{M}^{\mathrm{gp}}_{X_{nN}} \otimes \Lambda(-1) = (0)$.

*Proof:* It suffices to prove (2). Furthermore, by [Il02, Thm 5.2], we need only to calculate $\varinjlim_{p\nmid n} \overline{M}^{\mathrm{gp}}_{X_{nN}}$. At points $x$ where $\overline{M}^{\mathrm{gp}}_{X_N}$ is of rank 1 we have

$$\varinjlim_{p\nmid n} \overline{M}^{\mathrm{gp}}_{X_{nN},x} = \varinjlim_{p\nmid n} \big(\frac{1}{r_C}\mathbb{Z} \oplus_{\mathbb{Z}} \frac{1}{nN}\mathbb{Z}\big)/\mathrm{tors} = \varinjlim_{p\nmid n} \frac{1}{\mathrm{lcm}(r_C, nN)}\mathbb{Z} = \mathbb{Z}_{(p)}.$$

At a double points $x \in \mathrm{DP}(X_N)$ we have

$$\varinjlim_{p\nmid n} \overline{M}^{\mathrm{gp}}_{X_{nN},x} = \varinjlim_{p\nmid n} \Big( \big(\frac{1}{r_C}\mathbb{Z} \oplus \frac{1}{r_D}\mathbb{Z}\big) \oplus_{\mathrm{diag},\mathbb{Z}} \frac{1}{nN}\mathbb{Z} \Big)/\mathrm{tors} = \mathbb{Z} \oplus \mathbb{Z}_{(p)}.$$

Here $\oplus_{\mathbb{Z}}$ is the coproduct over $\mathbb{Z}$ in abelian groups, and $\mathbb{Z}_{(p)}$ is the localization at the prime ideal $(p)$. ☐

Thus the only non-vanishing $E_2$-terms in (2.3) in the limit over $p \nmid n$ are $E_2^{p0} = \mathrm{H}^p_{\mathrm{\acute{e}t}}\big(\overset{\circ}{X}_N, \Lambda\big)$ and $E_2^{01} = \bigoplus_{x \in \mathrm{DP}(X_N)} \Lambda(-1)$.

Let $U' = C' - \mathrm{DP}(X_N)$ be the complement of the set of double points in an irreducible component $C'$ of $X_N$. Let $U$ be the respective image of $U'$ under $\mathrm{pr} : X_N \to X_s$. The multiplicativity of the Euler characteristic in tame extensions (Hurwitz formula) the above yields the following:

$$\begin{aligned} \chi\big((X_{\tilde{s}})_{\mathrm{k\acute{e}t}}, \Lambda\big) &= \chi\big((\overset{\circ}{X}_N)_{\mathrm{\acute{e}t}}, \Lambda\big) - \#\mathrm{DP}(X_N) \\ &= \sum_{C'} \chi(U'_{\mathrm{\acute{e}t}}, \Lambda) = \sum_{C} \sum_{C' \mapsto C} \deg(C'/C) \cdot \chi(U_{\mathrm{\acute{e}t}}, \Lambda) \\ &= \sum_{C} -m_C \cdot \big(C \bullet Y_0 + H + \omega\big) = -\big(Y_1 \bullet Y_0 + H + \omega\big) \ . \end{aligned}$$

This proves (s). ☐

## 3 $p$-groups acting on $\ell$-groups

This section exploits only elementary group theory. We denote the maximal pro-$\ell$ quotient of a pro-finite group $G$ by $G^\ell$.

Let $L$ be a pro-$\ell$ group. Any action on $L$ induces an action on the maximal abelian $\ell$-elementary quotient $L^{\mathrm{ab}}/\ell$, which is an $\mathbb{F}_\ell$ vector space.

**Lemma 3.1.** *A continuous action of a pro-p group on a finitely generated pro-$\ell$ group $L$ factors through a finite quotient that maps isomorphically onto a subgroup of* $\mathrm{GL}(L^{\mathrm{ab}}/\ell)$.

*Proof:* This follows from the case of finite groups [Ha59, Thm 12.2.2]. ☐

**Corollary 3.2.** *(1) The monodromy action of the wild inertia in Theorem 1.1 factors through a finite quotient.*

*(2) Condition (c) of Theorem 1.2 is equivalent to the following:*

*(c') the restriction to $P$ of the natural $\ell$-adic exterior action*

$G_K \to \mathrm{Out}\big(\pi_1^{\log}(X_{\overline{\eta}})^\ell\big)$ *is trivial for some $p \neq \ell$.*

*Furthermore, if we are in the situation of Theorem 1.1, condition (a) is equivalent to (c) and (c').*

*Proof:* (1) The action of $P$ on $\mathrm{H}^1_{\text{két}}(X_{\overline{\eta}}, \mathbb{Q}_\ell)$ comes by scalar extension from an action on $\mathrm{H}^1_{\text{két}}(X_{\overline{\eta}}, \mathbb{Z}_\ell)$ which is a finitely generated pro-$\ell$ group with $\mathrm{H}^1_{\text{két}}(X_{\overline{\eta}}, \mathbb{F}_\ell)$ as maximal $\ell$-elementary abelian quotient. (We use $\mathrm{H}^*_{\text{két}}$, which in case of trivial log structure as in Theorem 1.1 coincides with $\mathrm{H}^*_{\text{ét}}$, because the argument also applies to the situation of Theorem 1.2.)

(2) The group $\mathrm{H}^1_{\text{két}}(X_{\overline{\eta}}, \mathbb{F}_\ell)$ is dual to the maximal $\ell$-elementary abelian quotient of $\pi_1^{\log}(X_{\overline{\eta}})^\ell$. Using a $p$-Sylow subgroup of $\mathrm{Aut}\big(\pi_1^{\log}(X_{\overline{\eta}})^\ell\big)$ we lift the exterior action of $P$ to a true action without changing the image. □

**Lemma 3.3.** *Let $\rho : P \to \mathrm{Out}(L)$ be the exterior action of a pro-p group $P$ on a pro-$\ell$ group $L$ constructed from an extension $1 \to L \to G \to P \to 1$.*

*Then $\rho$ is trivial if and only if $L$ is isomorphic to $G^\ell$ via the canonical map $L \to G^\ell$.*

*Proof:* Use a $p$-Sylow subgroup of $G$ to split the sequence (from the right). $L$ is isomorphic to $G^\ell$ if and only if the sequence also admits a retraction $G \to L$ if and only if the action $\rho$ is trivial. □

**Corollary 3.4.** *Condition (c') above is equivalent to the natural map $\pi_1^{\log}(X_{\overline{\eta}})^\ell \to \pi_1^{\log}(X_{\eta^t})^\ell$ inducing an isomorphism.*

*Proof:* The exterior action of $P$ comes from the natural short exact sequence

$$1 \to \pi_1^{\log}(X_{\overline{\eta}})^\ell \to \pi_1^{\log}(X_{\eta^t}) / \ker\left(\pi_1^{\log}(X_{\overline{\eta}}) \to \pi_1^{\log}(X_{\overline{\eta}})^\ell\right) \to P \to 1 \ .$$

□

# 4 Specialization and log smoothness

## 4.1 The 'easy direction'

In this section we use results about the logarithmic specialization map of logarithmic fundamental groups to prove the 'easy direction' of Theorem 1.2, that is '(d) implies (c)'.

*Proof:* (Vidal) (d) implies (c). Let $X/S$ be a log smooth, proper curve. By [Vi02, Thm I.2.2] the logarithmic specialization map induces an isomorphism

$$\pi_1^{\log}(X_{\overline{\eta}})^\ell \to \pi_1^{\log}(X_{\tilde{s}})^\ell$$

on pro-$\ell$ completions. Hence the $G_K$-action factors through $\pi_1^{\log}(S) \cong G_K/P$. Thus condition (c') is satisfied, and we are done by Corollary 3.2. □

*Remark* 4.1. Alternatively, we may also argue with the vanishing of the sheaf of log vanishing cycles for a log smooth, proper $X/S$, see [Na98, Thm 3.2]. Indeed, the sheaf $\mathrm{R}\Psi^{\log}(\mathbb{Q}_\ell)$ of nearby cycles being quasi-isomorphic to $\mathbb{Q}_\ell$ in this case, the spectral sequence of nearby cycles

$$E_2^{p,q} = \mathrm{H}^p_{\text{két}}\big(X_{\tilde{s}}, \mathrm{R}^q\Psi^{\log}(\mathbb{Q}_\ell)\big) \Longrightarrow \mathrm{H}^{p+q}_{\text{két}}(X_{\overline{\eta}}, \mathbb{Q}_\ell)$$

degenerates. Hence the Galois action on $\mathrm{H}^p_{\text{két}}(X_{\overline{\eta}}, \mathbb{Q}_\ell) \cong \mathrm{H}^p_{\text{két}}(X_{\tilde{s}}, \mathbb{Q}_\ell)$ factors through $\pi_1^{\log}(S) \cong G_K/P$.

### 4.2 Purity

In fact, the basic ingredient [Vi02, Thm I.2.2] of the proof above, like its classical counterpart, requires a logarithmic version of deformation and algebraization together with a purity assertion. These also yield the following proposition.

**Proposition 4.2.** *Let $X/S$ be a proper curve, $H \subset X$ a relative effective Cartier divisor. Assume that $X$ is regular and $X_{s,\mathrm{red}} + H$ is a normal crossing divisor on $X$. Endow $X$ with the log regular fs log structure $M(\log X_{s,\mathrm{red}} + H)$. Then the natural maps induce isomorphisms*

$$\pi_1^{\log}\big(X_{\eta^t}\big)^{\ell} \xrightarrow{\pi_1(j)} \pi_1^{\log}\big(X_{S^t}\big)^{\ell} \xleftarrow{\pi_1(i)} \pi_1^{\log}\big(X_{\bar{s}}\big)^{\ell} \ .$$

*Proof:* A close inspection of I. Vidal's proof of [Vi02, Thm I.2.2] shows that log regular instead of log smooth over $S$ is sufficient for the deformation and algebraization part. Thus $\pi_1(i)$ is an isomorphism. Again, log regularity is sufficient for the purity result of Fujiwara–Kato, cf. [Il02, Thm 7.6], and thus $\pi_1(j)$ is an isomorphism. □

## 5 Tame base extensions

The strategy for the proof of a theorem about good reduction is composed of three steps: (i) determine a good candidate to work with, (ii) analyze the good candidate's special fibre and come up with a cohomological condition that decides whether it is 'good', and (iii) find methods to finally enforce the cohomological condition.

### 5.1 The good candidate

In our case, the good candidate is provided by the theory of minimal models of surfaces equipped with a divisor. Let $S$ be a trait with its canonical log structure $M(\log s)$. As in Section 2.3, let $X/S$ be a proper curve such that the generic fibre $X_\eta$ is log smooth, the underlying scheme $\overset{\circ}{X}$ is regular and the log structure $M_X$ is induced by a normal crossing divisor $Y_0 + H \subset X$ where the horizontal part $H$ is finite and generically étale over $S$, and the vertical part $Y_0 = \sum C$ is the divisor associated to the reduced special fibre $X_{s,\mathrm{red}}$. Moreover, for $X/S$ to be our good candidate we ask $X/S$ to be relatively minimal subject to the above conditions.

### 5.2 The fs-base change

Let $S' \to S$ be a finite tame extension. When equipped with the canonical log structures it is finite log étale. We denote the fs log base change $X \times_S^{\mathrm{fs}} S'$ by $X'_{\mathrm{bc}}$. By [Ba95, Thm 3.3] we have:

$$X/S \ \text{ is log smooth} \quad \iff \quad X'_{\mathrm{bc}}/S' \ \text{ is log smooth.} \tag{5.1}$$

It remains to describe $X'_{\mathrm{bc}}$. But $X'_{\mathrm{bc}}$ is log regular itself being log étale over the log regular $X$. Thus $\overset{\circ}{X'_{\mathrm{bc}}}$ is normal [Ka94, Theorem 4.1] and $M_{X'_{\mathrm{bc}}} = M(\log (X'_{\mathrm{bc}})_{s,\mathrm{red}} + H')$ with $H'$ being the preimage of $H$ under the projection $X'_{\mathrm{bc}} \to X$, see [Ka94, Theorem 11.2]. So $X'_{\mathrm{bc}}$ is the normalization of $\overset{\circ}{X} \times_{\overset{\circ}{S}} \overset{\circ}{S'}$ and its singular set is contained in the set where the log structure is of rank 2. But $X'_{\mathrm{bc}}$ is not too far away from being the good candidate over $S'$. We will remedy the deficiencies subsequently.

### 5.3 Desingularisation by log blow-ups

The (classical) desingularisation of a log regular log scheme can be performed through subdividing the associated fans like with toric varieties, see [Ka94, Section 10]. In our case these fans are two-dimensional. Thus the subdivision is achieved by consecutive log blow-ups along ideals of $M_X$ generated by two elements. This is of importance as it implies that there is a log étale desingularisation map $X'_{\text{desing}} \to X'_{\text{bc}}$ such that the (reduced) fibres of positive dimension are chains of projective lines (log blow-ups are log étale). Moreover, $X'_{\text{desing}}$ is again log regular. On a log regular log scheme whose underlying scheme is regular, the log structure is induced by a normal crossing divisor.

### 5.4 Contraction of $(-1)$-curves

The underlying scheme of $X'_{\text{desing}}$ is regular but need not be relatively minimal. There might exist some $(-1)$-curves that we want to contract using Castelnuovo's Criterion. But only those $(-1)$-curves will be contracted that intersect the rest of the divisor inducing the log structure in at most two points. In fact, there are no hairs, i.e., $(-1)$-curves that intersect only once, for otherwise $X/S$ would not have been relatively minimal. On the other hand, the contraction is also a classical blow-up map whose structure is under full control. We easily see that it is the underlying map of a log blow-up. Therefore we obtain a log étale contraction map $X'_{\text{desing}} \to X'$ such that the (reduced) fibres of positive dimension are chains of projective lines and $X'$ is relatively minimal with respect to the usual requirements. So there emerges the good candidate $X'$ over $S'$. By applying the following Proposition 5.1 twice, we have:

$$X'_{\text{bc}}/S' \text{ is log smooth} \iff X'_{\text{desing}}/S' \text{ is log smooth,} \tag{5.2}$$
$$X'_{\text{desing}}/S' \text{ is log smooth} \iff X'/S' \text{ is log smooth.} \tag{5.3}$$

Combining (5.1) with (5.2), (5.3) we note that while working with our good candidate, we may perform finite tame extensions of our base trait $S$ without affecting log smoothness.

**Proposition 5.1.** *Let $S$ be a trait with its canonical log structure. Let $\sigma : X \to X^\diamond$ be a contraction of log regular, proper curves over $S$, such that the underlying scheme of $X$ is regular and $x^\diamond \in X^\diamond$ has* $\operatorname{rk} \overline{M}^{\text{gp}}_{X^\diamond,x^\diamond} = 2$.

*If $X/S$ is log smooth, then $X^\diamond/S$ is log smooth near $x^\diamond$ as well.*

For the proof we will need the following.

**Proposition 5.2.** *Let $S$ be a trait with its canonical log structure. Let $X/S$ be a log regular, proper curve over $S$ and $x \in X$ has* $\operatorname{rk} \overline{M}^{\text{gp}}_{X,x} = 2$*. Then the following are equivalent:*

*(a) $X/S$ is log smooth near $x$,*

*(b)* $\operatorname{coker}(\overline{M}^{\text{gp}}_{S,s} \to \overline{M}^{\text{gp}}_{X,x})$ *has no $p = \operatorname{char}(\kappa(x))$-torsion.*

*Proof:* If $X/S$ is log smooth near $x$, then

$$\Omega^1_{X/S} \otimes \kappa(x) \twoheadrightarrow \left(\overline{M}^{\text{gp}}_{X,x}/\overline{M}^{\text{gp}}_{S,s}\right) \otimes \kappa(x)$$

with one dimensional module of differentials. This proves (b).

On the other hand, let us assume (b). A simple argument using $\mathrm{Ext}^{\bullet}_{\mathbb{Z}}$ shows the existence of simultaneous charts for $X \to S$ based on $\mathbb{N} \to Q = \overline{M}_{X,x}$ sending $1 = \pi$ to $q$. The map $X/S$ factorizes étale locally as

$$\mathrm{Spec}\, \mathcal{O}^{\mathrm{sh}}_{X,x} \xrightarrow{j} \mathrm{Spec}\, R[Q]/(q = \pi) \xrightarrow{h} S.$$

Here $h$ is log smooth by assumtion (b) and [Ka89, Theorem 3.5], whereas $j$ is essentially étale being an isomorphism on completions after suitable finite étale extension of $R$. Indeed, both rings are normal of dimension 2 and the map on cotangent spaces is surjective: $Q - \{0\}$ generates the maximal ideal of $x \in X$ because $X$ is log regular with log structure of rank 2 at $x$. □

*Proof of Proposition 5.1:* Let $j : U \subset X$ (resp. $j^\diamond : U^\diamond \subset X^\diamond$) be the open set of trivial log structure for $X$ (resp. $X^\diamond$). Clearly $U = U^\diamond$. By [Ka94, Theorem 11.6] we have

$$\sigma_* M_X^{\mathrm{gp}} = \sigma_* j_* \mathbb{G}_{\mathrm{m},U} = j^\diamond_* \mathbb{G}_{\mathrm{m},U^\diamond} = M^{\mathrm{gp}}_{X^\diamond}.$$

By Zariski's Main Theorem and Connectedness Theorem ($\sigma_* \mathcal{O}_X = \mathcal{O}_{X^\diamond}$) the preimage of $x^\diamond$ is a connected Weil divisor $E$ with components $E_i, 1 \le i \le n$. If $n = 0$ there is nothing to prove. Let $E_{n+1}, \ldots, E_N$ be the other components of $X - U$ that meet $E$. Note that $N \ge 2$.

The valuation maps on $X$ lead to an exact sequence near $E$

$$1 \to \mathbb{G}_{\mathrm{m},X} \to M_X^{\mathrm{gp}} \to \bigoplus_{i=1}^{N} \mathbb{Z}_{E_i}.$$

Applying the left exact $\sigma_*$ yields $\overline{M}^{\mathrm{gp}}_{X^\diamond,x^\diamond} \subseteq \bigoplus_{i=1}^N \mathbb{Z} \cdot E_i$. Let $r_{E_i}$ be the multiplicity of $\pi$ along $E_i$. Then

$$\mathrm{coker}(\overline{M}^{\mathrm{gp}}_{S,s} \to \overline{M}^{\mathrm{gp}}_{X^\diamond,x^\diamond}) \subseteq \big(\bigoplus_{i=1}^{N} \mathbb{Z} \cdot E_i\big)/(r_{E_i})\mathbb{Z},$$

and thus all torsion is killed by $\gcd(r_{E_i})$. Proposition 5.2 shows that this gcd is prime to $p$. Hence we may deduce again from Proposition 5.2 that $X^\diamond/S$ is log smooth at $x^\diamond$. □

## 5.5 A shortcut

Using the theorem of semistable reduction we now get a quick proof of Theorem 1.2 in case $X_\eta$ has trivial log structure ($H = 0$).

*Proof:* (c) implies (d): The monodromy action of the inertia group on $\mathrm{H}^1_{\text{ét}}\big(X_{\overline{\eta}}, \mathbb{Q}_\ell\big)$ is known to be quasi-unipotent. As $P$ acts trivial by condition (c) the action is unipotent after a finite tame extension of $S$. The devissage argument above allows to assume that the monodromy action of the inertia group is already unipotent over $S$.

Then the good candidate $X/S$ is semistable by the theorem of semistable reduction in its precise form, cf. [Sai87, Thm 1]. Hence étale locally $X/S$ has the form $\mathrm{Spec}\, R[t]$ or $\mathrm{Spec}\, R[t_1, t_2]/(t_1 \cdot t_2 - \pi)$ and is therefore log smooth, cf. [Ka89, Ex 3.7(2)] or Prop. 5.2. So (d) holds for the good candidate. □

The argument above obscures that via logarithmic geometry one may actually reprove the theorem of semistable reduction. The subsequent section will achieve this.

# 6 Combinatorics within the special fibre

Let $X/S$ be the good candidate of Section 5.1. Let $\mathscr{B}$ be the set of irreducible components of the special fibre. As in Section 2.3 we introduce the divisors $Y_0 = \sum_{C\in\mathscr{B}} C$, $Y_1 = \sum_{C\in\mathscr{B}} m_C C$ and $Y = \sum_{C\in\mathscr{B}} r_C C$. Remember that $m_C$ was the prime-to-$p$ part of $r_C$, the multiplicity of $C$ in the special fibre $X_s$. As above, the relative dualizing sheaf of $\mathring{X}/\mathring{S}$ is denoted by $\omega$. Furthermore, we define the $\mathbb{Z}$-valued linear function $F(D) = \big(D\bullet Y_0 + H + \omega\big)$ on divisors $D$ with support in the special fibre.

**Theorem 6.1 (compare [Sai87, Section 2]).** *Let $X/S$ be a good candidate as above such that the horizontal part $H$ of the divisor inducing the log structure on $X$ is étale over $S$. If the Euler characteristic $\chi = -\frac{1}{2}F(Y)$ is negative, then*

$$F(Y) = F(Y_0) \qquad \text{if and only if} \qquad Y = Y_0 \ .$$

*Proof:* Clearly we assume that $F(Y) = F(Y_0)$ and have to show that all multiplicities $r_C$ equal 1.

*step 1.* If $Y = r_C \cdot C$ then the theorem follows from $F(Y) \neq 0$. We may therefore assume that $\#\mathscr{B} \geq 2$.

*step 2.* Now we locate the non-positive contributions to $F(Y)$.

**Lemma 6.2 ([Ab00, Lemma 2.4]).** *Let $\#\mathscr{B} \geq 2$ and $C \in \mathscr{B}$. Then $F(C) \leq 0$ occurs if and only if $\big(C \bullet C + \omega\big) = -2$ with $C$ of one of the following types:*

- *($\alpha$) $F(C) = -1$, $C$ intersects only one component $C'$ of $Y_0$ and avoids $H$. Moreover, $\big(C \bullet C'\big) = 1$ and $\big(C \bullet C\big)r_C + r_{C'} = 0$.*
- *($\beta$) $F(C) = 0$ and $C$ intersects only two components $C', C''$ of $Y_0 + H$. Moreover, $\big(C \bullet C'\big) = \big(C \bullet C''\big) = 1$ and $C'$ has support in $Y_0$. If $C''$ has support in $Y_0$ we have $\big(C \bullet C\big)r_C + r_{C'} + r_{C''} = 0$ while $\big(C \bullet C\big)r_C + r_{C'} = 0$ if $C''$ has support in $H$.*
- *($\gamma$) $F(C) = 0$, $C$ intersects only one component $C'$ of $Y_0$ and avoids $H$. Moreover, $\big(C \bullet C'\big) = 2$ and $\big(C \bullet C\big)r_C + 2r_{C'} = 0$.*

*Proof:* By the *Adjunction Formula* $\big(C\bullet C+\omega\big)$ equals $2g_C - 2$ where $g_C$ is the arithmetic genus of $C$. As $g_C \geq 0$ and $C$ has to intersect at least one other component of $Y_0$ by Zariski connectedness, we have

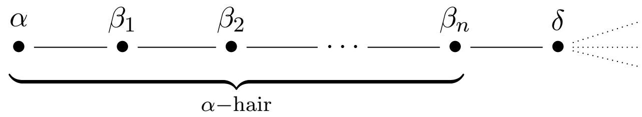
$$F(C) = \big(C \bullet C + \omega\big) + \sum_{C' \neq C} \big(C \bullet C'\big) + \big(C \bullet H\big) \geq -1$$

Using that $\big(C \bullet C + \omega\big)$ is even and $\#\mathscr{B} \geq 2$ one derives the assertion of the lemma. The equation for the multiplicities follows from $\big(C \bullet Y\big) = 0$. □

The curves dealt with in Lemma 6.2 will be called **of type** $(\alpha), (\beta)$ **or** $(\gamma)$ respectively or of **non-positive type**. Due to relative minimality, in all cases $(\alpha)$, $(\beta)$ or $(\gamma)$ the selfintersection is $\big(C \bullet C\big) \leq -2$.

*step 3.* To balance the negative contributions to $F(Y)$ we build clusters coming from the combinatorial structure of the dual graph of the special fibre.

First, not all components can be of non-positive type for otherwise $F(Y)$ is non-positive contradicting the assumption on the Euler characteristic. Secondly, we define an **$\alpha$-hair** as a maximal connected subgraph of the dual graph of $Y_0$ that contains one curve of type $(\alpha)$ and all other curves are of type $(\beta)$:




The other neighbour $\delta$ of $\beta_n$ is called the **neighbour of the $\alpha$-hair**. The neighour has support in $Y_0$ and is not of non-positive type, for otherwise all components of $Y_0$ were of non-positive type.

Let $\mathscr{B}_\alpha$ be the set of components that belong to some $\alpha$-hair, let $\mathscr{B}_\delta$ be the set of all neighbours of $\alpha$-hairs, and let $\mathscr{B}_\rho$ be the set of all other components. Then $\mathscr{B} = \mathscr{B}_\alpha \cup \mathscr{B}_\delta \cup \mathscr{B}_\rho$ yields a disjoint partition of the set of irreducible components of $Y_0$.

Let $\pi : \mathscr{B}_\alpha \to \mathscr{B}_\delta$ map a component to the neighbour of the $\alpha$-hair that it is part of. Furthermore, for $D \in \mathscr{B}_\delta$ let

$$\varphi(D) = (r_D - 1)F(D) + \sum_{\{C \in \mathscr{B}_\alpha \,|\, \pi(C) = D\}} (r_C - 1)F(C)$$

be the part in $F(Y) - F(Y_0)$ that is 'combinatorially related' to $D$.

**Lemma 6.3.** *Let $D$ be a neighbour. If $F(Y) > 0$ then $\varphi(D) > 0$.*

*Proof:* Let us write $R = Y_0 + H - D - \sum_{\{C \in \mathscr{B}_\alpha \,|\, \pi(C) = D\}} C$. Then

$$\varphi(D) = \frac{r_D - 1}{2}\Big(F(D) + \big(D \bullet D + \omega\big) + \big(D \bullet R\big)\Big) + \sum_{\substack{C \text{ of type } (\alpha), \\ \pi(C) = D}} \frac{r_D - 1}{2} - (r_C - 1)\ .$$

The following lemma shows that the second summand is positive. So if we assume that $\varphi(D) \leq 0$ then $F(D) + \big(D \bullet D + \omega\big) + \big(D \bullet R\big)$ has to be negative. But the minimal values are $F(D) = 1$, $\big(D \bullet D + \omega\big) = -2$, and $\big(D \bullet R\big) = 0$ which therefore must be attained. Hence the fibre consists only of $D$ together with at least three $\alpha$-hairs neighbouring $D$, for otherwise $D$ were of type $(\alpha)$ or $(\beta)$. Consequently,

$$F(Y) = \varphi(D) + F(Y_0) \leq F(Y_0) = F(D) - \#\{\alpha\text{-hairs}\} \leq -2$$

leads to a contradiction. $\square$

**Lemma 6.4.** *(1) For an $\alpha$-hair together with its neighbour the multiplicity $r_C$ is a convex function on the distance from the terminal curve of type $(\alpha)$.*

*(2) Let $r_\alpha$, resp. $r_\delta$, be the multiplicity of the terminal curve of type $(\alpha)$, resp. the neighbour, of the same $\alpha$-hair. The convex function of (1) is strict monotone increasing and in particular $r_\alpha \leq \frac{1}{2} r_\delta$.*

*Proof:* This follows from the relations between the multiplicities and the bound on the selfintersection numbers listed in Lemma 6.2. $\square$

*step 4.* We claim that under the assumptions of $F(Y) = F(Y_0)$ there is no curve of type $(\alpha)$. Indeed, we have only non-negative summands in

$$0 = F(Y) - F(Y_0) = \sum_{D \in \mathscr{B}_\delta} \phi(D) + \sum_{C \in \mathscr{B}_\rho} (r_C - 1)F(C)$$

forcing all summands to vanish. Hence all components of $Y$ which are neither of type $(\beta)$ nor $(\gamma)$ are reduced and there do not exist any $\alpha$-hairs by step 3.

*step 5.* It remains to deal with curves of type $(\beta)$ or $(\gamma)$. As not all components are of non-positive type the curves of type $(\beta)$ organize in maximal chains in the dual graph of $Y_0$ and no chain forms a closed circle. At both ends these chains terminate: either in a curve which is not of non-positive type and thus is reduced, or in a component $C$ of type $(\beta)$ which intersects $H$. Because $H/S$ is étale, a local equation for $C$ at $C \cap H$ is given by $\pi \cdot$unit and thus $r_C = 1$. An adaptation of Lemma 6.4 shows that the multiplicity, being a convex function on the position of the component in the chain, must be a constant equal to 1 along such a $\beta$-chain.

A curve $C$ of type $(\gamma)$ intersects with a curve $C'$ which is not of non-positive type. Thus $\big(C \bullet C\big) r_C + 2r_{C'} = 0$, and $r_{C'} = 1$ shows $r_C = 1$ for not to contradict relative minimality. This completes the proof of Theorem 6.1. □

# 7 Proof of the main theorem

We may now complete the proof of Theorem 1.2 apart from the special properties of the special fibres in the case of logarithmic smooth reduction. Those will be discussed in Section 8.

*Proof:* (c) implies (d). Let $X/S$ be the good candidate as in Section 5.1. By Corollary 3.2, Corollary 3.4 and Proposition 4.2, condition (c) implies that $\pi_1^{\log}(X_{\overline{\eta}})^{\ell} \cong \pi_1^{\log}(X_{\bar{s}})^{\ell}$ and via (2.1) thus $\chi\big((X_{\overline{\eta}})_{\text{két}}, \mathbb{Z}/\ell\big) = \chi\big((X_{\bar{s}})_{\text{két}}, \mathbb{Z}/\ell\big)$ for that particular prime $\ell \neq p$.

From the 5-term exact sequence (2.2) we obtain a Galois sequence

$$\mathrm{H}^1_{\text{két}}\big(X_{\overline{\eta}}, \mathbb{Z}_\ell\big) \to \mathbb{Z}_\ell(-1)[H(\overline{K})] \to \mathbb{Z}_\ell(-1)$$

so that trivial action of $P$ on $\mathrm{H}^1_{\text{két}}$ yields that $H(\overline{K}) = H(K^{\text{sep}})$ is $P$-stable. Thus $H/S$ is at most tamely ramified and $H$ splits after an apropriate finite tame extension of the base trait $S$.

The base change argument of Section 5 allows us to assume that $H/S$ is étale and that the support of $Y_1 - Y_0$ is contained in chains of projective lines, i.e., $\mathbb{P}^1$'s that intersect the rest of the fibre in two points. Indeed, by a suitable tame base change we kill the prime to $p$ part of the existing multiplicities. However, new components arise in the new good candidate: chains of $\mathbb{P}^1$'s. In particular $F(Y_1) = F(Y_0)$. By Theorem 2.3 we conclude further that

$$F(Y) = -\chi\big((X_{\overline{\eta}})_{\text{két}}, \mathbb{Z}/\ell\big) = -\chi\big((X_{\bar{s}})_{\text{két}}, \mathbb{Z}/\ell\big) = F(Y_1) = F(Y_0)\ .$$

The combinatorial reasoning of Theorem 6.1 shows that $Y = Y_0$ or that the special fibre is a reduced normal crossing divisor that meets the relative étale $H$ with normal crossing. It remains to argue that such an $X/S$ is actually log smooth.

At regular points of $Y_0 + H$ the map $X \to S$ is strict and even smooth in the classical sense ($k$ is perfect). At a double point $x$, we choose regular parameters $u, v$ such that $\{u \cdot v = 0\} = Y_0 + H$ locally at $x$. We may impose $uv^a = \pi$ with either $a = 1$ (actual double point) or $a = 0$ (tail). In both cases $X/S$ factors étale locally as

$$\operatorname{Spec} \mathcal{O}^{\text{sh}}_{X,x} \xrightarrow{j} W = \operatorname{Spec} R[u,v]/(uv^a = \pi) \xrightarrow{h} S$$

with log structure on $W$ induced by $u^{\mathbb{N}}v^{\mathbb{N}}$. The map $h$ is log smooth being the fs-base change of a map induced by $(1, a) : \mathbb{N} \to \mathbb{N}^2$. The map $j$ is essentially étale as can be checked on completions which are regular rings of dimension 2. Indeed, by the choice of $u, v$ the map $j$ is surjective on the Zariski cotangent space after scalar extension. (Compare Proposition 5.2) □

**Corollary 7.1 (semistable reduction of curves).** *Let $S$ be a trait. Any proper, smooth curve of genus $\geq 2$ over the generic point $\eta \in S$ admits a semistable model over some finite extension of $S$.*

*Proof:* By Corollary 3.2 the wild inertia action vanishes after some finite extension. Thus we are reduced to the above proof of (c) implies (d). □

This Corollary 7.1 is actually a powerful theorem of Deligne/Mumford, Grothendieck, Artin/Winters, ..., see [Ab00] for references.

# 8 Multiplicities divisible by $p$

In this section we address the addendum to Theorem 1.2 that describes the special fibre in the case of log smooth reduction. We emphasize geometric reasoning thereby providing a link between the conditions (b) and (d).

Let $X/S$ be log smooth such that the underlying scheme of $X$ is relatively minimal with respect to being regular and the divisor inducing the log structure being normal crossing. This was our good candidate in Section 5.1.

Let $E$ be a component of $X_s$ with multiplicity divisible by $p$. The reasoning of Sections 5 & 6 shows that $E$ is a projective line that intersects the rest of the special fibre in exactly two points. By Proposition 5.2 of two intersecting components of the special fibre, not both components have multiplicity divisible by $p$. This shows that (d) implies (b).

For the converse that (b) implies (d) we contract an annoying projective line $E$ with multiplicity divisible by $p$ and intersecting $C, D$ transversally. The resulting normal $S$-curve $X^\diamond/S$ has a rational singularity in $x^\diamond$, the image of $E$. With the help of M. Artin's work on rational singularities we determine the étale local structure: from the monoid $Q = \{(c, d, e) \in \mathbb{N}^3 \,|\, c + d + (E \bullet E)e \geq 0\}$ with $(r_C, r_D, r_E) =: q \in Q$ we form the log smooth $S$-scheme $\operatorname{Spec} R[Q]/\pi = q$ that is étale locally isomorphic at '$Q = 0$' to $X^\diamond$ in $x^\diamond$, see [Sx02, I.3.4.3]. In general, after the contraction the curve is not regular any more. Nevertheless, it is log smooth over $S$, see Proposition 5.1, thus proving the addendum of Theorem 1.2.

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