Projective Anabelian Curves in Positive Characteristic and Descent Theory for Log-Étale Covers

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Introduction

Anabelian Geometry and its motivation

The fundamental group of a connected variety X/K classifies finite étale covers of X. It is a pro-finite group $\pi_1(X, \overline{x})$ that depends on the choice of a base point $\overline{x} \in X$ up to an inner automorphism. Let $\overline{K}$ be a fixed algebraic closure of K, $G_K = \mathrm{Aut}(\overline{K}/K)$ its absolute Galois group, and assume that $X_{\overline{K}} = X \times_K \overline{K}$ is still connected. There is a natural short exact sequence, cf. IX 6.1 [SGA 1],

$$1 \to \pi_1(X_{\overline{K}}, \overline{x}) \to \pi_1(X, \overline{x}) \to G_K \to 1$$

that gives rise by conjugation to an outer Galois representation

$$\rho_X : G_K \to \mathrm{Out}\big(\pi_1(X_{\overline{K}})\big) \ .$$

Here Out means the group of automorphisms modulo inner automorphisms. Clearly the choice of a base point becomes irrelevant for the representation ρ_X.

Anabelian Geometry deals with the problem of deciphering geometric information about X/K from the group-theoretical properties of $\pi_1(X, \overline{x})$ or the associated representation ρ_X. One calls a variety anabelian if it belongs to a category of varieties Anab_K such that the restriction of the fundamental group functor π_1 to Anab_K yields an embedding into a category of purely group-theoretical objects. The term *anabelian* reflects the broadly accepted opinion that varieties belonging to Anab_K possess "very" non-abelian fundamental groups.

To illustrate the above let us draw an intuitive "conclusion". For an anabelian variety X/K its automorphism group $\mathrm{Aut}_K(X)$ should be mapped isomorphically by π_1 to the outer isomorphisms $\mathrm{Out}_{G_K}\big(\pi_1(X_{\overline{K}})\big)$ of its representation, i.e., the centralizer of the image $\rho_X(G_K)$ in $\mathrm{Out}\big(\pi_1(X_{\overline{K}})\big)$. Under some additional hypothesis $\mathrm{Aut}_K(X)$ is the group of K-rational points of an algebraic group, hence discrete in flavour, whereas $\mathrm{Out}_{G_K}\big(\pi_1(X_{\overline{K}})\big)$ is a pro-finite group and therefore compact. Consequently, X/K had better have a finite automorphism group.

There is an analogy in homotopy theory of CW-complexes. Let $K(\pi, 1)$ be an Eilenberg-MacLane space for the group π in dimension 1. It represents the following functor

$$\mathrm{Hom}\big(X, K(\pi, 1)\big) = \mathrm{Hom}\big(\pi_1^{\mathrm{top}}(X), \pi\big)$$

where we mean homomorphisms up to homotopy, respectively up to composition with inner automorphisms of π. So the group theory of the fundamental group controls the homotopy classes of maps into a $K(\pi, 1)$ space. One may characterise cohomologically the fact that a CW-space X is a $K(\pi, 1)$. For any $\pi_1^{\mathrm{top}}(X)$-module A we form the associated locally constant system $\mathscr{A}$ on X. Being a $K(\pi, 1)$ space is equivalent to the natural map

$$\mathrm{H}^*\big(\pi_1(X), A\big) \to \mathrm{H}^*\big(X, \mathscr{A}\big) \tag{0.0.1}$$

being an isomorphism. The étale analogue of condition (0.0.1) gives rise to the notion of an algebraic $K(\pi, 1)$ space analysed in Appendix A.

All the above applies to hyperbolic curves, i.e., smooth curves with negative Euler-characteristic. Their fundamental group is highly non-abelian, their automorphism group is finite, and their étale cohomology with locally constant finite coefficients coincides with the respective group cohomology of the fundamental group. Hence, it is very natural to conjecture hyperbolic curves to be anabelian.

A brief history

The notion of anabelian varieties dates back to Grothendieck. In 1983, within a letter to Faltings, he announced a list of conjectures in what we now call *Anabelian Geometry*, cf. [Gr83]. His "Yoga der anabelschen Geometrie" declares suitable varieties — including hyperbolic curves — over fields of absolutely finite type to be anabelian. For a survey see [Fa98] and [NTM01].

One distinguishes between three different kinds of Grothendieck Conjectures (GC) in Anabelian Geometry: the Hom-form, the Isom-form and the section conjecture. The Hom-form deals with categories of varieties with dominant maps, whereas the Isom-form only considers isomorphisms of the varieties in question. Hence, the Isom-form basically amounts to recover the isomorphism type of the variety from the isomorphism type of its fundamental group together with a discussion of automorphisms of the respective objects as above. The section conjecture is of a different nature in that it tries to compare K-rational points of a geometrically connected variety X/K with sections of the canonical surjective map $\pi_1(X) \twoheadrightarrow G_K$. We will focus on the Isom-form of GC.

As early as in 1969, there where glimpses of anabelian phenomena (in the Isom-form) in the case of absolute Galois groups of number fields in the work of Neukirch, Uchida, Iwasawa and others. The birational case for finitely generated fields of higher transcendence degree was finally settled by Pop in the 1990's. Only shortly afterwards Tamagawa came with a break through in [Ta97]. He was able to prove the Isom-form of GC for affine hyperbolic curves over finitely generated extensions of $\mathbb{Q}$. Even more important was his theorem that GC in the Isom-form holds for affine hyperbolic curves over finite fields in a modified absolute version if one deals with tame fundamental groups, see [Ta97]. This result is contrasted by the fact that the absolute Galois group of a finite field contains not so much arithmetic as previously supposed to be necessary for the ground field. In particular, finite fields themselves are not anabelian as their π_1 is always isomorphic to $\hat{\mathbb{Z}}$.

In [Mz96] Mochizuki removed the assumption that the hyperbolic curves need to be affine (still in characteristic 0). This was accomplished by a clever use of Tamagawa's finite field case. What is more, in [Mz99] Mochizuki even derived a Hom-form of GC for hyperbolic curves over sub-p-adic fields. There he used totally different methods from p-adic Hodge-theory. The last result hints at that there might be results of different flavours in Anabelian Geometry: ℓ-adic, p-adic Hodge, logarithmic, crystalline, etc.

Positive characteristic

Now we turn our attention to geometrically connected varieties over a field K of positive characteristic. First of all, an Isom-form of the Grothendieck conjecture can not be true. By topological invariance the geometric Frobenius F_K induces isomorphisms of fundamental groups of varieties that need not be isomorphic. There is a natural solution of how to deal with this problem: we formally invert geometric Frobenii. So anabelian geometry in positive characteristic deals with varieties in a category $\mathsf{Var}_{K,F_K^{-1}}$ where F_K has become an isomorphism. Let $\mathrm{Isom}_{K,F_K^{-1}}(\cdot,\cdot)$ denote the set of isomorphisms in $\mathsf{Var}_{K,F_K^{-1}}$. For the structure of automorphism groups in $\mathsf{Var}_{K,F_K^{-1}}$ see Appendix B. For *affine*, hyperbolic curves over a finitely generated extension of $\mathbb{F}_p$ the Isom-form (with geometric Frobenii inverted) was proven in the author's Diplomarbeit, cf. [Sx02], under the assumption that the curves in question are not isotrivial. That means that we ask the curves not to be defined over a finite field and even not after performing a finite extension of the field of constants.

The main result

The objective of this thesis consists of proving the Grothendieck Conjecture in Anabelian Geometry also for *projective*, hyperbolic curves *in positive characteristic* over finitely generated field extensions of $\mathbb{F}_p$. We have the following result:

Theorem 5.1.1. *Let K be a finitely generated field of characteristic $p \geq 0$ with algebraic closure $\overline{K}$. Let X, X' be smooth, hyperbolic, geometrically connected curves over K. Assume that at least one of them is not isotrivial. Then the functor π_1 induces a natural bijection*

$$\pi_1 : \mathrm{Isom}_{K,F_K^{-1}}\big(X,X'\big) \xrightarrow{\sim} \mathrm{Isom}_{G_K}\big(\pi_1(X_{\overline{K}}),\pi_1(X'_{\overline{K}})\big)$$

of finite sets.

The proof follows in characteristic 0 from Tamagawa's and Mochizuki's theorems, and in positive characteristic for affine, hyperbolic curves from the author's Diplomarbeit. Here we treat the case of projective, hyperbolic curves.

To accomplish this task, we try to imitate as many elementary geometric phenomena as possible in group-theoretical terms. For example, there is Galois descent for isomorphisms between varieties that governs the behaviour under constant field extensions. We can easily mimic this for the respective isomorphisms of exterior Galois representations (contrary to the respective question for $\pi_1(X)$). Consequently, it is not a problem to enlarge the field of constants. Secondly, we would like to restrict considerations to finite étale G-covers, i.e., étale G-torsors. Therefore we define the same notion for an exterior Galois representation. Thirdly, dominant maps between varieties are epimorphisms, and thus we want to show the same property for the respective maps between fundamental groups. We furthermore try to derive the essential group-theoretical properties exploited in the formal part of the proof of Theorem 5.1.1 from reasonable geometric assumptions. All this is contained in Chapter 4. The proof of Theorem 5.1.1 will be given in Chapters 5, 6 and 7.

Structure and strategy

This thesis is organised in two parts. Anabelian Geometry is the topic of the second part. Part 1 is of an auxillary nature, at least from the point of view of Anabelian Geometry. We will describe the interrelations of the two parts when we describe how to overcome the difficulties with the stategy for the proof of Theorem 5.1.1.

Formal considerations reduce the proof of Theorem 5.1.1 to the problem of constructing geometric isomorphisms from group-theoretical ones such that both coincide on cohomology.

The idea we are going to embark on is the following. If a *fine* moduli space $\mathscr{M}$ is available then isomorphisms of curves correspond to coincidence of representing maps $\xi : S \to \mathscr{M}$ from the base S of the curves to the moduli space $\mathscr{M}$. Hence we choose a suitable model S of finite type over $\mathbb{Z}$ of our base field K such that the curves in question have good reduction $\mathscr{X}$ over S. We are then left with the task of decoding ξ from the π_1 of the generic fibre.

By specialisation techniques we know the π_1 of the special fibres. We would like to use the GC for hyperbolic curves over finite fields to characterise the image $\xi(s)$ at a closed point $s \in S$. The problems with this strategy are threefold.

(1) The moduli space $\mathscr{M}_g$ of smooth, proper curves of given genus is a stack but not a scheme.

(2) The GC for hyperbolic curves over finite fields is only known for *affine*, hyperbolic curves.

(3) The GC for hyperbolic curves over finite fields is of an absolute flavour in that it only yields isomorphisms of curves up to Frobenius twist. This twist by Frobenius may a priori depend on the closed point $s \in S$.

We overcome these problems as follows. (1) We use level n structures on smooth projective curves the moduli space of which is an étale cover $\mathscr{M}_g[n]$ of $\mathscr{M}_g$ that is a scheme for $g \geq 1, n \geq 3$.

(2) The fact that the GC for hyperbolic curves over finite fields is available only for affine curves constitutes the main obstacle. We use degenerating G-Galois covers with stable but non smooth reduction above closed points and work G-equivariantly. One is naturally lead to prove a Van Kampen Theorem (Chapter 2) in logarithmic geometry (Chapter 3). Recall that the Kummer fundamental group π_1^{kum} of [Saï97] is an amalgam of the respective π_1^{tame} of the smooth parts of the components of the special fibre along the dual graph. These π_1^{tame} are precisely the groups we are interested in. However, [Saï97] does not work equivariantly with Galois action. We use the logarithmic fundamental group $\pi_1^{\log}$ for a certain fs-log structure as a replacement and thereby prove a vast generalisation of the amalgamation formula from [Saï97].

At a certain stage of the discussion of the $\pi_1^{\log}$ of the stable special fibre one needs to recover a filtration in ℓ-adic cohomology. We do this using Frobenius weights which forces us to work over a discrete valuation ring with a finite residue field. But possibly the recovering of the filtration can be performed by the action of an inertia generator on the respective logarithmic ℓ-adic cohomology. This amounts to profound knowledge of the monodromy and weight filtrations in the logarithmic context.

(3) We recover the isomorphism type of the special fibres of our family $\mathscr{X}/S$ up to Frobenius twist. That means that we only describe the topological part of the corresponding representing map $\xi : S \to \mathscr{M}$. The discrepancy between topological coincidence and coincidence as maps of schemes is dealt with in Chapter 1.

Part 1

The main results of Part 1 are as follows. In Chapter 1 we treat difficulty (3) of the above list by proving the following theorem (already contained in [Sx02]).

Theorem 1.2.1. *Let $S, \mathscr{M}$ be of finite type over $\mathrm{Spec}(\mathbb{F}_p)$, S be irreducible and reduced, and let $f, g : S \to \mathscr{M}$ be maps such that $f^{\mathrm{top}} = g^{\mathrm{top}}$, i.e., f and g coincide topologically.*

Then f and g differ only by a power of the Frobenius map. Uniqueness of the exponent is equivalent to $f^{\mathrm{top}} = g^{\mathrm{top}}$ not being constant.

In Chapter 2 we accomplish with an exercise from IX §5 [SGA 1]. We prove an abstract Van Kampen Theorem that describes how the π_1 of a space may be built from the respective π_1's of components of a covering in a good topology and the corresponding intersection configuration of these components.

Finally, logarithmic geometry in Chapter 3 became the second main topic of this thesis. Originally, it was only meant to serve for detecting π_1^{tame} of the smooth locus of irreducible components of a stable curve as in difficulty (2) above, but in a Galois equivariant way. Then it became a treatise on descent theory of logarithmic étale covers in almost full (log-)SGA 1 style. We prove the following descent result for log-étale covers.

Theorem 3.2.20 (log SGA1 IX 4.7). *Let $f : S' \to S$ be a finite, surjective and exact morphism of finite presentation between fs-log schemes. Then f satisfies universally effective descent for $\mathsf{Rev}^{\log}$.*

Logarithmic descent theory allows in particular to reprove that logarithmic blow-up maps induce isomorphisms on logarithmic fundamental groups. We were then lead to a criterion for log-smooth reduction of smooth proper curves as follows (only (c) $\Rightarrow$ (d) is new).

Theorem 3.4.8 (criterion for log-smooth reduction). *Let S be the spectrum of an excellent henselian discrete valuation ring with fraction field K and perfect residue field k of characteristic $p > 0$.*

Let X_K be a smooth proper curve over $\mathrm{Spec}(K)$ of genus $g \geq 2$, and let ℓ be a prime number different from p. Then the following are equivalent:

- *(a) The wild inertia group $P < G_K$ acts trivially on $\pi_1(X_{\overline{K}})^{\ell}$, i.e., the restriction of $\rho_{X_K} : G_K \to \mathrm{Out}\big(\pi_1(X_{\overline{K}})^{\ell}\big)$ to P is trivial.*
- *(b) P acts trivially on $\pi_1(X_{\overline{K}})^{\mathrm{ab},\ell}$.*
- *(c) The minimal regular model of X_K over S such that the reduced special fibre is a divisor with normal crossings satisfies the following property: any component of the special fibre with multiplicity divisible by p is isomorphic to $\mathbb{P}^1_k$ and meets the rest of the special fibre in exactly two points lying on components of multiplicity prime to p.*

(d) *X_K has log-smooth reduction over S, i.e., there is a model X/S such that $M(\log X_s)$ is a fs-log structure on X that turns it into a log-smooth fs-log scheme over S.*

As an application we find a criterion for logarithmic good reduction for covers of curves that is almost group-theoretical. An important insight of Chapter 6 says that the criterion is actually group-theoretical, at least in the situations we encounter in Part 2.

Theorem 3.4.14 (log good reduction of covers). *Let S be as above. Let X/S be a proper, generically smooth, but not smooth, semistable curve of genus $g \geq 2$ equipped with the canonical fs-log structure $M(\log X_s)$. Let $\mathrm{pr} : \pi_1(X_K) \to G_K$ be the canonical map, and let H be an open normal subgroup of $\pi_1(X_K)$. Then the following are equivalent:*

(a) *H contains the kernel of the log specialisation map*

$$\mathrm{sp}_{\log} : \pi_1^{\log}(X_K) \twoheadrightarrow \pi_1^{\log}(X_s) \ .$$

(b) *There exists a finite tamely ramified extension K'/K (corresponding to an extension of spectra of valuation rings $S' \to S$) with inertia group $I' = I \cap G_{K'}$ such that if we set $H' = H \cap \mathrm{pr}^{-1}(G_{K'})$ and $\overline{H}' = H' \cap \pi_1(X_{\overline{K}})$ the following holds:*

   (i) $\mathrm{pr}(H') = G_{K'}$,

   (ii) *I' acts unipotently on $\left(\overline{H}'\right)^{\mathrm{ab},\ell}$ for some prime number $\ell \neq p$,*

   (iii) *For a $G = \mathrm{Aut}(Y'_{K'}/X_{K'})$-equivariant semistable model Y'/S' of the covering space $Y'_{K'}$ over $X_{K'}$ associated to H', the stabilizers of the G-action on the set of geometric double points of Y' have orders prime to p.*

Instructions for the reader

The three chapters of Part 1 are independent and may be read in an arbitrary order. The reader who is primarily interested in the Grothendieck conjecture in Anabelian Geometry is even advised to skip Part 1 in a first reading and start immediately with Part 2. The two parts interfer only in Chapter 6 through the Van Kampen formula and certain logarithmic structures on stable curves. Furthermore, Chapter 7 exploits Theorem 1.2.1 on topological coincidence of maps.

For references to the literature we have adopted the following style. To point at a specific position we first give the section or theorem number according to the style of the respective source followed by the abbreviation of the latter in [ ] as used in the bibliography. These abbreviations contain mainly initials of last names of authors and the last two digits of the year of publication.

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Part I

Geometry

Chapter 1

Topological Coincidence of Maps

This chapter is motivated by the requirements of our strategy towards an Isom-version of Anabelian Geometry for curves. Theorem 1.2.1 yields a usefull globalisation statement: two maps between varieties over finite fields that differ pointwise by a power of Frobenius in fact globally coincide up to a power of Frobenius. The exponent that a priori depends on the point must be a global constant. This result will be a key ingredient for the proof of the main theorem of this thesis. Theorem 1.2.1 is already contained in the author's Diplomarbeit [Sx02]. We recall it here for the sake of completeness and to provide more details. Moreover, we prove a generalisation that, however, is not used in the sequel.

The proof of Theorem 1.2.1 requires the existence of sufficiently many irreducible curves on varieties — which is well known. We get these curves by Bertini arguments in Section 1.1.

1.1 Bertini revisited

This section deals with elementary geometric statements that are proved using Bertini–Lefschetz techniques. That means the dimension reduction due to an appropriatly general hypersurface section and the stability of certain properties under this process.

Let k be a field. The projective space $P = \mathbb{P}\big(\Gamma(\mathbb{P}^n_k, \mathcal{O}(d))\big)$ parametrises hypersurfaces of degree d in $\mathbb{P}^n_k$. Let

$$\mathscr{H} := \{\ (x,H) \in \mathbb{P}^n_k \times P \mid x \in H\ \} \subset \mathbb{P}^n_k \times P \tag{1.1.1}$$

be the universal family of degree d hyperplanes. Via the first projection $\mathscr{H}$ is a projective space bundle over $\mathbb{P}^n_k$.

Lemma 1.1.1. *Let $X \subset \mathbb{P}^n_k$ be a connected, normal, closed subvariety of* $\dim(X) \geq 2$. *For $d \gg 0$, the space of degree d hypersurfaces $H \subset \mathbb{P}^n_k$ such that $X \cap H$ is geometrically irreducible of dimension* $\dim(X) - 1$ *consists of a dense constructible subset $G \subset P$.*

Proof: The universal intersection $\mathscr{H} \cap X \times P$ is a closed subscheme $\mathrm{Int} \subset X \times P$ that is normal and connected being a projective space bundle over the normal and connected X. Its sheaf of ideals $\mathscr{I}_{\mathrm{Int}}$ is a linebundle because the map $\mathscr{I}_{\mathscr{H}} \otimes \mathcal{O}_{X\times P} \twoheadrightarrow \mathscr{I}_{\mathrm{Int}} \subset \mathcal{O}_{X\times P}$ of linebundles on the normal $X \times P$ is injective as it is not the zero map. Hence $\mathscr{I}_{\mathrm{Int}}$ is flat over P. The short exact sequence

$$0 \to \mathscr{I}_{\mathrm{Int}} \to \mathcal{O}_{X\times P} \to \mathcal{O}_{\mathrm{Int}} \to 0$$

yields after applying $\mathrm{R}^\bullet \mathrm{pr}_{2,*}$ the exact sequence

$$\mathcal{O}_P \to \mathrm{pr}_{2,*}\mathcal{O}_{\mathrm{Int}} \to \mathrm{R}^1 \mathrm{pr}_{2,*}\mathscr{I}_{\mathrm{Int}} \ ,$$

where by flat base change $\mathcal{O}_P \cong \mathrm{pr}_{2,*}\mathcal{O}_{X\times P}$. The fibrewise first cohomology of $\mathscr{I}_{\mathrm{Int}} \cong \mathscr{I}_{\mathscr{H}} \otimes \mathcal{O}_{X\times P}$ is $\mathrm{H}^1(X, \mathcal{O}(-d))$ which vanishes for $d \gg 0$ by III 9.7 [Ha77]. The theorem on Cohomology and Base Change then implies $\mathrm{R}^1 \mathrm{pr}_{2,*}\mathscr{I}_{\mathrm{Int}} = 0$ and hence $\mathcal{O}_P \cong \mathrm{pr}_{2,*}\mathcal{O}_{\mathrm{Int}}$ as all fibres are nonempty. Consequently, by the Zariski Connectedness Theorem the second projection $\mathrm{pr}_2 : \mathrm{Int} \to P$ has geometrically connected fibres for $d \gg 0$.

The space of good parameters $G \subset P$ consists of

$$\left\{ H \in P \mid \mathrm{pr}_2|_{\mathrm{Int}}^{-1}(H) = X \cap H \text{ is geom. irreducible of dimension } \dim(X) - 1 \right\}$$

which is constructible by 9.7.7 [EGA$_{\mathrm{IV}}$] and Chevalley's Theorems 1.8.4, 13.1.3 [EGA$_{\mathrm{IV}}$]. Being a localisation of a normal scheme the generic fibre of $\mathrm{pr}_2|_{\mathrm{Int}}$ is normal itself. As normal and geometrically connected imply geometrically irreducible and as furthermore X is not contained in all degree d hypersurfaces G contains the generic point. □

Lemma 1.1.2. *Let $Y \subset X \subset \mathbb{P}^n_k$ be closed subvarieties such that $\dim(Y) < \dim(X)$. For any d, the space of degree d hypersurfaces $H \subset \mathbb{P}^n_k$ such that $X \cap H$ is not contained in Y consists of a dense constructible subset of $G \subset P$.*

Proof: In the setting of (1.1.1) over P consider $\mathrm{Int}(Y) = (Y \times P) \cap \mathscr{H}$ and $\mathrm{Int}(X) = (X \times P) \cap \mathscr{H}$ as closed subsets of $X \times P$. The locus of degree d hypersurfaces such that $X \cap H$ is not contained in Y consists of $G = \mathrm{pr}_2\big(\mathrm{Int}(X) - \mathrm{Int}(Y)\big)$. Hence, G is constructible. Since generically the dimension drops by 1, we have generically $\dim(Y \cap H) < \dim(X \cap H)$, which shows that G is dense. □

The following lemma is certainly well known and should admit a more elementary proof.

Lemma 1.1.3. *Let X/k be irreducible, of finite type and of dimension $\dim(X) \geq 1$. Then any two closed points of X lie on an irreducible curve on X.*

Proof: Instead of being focused on closed points we prove the following. For every two locally closed nonempty subsets Z_1, Z_2 of X there is an irreducible curve on X that hits both of them. This assertion can be pulled back by a surjection $X' \to X$. Just consider the preimages of the Z_i. Or, we may enlarge X not changing the Z_i in case $X \subset X'$ is open.

By base extension (restrict to a dominating irreducible component of $X_{\bar{k}}$), the Chow Lemma, taking the reduced closure in a projective embedding, blow-up and normalisation we may therefore assume that X is connected, normal and projective, the Z_i are open parts of Divisors $\overline{Z_i}$, and k is algebraically closed.

Lemma 1.1.1 says that the intersection $X \cap H$ with a suitably generally chosen hypersurface H in a fixed embedding into $\mathbb{P}^n_k$ is again irreducible but of dimension $\dim(X) - 1$ (observe that as k is algebraically closed, the good locus of parameters G has k-rational points). By Lemma 1.1.2 applied to the boundary $\overline{Z_i} \setminus Z_i = \partial Z_i \subset \overline{Z_i}$ the intersection moreover still contains part of the Z_i. Finally, we may conclude by induction on $\dim(X)$ as the above reductions by Chow Lemma, blow-up and normalisation do not increase the dimension. The case of $\dim(X) = 1$ is trivial. □

There is no true restriction of the argument to the case of two points. Hence we immediately get the following corollary.

Corollary 1.1.4. *Let X/k be irreducible, of finite type and of dimension* $\dim(X) \geq 1$. *Then any finite set of closed points of X lies on an irreducible curve on X.* □

Furthermore, we may generically avoid a closed, properly contained subset.

Lemma 1.1.5. *Let X/k be irreducible, of finite type and of dimension* $\dim(X) \geq 1$. *Let $W \subset X$ be a subset that does not contain all closed points. Then any finite set of closed points of X lies on an irreducible curve on X which is not contained in W.*

Proof: The consideration of W only requires to take into account one further closed point of X from the complement of W and to apply Corollary 1.1.4. □

Corollary 1.1.6. *Let X/k be irreducible, of finite type and of dimension* $\dim(X) \geq 1$. *Let $f : X \to Y$ be a nonconstant map of varieties over k. Then any finite set M of closed points of X lies on an irreducible curve C such that the restriction $f|_C : C \to Y$ is also nonconstant.*

Proof: Use Lemma 1.1.5 with $W = f^{-1}\big(f(x)\big)$ for one point $x \in M$. □

1.2 Counting points

The main result of this chapter is Theorem 1.2.1 which is used as an important step in the proof of the main result of this thesis. A first version of Theorem 1.2.1 may be found in the author's Diplomarbeit.

Let f^{top} denote the topological component of a map f of schemes. In positive characteristic p there is the Frobenius map F which is raising to p^{th} power and has $F^{\text{top}} = \text{id}$.

Theorem 1.2.1. *Let X, Y be of finite type over* $\operatorname{Spec}(\mathbb{Z})$, *$X$ be irreducible and reduced, and $f, g : X \to Y$ be maps such that $f^{\text{top}} = g^{\text{top}}$.*

Then $f = g$ or $X/\mathbb{F}_p$. If $X/\mathbb{F}_p$ then f and g differ only by a power of the Frobenius map. Uniqueness of the exponent is equivalent to $f^{\text{top}} = g^{\text{top}}$ not being constant.

We sketch the key idea of the proof. We represent maps as graphs such that a relation as claimed by the theorem becomes an incidence relation. The size of the respective intersection can be determined by counting $\mathbb{F}_{p^r}$-rational points. We let r go to infinity and use the Hasse–Weil bound for the number of points on a curve.

Proof: The assertion on uniqueness is clear as $f = f \circ F^m$ implies that the residue field at the image of the generic point of X is fixed by F^m. From now on we assume that f^{top} is nonconstant.

In view of uniqueness we may assume X, Y affine. Now the locus of coincidence $\{f \equiv g\}$ is closed in X and contains $\bigcup_p X(\mathbb{F}_p)$. Here for any prime number p we consider $X(\mathbb{F}_p)$ as a subset of the topological space underlying X. If $X_{\mathbb{Q}} \neq \emptyset$ we are done by the following lemma.

Lemma 1.2.2. *Let X be irreducible and of finite type over* $\operatorname{Spec}(\mathbb{Z})$, *such that $X_{\mathbb{Q}}$ is not empty. Then $\bigcup_p X(\mathbb{F}_p)$ is Zariski-dense in X.*

Proof: The generic point of X is in the generic fiber $X_{\mathbb{Q}}$ which is the closure of its closed points. Any of these points has a number field as residue field and defines a closed subscheme Z of dimension 1. By the Čebotarev Theorem Z is the closure of $\bigcup_p Z(\mathbb{F}_p) \subseteq \bigcup_p X(\mathbb{F}_p)$. □

From now on assume $X/\mathbb{F}_p$. For $m \in \mathbb{Z}$ define $X_m = \{f \circ F^m \equiv g\}$ or $\{f \equiv g \circ F^{-m}\}$ depending on the sign of m. Topological coincidence implies that

$$X(\mathbb{F}_q) \subseteq \bigcup_{m \in \mathbb{Z}} X_m(\mathbb{F}_q)$$

since $\mathbb{F}_q$-points are topologically identical if and only if they are $G(\mathbb{F}_q/\mathbb{F}_p)$ conjugate, and this Galois group is generated by Frobenius. If $q = p^r$ then it is sufficient to allow m to vary over representatives of $\mathbb{Z}/r\mathbb{Z}$. To keep things small we choose representatives with minimal absolute value and thereby conserve symmetry:

$$X(\mathbb{F}_{p^r}) \subseteq \bigcup_{-r/2<m\leq r/2} X_m(\mathbb{F}_{p^r}) \ . \tag{1.2.2}$$

The case of $\dim X = 1$. First consider the following lemma:

Lemma 1.2.3. *If* $\dim X = 1$ *there is a constant* c *such that for all* m *with* $X \neq X_m$ *the bound* $\#X_m(\mathbb{F}_q) \leq c \cdot p^{|m|}$ *holds.*

Proof: Choose closed immersions $X \subseteq \mathbb{A}^d_{\mathbb{F}_p}, Y \subseteq \mathbb{A}^n_{\mathbb{F}_p}$ and consider the graph Γ of (f, g)

$$X \xleftarrow[\sim]{\mathrm{pr}_1} \Gamma \subseteq X \times Y \times Y \subseteq \mathbb{A}^{d+2n}_{\mathbb{F}_p} \subseteq \mathbb{P}^{d+2n}_{\mathbb{F}_p} \ .$$

Let $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ be coordinates of the factor $\mathbb{A}^{2n}_{\mathbb{F}_p}$. For $m \in \mathbb{Z}$ let R_m be the vanishing locus in $\mathbb{P}^{d+2n}_{\mathbb{F}_p}$ of those n sections of $\mathcal{O}(p^{|m|})$ described by $x_i^{p^m} - y_i$ or $y_i^{p^{-m}} - x_i$ (depending on the sign of m) on the affine part $\mathbb{A}^{d+2n}_{\mathbb{F}_p}$. This R_m is the closure of the product of $\mathbb{A}^d_{\mathbb{F}_p}$ with the graph of F^m, and $\mathrm{pr}_1(\Gamma \cap R_m) = X_m$.

If $X_m \neq X$ then a hypersurface H_m of degree $p^{|m|}$ defined by a single suitably chosen such section suffices to cut down Γ to dimension 0.

An easy intersection theoretic estimate in $\mathbb{P}^{d+2n}_{\mathbb{F}_p}$ with the closure $\bar{\Gamma}$ of Γ gives

$$\#X_m(\mathbb{F}_q) \leq \deg(\bar{\Gamma} \cap H_m) = \deg(\bar{\Gamma}) \cdot \deg(H_m) = \deg(\bar{\Gamma}) \cdot p^{|m|} \ .$$

□

If $X_m = X$ for no $m \in \mathbb{Z}$ then by Lemma 1.2.3 and (1.2.2)

$$\#X(\mathbb{F}_{p^r}) \leq \sum_{-r/2<m\leq r/2} \#X_m(\mathbb{F}_{p^r}) \leq c \cdot r \cdot p^{r/2} \tag{1.2.3}$$

for some constant c. But (1.2.3) contradicts the "Weil conjectures" (vary $\mathbb{F}_{p^r}$ within the finite fields containing the field of constants of the smooth part of X, then $\#X(\mathbb{F}_{p^r})$ has order of magnitude p^r). In fact, we need only the case of algebraic curves which is known by Hasse-Weil.

The case of $\dim X > 1$. Take $C_1, C_2 \subseteq X$ irreducible, horizontal curves, i.e., such that $f|_{C_i}$ is not constant. By the one dimensional case there are unique $m_i \in \mathbb{Z}$ such that

$C_i \subseteq X_{m_i}$. For closed points $x_i \in C_i$ choose an irreducible curve $C \subset X$ passing through x_1, x_2, cf. Lemma 1.1.3. If $f(x_1) \neq f(x_2)$ then $C \subseteq X_m$ for a unique $m \in \mathbb{Z}$.

If a closed point y lies in $X_r \cap X_{r'}$ then $\deg(f(y))|r - r'$. Consequently, $\deg(f(x_i))$ divides $m_i - m$. Thus

$$\gcd\Big(\deg(f(x_1)), \deg(f(x_2))\Big) \mid m_1 - m_2 \ . \tag{1.2.4}$$

Varying x_1, x_2, the condition (1.2.4) shows that $m_1 - m_2$ has arbitrary large divisors, hence $m_1 = m_2$. Finally X is the closure of the union of such horizontal curves and so $X = X_m$ for some m, as desired. □

The following generalisation of Theorem 1.2.1 exploits more effectively its method of proof and might be useful in the future.

Proposition 1.2.4. *Let X, Y be of finite type over* $\mathrm{Spec}(\mathbb{F}_p)$, *$X$ be irreducible and reduced, and $Y/\mathbb{F}_p$ be separated. Let $f_i, g_j : X \to Y$ for $i \in I, j \in J$ be two finite collections of maps.*

Assume that for all closed points $x \in X$ there is a pair of indices $(i,j) \in I \times J$ such that topologically $f_i^{\mathrm{top}}(x) = g_j^{\mathrm{top}}(x)$. Then there is a pair of indices (i,j) such that f_i and g_j differ only by a power of the Frobenius map.

Proof: First we consider the case $\dim(X) = 1$. Let Y^I denote the product with index set I, i.e., $Y^I(T) = \{(t_i)_{i \in I} \mid t_i \in Y(T)\ \}$. Define relation schemes $R_m(i,j) \subseteq X \times Y^I \times Y^J$ for all $m \in \mathbb{Z}$ and $(i,j) \in I \times J$ as the product of the remaining factors with the graph of F^m between the i^{th} and j^{th} copy of Y. These $R_m(i,j)$ and the graph Γ of $(f_i, g_j)_{I \cup J}$

$$X \xleftarrow[\sim]{\mathrm{pr}_1} \Gamma \subseteq X \times Y^I \times Y^J$$

are closed subvarieties. Hence either $\Gamma \subseteq R_m(i,j)$ whence a relation between f_i and g_j as prescribed by the Proposition or $\Gamma \cap R_m(i,j)$ is a finite set of points. Assuming the latter leads to a contradiction as follows. The assumptions of the proposition imply

$$\Gamma(\mathbb{F}_{p^r}) \subseteq \bigcup_{\substack{(i,j) \in I \times J \\ -r/2 < m \leq r/2}} \big(\Gamma \cap R_m(i,j)\big)(\mathbb{F}_{p^r})$$

and the argument of Lemma 1.2.3 still yields a contradiction to the expected number of points by the “Weil conjectures”.

Now return to the general case of $\dim(X)$ arbitrary. Keep the notations of the above argument and set $\Gamma_m(i,j) = \Gamma \cap R_m(i,j)$. Let $M \subset I \times J$ be a minimal set of pairs of indices such that for all q

$$\Gamma(\mathbb{F}_q) \subseteq \bigcup_{\substack{(i,j) \in M \\ m \in Z}} \big(\Gamma_m(i,j)\big)(\mathbb{F}_q) \ .$$

We now choose for $(i,j) \in M$ a closed point

$$x_{i,j} \in \Gamma \setminus \bigcup_{\substack{(i',j') \in M \setminus \{(i,j)\} \\ m \in \mathbb{Z}}} \Gamma_m(i',j') \ .$$

By Corollary 1.1.4 there is an irreducible curve $C \subset \Gamma$ passing through all $x_{i,j}$. If we apply the argument of the one dimensional case for the restrictions to $\mathrm{pr}_1(C) \subset X$ with the restriction to pairs of indices in M we find that C must be contained in a single $\Gamma_m(i,j)$. Hence the cardinality of M is one and we are reduced to Theorem 1.2.1. □

Chapter 2

The Van Kampen Theorem

Beginning with the original paper of Van Kampen [VK33] there have been a large variety of papers dealing with the problem of computing the fundamental group of a space by those of the parts of an open cover. Most of these contributions deal with topological fundamental groups, i.e., discrete groups. The account in IX §5 [SGA 1] however treats algebraic fundamental groups and thus pro-finite groups. More importantly, this source uses the "tautological proof": fundamental groups are equivalently described by the respective category of coverings for which a glueing property can be established. This glueing property — which in the topological context is obvious for open coverings — transforms abstractly into a formula for the fundamental groups. The established name for this glueing procedure is *Descent Theory* and a Van Kampen Theorem is obtained by combining an abstract Van Kampen Theorem, cf. Theorem 2.2.9, with a result about effective descent, cf. Theorem 2.1.3 and Section 3.2.4.

The purpose of this chapter is to accomplish with an exercise left to the reader in a footenote of IX §5 [SGA 1] which asks for an obvious generalisation. Therefore the essential content of the chapter is not new. Moreover, the reasoning is inspired by that source. We achieve an abstract Van Kampen Theorem which constitutes in a amalgamation formula for the fundamental group of a 2-complex with group data in terms of the respective vertex groups and the configuration of the 2-complex. The methods are purely combinatorial.

We will apply in (6.2.5) the Van Kampen Theorem in the case of the logarithmic fundamental group of a stable curve. We find an amalgamation formula in terms of the logarithmic fundamental groups of the irreducible components and their intersection configuration. Chapter 3 provides the relevant effective descent result that serves as a bridge between the abstract Van Kampen Theorem and the geometry.

We assume that the reader is familiar with the concept of a Galois category and its properties as in V §4 [SGA 1]. In particular we call the automorphism group of a fibre functor F on a Galois category $\mathscr{C}$ its fundamental group $\pi_1(\mathscr{C}, F)$ with base point F. This enhances F to an equivalence of $\mathscr{C}$ with the category of finite, discrete, continuous $\pi_1(\mathscr{C}, F)$-sets.

2.1 Descent data

This section contains mainly definitions. We set the stage for the Van Kampen Theorem by defining descent data.

The simplicial category $\mathbf{\Delta}$ consists of objects $[n]$ for each natural number including 0 and has morphisms

$$\mathrm{Hom}_{\Delta}([n],[m]) = \big\{\alpha : \{0,\ldots,n\} \to \{0,\ldots,m\} \;\big|\; \alpha \text{ non decreasing } \big\}\ .$$

There are face maps $\partial_i : [n-1] \to [n]$ for $0 \le i \le n$ which are strict monotone but omit the value i, and degeneration maps $\sigma_i : [n+1] \to [n]$ for $0 \le i \le n$ which are surjective but take the value i twice. Sometimes one only uses those morphisms which are generated by the ∂_i. We will do so, and let Δ again denote the resulting category. The set $\mathrm{Hom}([0],[n])$ of "vertices" of n simplices consists of maps v_i with image i.

Let $\mathscr{C}$ be a category. The category of simplicial objects in $\mathscr{C}$ is the category of contravariant functors $C_\bullet : \Delta \to \mathscr{C}$. The value $C_\bullet([n]) =: C_n$ is called the object in degree n or "the n simplices" of $C_\bullet$. The value of ∂_i is called the i^{th} boundary map. For our purpose the truncation $C_{\le 2}$ of $C_\bullet$ at degree 2 suffices. We call such a diagram of objects in $\mathscr{C}$ simply a 2-complex.

Example. Let $h : S' \to S$ be a map of schemes. There is an associated simplicial scheme $S_\bullet(h)$ where $S_n(h) = S' \times_S \cdots \times_S S'$ with $n+1$ factors and ∂_i is the projection under omission of the i^{th} component, whereas σ_i is built upon the diagonal and doubles the i^{th} component.

Let $T_{\le 2}$ be a 2-complex of schemes, i.e., the truncation at degree 2 of a simplicial scheme. Furthermore, let $\mathscr{F} \to \mathrm{Sch}$ be a category fibred over the category of schemes, cf. VI [SGA 1] and let $\mathscr{F}(S)$ denote the category of sections of $\mathscr{F}$ above a scheme S. Our main example will be Rev such that $\mathsf{Rev}(S)$ is the category of finite étale covers over S.

Definition 2.1.1. *The category* $\mathsf{DD}(T_{\le 2},\mathscr{F})$ *of* **descent data** *for* $\mathscr{F}/\mathrm{Sch}$ *relative* $T_{\le 2}$ *is the following. Its objects consist of pairs* (X',φ) *where* $X' \in \mathscr{F}(T_0)$ *and* φ *is an isomorphism* $\partial_0^* X' \xrightarrow{\sim} \partial_1^* X'$ *in* $\mathscr{F}(T_1)$ *such that the cocycle condition holds, i.e., the following commutes in* $\mathscr{F}(T_2)$ *:*

$$\begin{array}{ccc} v_2^*X' & \xrightarrow{\partial_0^*\varphi} & v_1^*X' \\ & {\scriptstyle \partial_1^*\varphi}\searrow \quad \swarrow{\scriptstyle \partial_2^*\varphi} & \\ & v_0^*X' & \end{array} \tag{2.1.1}$$

Morphisms $f : (X',\varphi) \to (Y',\psi)$ *in* $\mathsf{DD}(T_{\le 2},\mathscr{F})$ *are morphisms of* $f : X' \to Y'$ *in* $\mathscr{F}(T_0)$ *such that its two pullbacks* $\partial_0^* f$ *and* $\partial_1^* f$ *are compatible with* φ,ψ*, i.e.,* $\partial_1^* f \circ \varphi = \psi \circ \partial_0^* f$.

Descent data for $\mathscr{F}$ relative a morphism $h : S' \to S$ are defined as the particular case of $\mathsf{DD}(h,\mathscr{F}) := \mathsf{DD}(S_{\le 2}(h),\mathscr{F})$ which is the category of pairs of an object $X' \in \mathscr{F}(S')$ and an isomorphism $\varphi : \mathrm{pr}_2^* X' \xrightarrow{\sim} \mathrm{pr}_1^* X'$ in $\mathscr{F}(S' \times_S S')$ such that the following commutes in $\mathscr{F}(S' \times_S S' \times_S S')$:

$$\begin{array}{ccc} \mathrm{pr}_3^*X' & \xrightarrow{\mathrm{pr}_{23}^*\varphi} & \mathrm{pr}_2^*X' \\ & {\scriptstyle \mathrm{pr}_{13}^*\varphi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_{12}^*\varphi} & \\ & \mathrm{pr}_1^*X' & \end{array} \tag{2.1.2}$$

Here pr_i (resp. pr_{ij}) is the projection onto the i^{th} (resp. the i^{th} and j^{th}) component.

Pullback by h induces a functor $h^* : \mathscr{F}(S) \to \mathsf{DD}(h,\mathscr{F})$.

Definition 2.1.2. *In the above context h is called a* **faithful** *(resp.* **descent***, or even* **effective descent***) morphism for $\mathscr{F}$ if h^* is a faithful functor (resp. fully faithful functor, or even an equivalence of categories) between $\mathscr{F}(S)$ and $\mathsf{DD}(h, \mathscr{F})$. The adjective universal may be added as usual in case the respective property persists to hold after arbitrary base extension.*

Example. Instead of schemes any geometric category with products is sufficient for the above formalism of descent data relative to a map. Let $X = \bigcup X_i$ be an open covering of a topological space, a manifold, ... Then $f : \coprod X_i \to X$ is an effective descent morphism for linebundles, vectorbundles, spaces above X, covering spaces, ... These examples show that descent data merely formalise the prerequisits for glueing. Being an effective descent morphism reflects the fact that the respective objects can be glued. Thus objects are given by local data plus glueing description and its properties are local.

2.1.1 Descent theory for finite étale covers

From [SGA 1] we have the following theorem.

Theorem 2.1.3. *Let $h : S' \to S$ satisfy one of the following properties:*

(1) h is proper, surjective and of finite presentation,

(2) h is fppf, i.e., faithfully flat and of finite presentation,

(3) h is fpqc, i.e., faithfully flat and quasi-compact.

Then h is a universal effective descent morphism for finite étale covers, i.e., for Rev.

Proof: IX 4.12, IX 4.9 and IX 4.1 [SGA 1]. □

2.1.2 Ordered descent data

For an open covering $\coprod X_i \to X$ of a topological space one is used to give a glueing isomorphism φ only once for an intersection $X_i \cap X_j$ and not twice (also for $X_j \cap X_i$) as the above formalism requires. The reason is that the above covering is composed as a coproduct of monomorphisms. We return to the category of schemes.

Definition 2.1.4. *A map $f : X \to Y$ is called a* **monomorphism** *if the relative diagonal $X \to X \times_Y X$ is an isomorphism. Equivalently, it defines an injective map $X(T) \hookrightarrow Y(T)$ on T valued points for arbitrary T.*

Every monomorphism is automatically a universal monomorphism. Arbitrary immersions provide examples for monomorphisms. And direct limits of monomorphisms, like localisations, are monomorphisms again.

Let $h = \coprod h_i : S' = \coprod_{i \in I} S'_i \to S$ be a map of schemes and $\prec$ an ordering on the set of indices I. Then consider the following simplicial scheme $S^{\prec}_\bullet(h)$ of ordered partial products (we have suppressed the $\coprod$ decomposition in the notation). It is the closed simplicial subscheme $S^{\prec}_\bullet(h) \subset S_\bullet(h)$ such that $S^{\prec}_n(h) = \coprod_{i_0 \prec \ldots \prec i_n} S'_{i_0} \times_S \ldots \times_S S'_{i_n}$. This already fixes the boundary maps ∂_i for $S^{\prec}_\bullet(h)$.

If S is locally noetherian then in particular $S = \coprod S_i$ where $\{S_i\} = \pi_0(S)$ is the set of connected components. We get an example of a map for the following proposition if we take $S' = \coprod S_i$ and h_i is the respective closed immersion $S_i \hookrightarrow S$.

Proposition 2.1.5. *Let $h_i : S'_i \to S$ be monomorphisms, let h be the map $\coprod h_i : S' = \coprod S'_i \to S$ and choose an ordering $\prec$ on the set of indices. Then the natural open immersion $j : S^{\prec}_{\bullet}(h) \hookrightarrow S_{\bullet}(h)$ induces an equivalence of categories of descent data*

$$j^* : \mathsf{DD}(h, \mathscr{F}) \xrightarrow{\cong} \mathsf{DD}(S^{\prec}_{\leq 2}(h), \mathscr{F}) \ . \tag{2.1.3}$$

Proof: Let $S''_{ij} = S'_i \times_S S'_j$ and $S'''_{ijk} = S'_i \times_S S'_j \times_S S'_k$. The "glueing" isomorphisms φ of descent data decomposes as $\varphi = \coprod_{i,j} \varphi_{ij}$ (resp. $\varphi^{\prec} = \coprod_{i \prec j} \varphi_{ij}$) according to the decomposition $S_2(f) = \coprod_{i,j} S''_{ij}$ (resp. $S^{\prec}_2(f) = \coprod_{i \prec j} S''_{ij}$).

Due to h_i being monomorphic we have $S'''_{iii} = S'_i$ and $S'''_{iji} = S''_{ij}$.For these indices the cocycle condition (2.1.1) yields $\varphi_{ii} \circ \varphi_{ii} = \varphi_{ii}$ and $\varphi_{ji} \circ \varphi_{ij} = \varphi_{ii}$. Hence $\varphi_{ii} = \mathrm{id}$ and $\varphi_{ji} = \varphi_{ij}^{-1}$. Thus j^* in (2.1.3) is an equivalence for φ and $\varphi^{\prec}$ carrying the same information. □

2.2 The abstract Van Kampen Theorem

This section ows much to the content of IX §5 [SGA 1]. See also [Se80] for the discrete situation without 2-simplices, and [We61] for the topological situation.

In the following, we will first describe group data along a 2-complex. The associated category of locally constant systems is then built upon combinatorial data imitating covering spaces. Locally constant systems form a Galois category. The abstract Van Kampen Theorem provides an isomorphism of an amalgamated product of the group data along the 2-complex with the fundamental group of the respective category of locally constant systems.

2.2.1 2-complexes with group data

By a 2-complex E we mean a simplicial set in degrees up to 2 with boundary maps ∂_i, i.e., a 2-complex of sets in the notation of the preceeding section. One may consider E as a small category: its objects are the elements of the E_n for $n = 0, 1, 2$ and its morphisms are formally obtained as $\partial : s \to t$ where $s \in E_n$ and $t = \partial(s)$.

Let Δ_n denote the topological n-simplex. Then we have a well known topological realisation functor $|E| = \coprod E_n \times \Delta_n / \sim$. We call E connected if $|E|$ is a connected topological space.

Definition 2.2.1. *A* **group data** $(\mathscr{G}, \alpha)$ *on E consists of the following.*

(i) A mapping (not a functor!) $\mathscr{G}$ from E considered as a category to the category of pro-finite groups: to a simplex $s \in E_n$ is attributed a pro-finite group $\mathscr{G}(s)$ and to a map $\partial : s \to t$ is attached a morphism $\mathscr{G}(\partial) : \mathscr{G}(s) \to \mathscr{G}(t)$.

(ii) For every 2-simplex (vef) of the barycentric subdivision ($v \in E_0, e \in E_1, f \in E_2$ and boundary maps ∂', ∂ such that $\partial'(f) = e, \partial(e) = v$) we fix an element $\alpha_{vef} \in \mathscr{G}(v)$ which controls the deviation from $\mathscr{G}$ being a functor, i.e., the following diagram commutes.

$$\begin{array}{ccc} \mathscr{G}(f) & \xrightarrow{\mathscr{G}(\partial')} & \mathscr{G}(e) \\ \downarrow{\scriptstyle \mathscr{G}(\partial\partial')} & & \downarrow{\scriptstyle \mathscr{G}(\partial)} \\ \mathscr{G}(v) & \xrightarrow{\alpha()\alpha^{-1}} & \mathscr{G}(v) \end{array} \tag{2.2.4}$$

A group data $(\mathscr{G}, \alpha)$ simply describes a lift of a functor from E to the category of pro-finite groups with exterior morphisms to a mapping into the category of pro-finite groups with true morphisms.

2.2.2 Locally constant systems

Definition 2.2.2. *A* $(\mathscr{G}, \alpha)$**-system** M *on* E *consists of the following:*

(i) for every simplex $s \in E$ *a finite discrete continuous* $\mathscr{G}(s)$*-set* M_s*,*

(ii) for every $\partial : s \to t$ *a* $\mathscr{G}(s)$*-equivariant boundary map* $m_\partial : M_s \to \mathscr{G}(\partial)^* M_t$*, such that*

(iii) for every 2*-simplex* (vef) *of the barycentric subdivision with* $\partial'(f) = e, \partial(e) = v$ *the following commutes.*

$$\begin{array}{ccc} M_f & \xrightarrow{m_{\partial'}} & M_e \\ {\scriptstyle m_{\partial\partial'}}\downarrow & & \downarrow{\scriptstyle m_\partial} \\ M_v & \xrightarrow{\alpha.} & M_v \end{array} \tag{2.2.5}$$

Diagram (2.2.5) is a little inaccurate in that it does not record the group action. The map $(\alpha.)$ is merely a $\mathscr{G}(v)$-equivariant map $M_v \to \big(\alpha()\alpha^{-1}\big)^* M_v$, because $\alpha\gamma m = (\alpha\gamma\alpha^{-1})\alpha m$ by (2.2.4). But thus diagram (2.2.5) is $\mathscr{G}(f)$-equivariant as $\mathscr{G}(\partial')^*\mathscr{G}(\partial)^* = \mathscr{G}(\partial\partial')^*\big(\alpha()\alpha^{-1}\big)^*$.

Morphisms of $(\mathscr{G}, \alpha)$ systems on E are defined in the obvious way: a collection of $\mathscr{G}(s)$-equivariant maps that commute with the m's.

Definition 2.2.3. *A* $(\mathscr{G}, \alpha)$ *system* M *on* E *is called* **locally constant** *if all of its boundary maps* m_∂ *are bijective. The category of all locally constant systems on* E *is denoted by* $\mathsf{lcs}\big(E, (\mathscr{G}, \alpha)\big)$*.*

Proposition 2.2.4. *Let* $\big(E, (\mathscr{G}, \alpha)\big)$ *be a connected* 2*-complex with group data. Then its category of locally constant systems* $\mathsf{lcs}\big(E, (\mathscr{G}, \alpha)\big)$ *is a Galois category.*

Proof: We need to check the axioms (G1) - (G6) of V §4 [SGA 1]. The existence of fibre products, final object, coproducts, initial object and quotients by finite group actions are established by constructing the relevant locally constant systems "simplexwise" by the respective property in the Galois categories of $\mathscr{G}(s)$-sets. This works as the pullback functors $\mathscr{G}(\partial)$ respect these properties.

The existence of a factorisation in a strict epimorphism composed with a monomorphism of a direct summand is obtained as the factorisation over the simplexwise image which is easily verified to be a locally constant system. One checks that the simplexwise complement of the image furnishes a complementing direct summand. Furthermore, a simplexwise surjection is a strict epimorphism as in the category of $\mathscr{G}(s)$-sets strict epimorphisms coincide with surjective maps.

Now we need to construct a fibre functor. For any simplex s the evaluation $F_s : \mathsf{lcs}\big(E, (\mathscr{G}, \alpha)\big) \to \mathsf{sets}$ which maps M to the underlying set of M_s yields a fibre functor. This follows as the categorical constructions of the above argument have been performed "simplexwise". Furthermore F_s is conservative because $|E|$ is connected. $\square$

Corollary 2.2.5. *For every simplex $s \in E$ there is a canonical exact functor*

$$F_s : \mathsf{lcs}\big(E, (\mathscr{G}, \alpha)\big) \to \mathscr{G}(s)\text{-}\mathsf{sets}$$

of Galois categories. □

2.2.3 The topological component

For $M \in \mathsf{lcs}\big(E, (\mathscr{G}, \alpha)\big)$ we construct a simplicial set $M_{\leq 1}$ in degrees up to 1 over $E_{\leq 1}$ (the data of a simplicial set in degrees up to 1 simply describes an oriented graph): $(M_{\leq 1})_0 = \coprod_{v \in E_0} M_v$ and $(M_{\leq 1})_1 = \coprod_{e \in E_1} M_e$. The boundary maps are induced by the m_∂ and the map $M_{\leq 1} \to E_{\leq 1}$ is the canonical one.

This assignement is certainly functorial. Its composition with the topological realisation functor $|\cdot|$ is denoted by $|\cdot_{\leq 1}|$ and yields a finite topological covering $|M_{\leq 1}| \to |E_{\leq 1}|$.

Let $\mathsf{cov}(X)$ be the category of finite topological covers of a topological space X. Then the following is immediate.

Proposition 2.2.6. *The functor $|\cdot_{\leq 1}| : \mathsf{lcs}\big(E, (\mathscr{G}, \alpha)\big) \to \mathsf{cov}(|E_{\leq 1}|)$ is an exact functor between Galois categories.* □

As usual, to a maximal subtree $T \subset E_{\leq 1}$ is attributed a fibre functor $F_T : \mathsf{cov}(|E_{\leq 1}|) \to \mathsf{sets}$ by assigning to $p : X \to |E_{\leq 1}|$ the set $\pi_0\big(p^{-1}(|T|)\big)$. The resulting fundamental group $\pi_1(\mathsf{cov}(|E_{\leq 1}|, F_T)$ equals the pro-finite completion of the topological fundamental group of $|E_{\leq 1}|$. It furthermore is canonically isomorphic to the pro-finite completion $\widehat{\pi}_1(E_{\leq 1}, T)$ of the combinatorial fundamental group of the graph $E_{\leq 1}$ which in turn is canonically isomorphic to $\widehat{Fr}(E_1)/\langle\langle \vec{e} \mid e \in T \rangle\rangle$. Here $\widehat{Fr}(S)$ denotes the free pro-finite group on a set S, $\vec{s}$ is the symbol for the generator corresponding to $s \in S$, and $\langle\langle I \rangle\rangle$ means the minimal normal subgroup containing I.

Let F_T also denote the resulting fibre functor $F_T(|\cdot_{\leq 1}|)$ on $\mathsf{lcs}\big(E, (\mathscr{G}, \alpha)\big)$. For every simplex $s \in E_0$ the group $\mathscr{G}(s)$ canonically acts on $F_T(M)$ for $M \in \mathsf{lcs}\big(E, (\mathscr{G}, \alpha)\big)$ via $M_s = \pi_0\big(p^{-1}(s)\big) = \pi_0\big(p^{-1}(T)\big)$.

Let $*$ be the notation for pro-finite amalgamation, cf. [RZ00].

Proposition 2.2.7. *The functor*

$$Q : \mathsf{lcs}\big(E, (\mathscr{G}, \alpha)\big) \to \Big(\mathop{\ast}_{v \in E_0} \mathscr{G}(v) * \widehat{\pi}_1(E_{\leq 1}, T) \Big)\text{-}\mathsf{sets}$$

encapsulating the above operations on $F_T(\cdot)$ is fully faithful.

Proof: We prove that Q conserves connectedness of objects which amounts to transitivity of the group action. We argue by contradiction. Let $M \in \mathsf{lcs}\big(E, (\mathscr{G}, \alpha)\big)$ be connected and let $N \subset F_T(M)$ be a nonempty proper subset stable under the action of $\widehat{\pi}_1(E_{\leq 1}, T)$ and $\mathscr{G}(v)$ for $v \in E_0$. The stability under $\widehat{\pi}_1(E_{\leq 1}, T)$ implies that N is respected by the boundary maps m_∂ and thus extends to a subgraph $N_{\leq 1} \subset M_{\leq 1}$.

We need to show that it extends further over 2-simplices. This is a local question that only depends on the simplices in the boundary of a given face $f \in E_2$, i.e., images $\partial(f)$. We define N_f as the preimage under m_∂ of $N_s \subset M_s$ for any $\partial : f \to s$ and need to verify that this subset of M_f is well defined. It clearly suffices for any 2-simplex (vef) of the barycentric subdivision to compare $\partial' : f \to e$ with the composition $\partial\partial' : f \to e \to v$. But as $\alpha_{vef} \in \mathscr{G}(v)$ respects N_v we have by (2.2.5)

$$m_{\partial\partial'}^{-1}(N_v) = m_{\partial\partial'}^{-1}\big((\alpha\cdot)^{-1}(N_v)\big) = m_{\partial'}^{-1} m_{\partial}^{-1}(N_v) = m_{\partial}^{-1}(N_e)$$

and thus N originates from an object $N \in \mathsf{lcs}\big(E, (\mathscr{G}, \alpha)\big)$ which is a subobject of M and thus contradicts the connectedness of M. □

2.2.4 $\mathscr{G}$-doted paths and homotopy relations

Let $\big(E, (\mathscr{G}, \alpha)\big)$ be a connected 2-complex with group data. As above, F_v and F_T denote the respective fibre functors of $\mathsf{lcs}\big(E, (\mathscr{G}, \alpha)\big)$ assigned to a vertex $v \in E_0$ and a maximal subtree $T \subset E_{\leq 1}$. The respective fundamental groups are abbreviated as $\pi_1(E, \mathscr{G}; v)$ and $\pi_1(E, \mathscr{G}; T)$.

Definition 2.2.8. *Let $\vec{e} \in \widehat{Fr}(E_1)$ be the symbol for the element $e \in E_1$. A* **$\mathscr{G}$-doted path** *from $i \in E_0$ to $f \in E_0$ is a pro-word γ in $\mathop{\ast}_{v\in E_0} \mathscr{G}(v) * \widehat{Fr}(E_1)$ such that successive letters are always geometrically adjacent and whose first (resp. last) letter is localised in i (resp. f), i.e., at every finite level γ is represented by a word such that consecutive letters consist in one of the following options:*

(i) $\ldots \vec{e}\vec{\varepsilon} \ldots$ *for $e, \varepsilon \in E_1$ such that $\partial_0(e) = \partial_1(\varepsilon)$,*

(ii) $\ldots \vec{e}g \ldots$ *for $e \in E_1$ and $g \in \mathscr{G}\big(\partial_0(e)\big)$,*

(iii) $\ldots g\vec{e} \ldots$ *for $e \in E_1$ and $g \in \mathscr{G}\big(\partial_1(e)\big)$,*

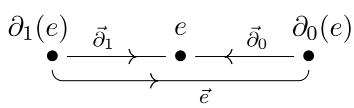
(iv) $\ldots gg' \ldots$ *for $g, g' \in \mathscr{G}(v)$ for some $v \in E_0$,*

and its first letter is $g \in \mathscr{G}(i)$ or $\vec{e}$ corresponding to $e \in E_1$ with $\partial_1(e) = i$ (resp. its last letter is $g \in \mathscr{G}(f)$ or $\vec{e}$ corresponding to $e \in E_1$ with $\partial_0(e) = f$).

The subset of $\mathscr{G}$-doted paths in $\mathop{\ast}_{v\in E_0} \mathscr{G}(v) * \widehat{Fr}(E_1)$ is denoted by $\Omega(E, \mathscr{G}; i, f)$. If $i = f = v$ we call them $\mathscr{G}$-doted loops based at v. They form a subgroup $\Omega(E, \mathscr{G}; v)$.

The intuition behind the following is to interpret $m_\partial : M_s \to M_t$ for some $M \in \mathsf{lcs}\big(E, (\mathscr{G}, \alpha)\big)$ and boundary map $\partial : s \to t$ as the continuation of fibres of a covering along a path $\vec{\partial}$ of the base which lead in the *opposite* direction, namely $\vec{\partial} : t \rightarrowtail s$. The map m_∂ will be called the *monodromy* action along the "virtual" path $\vec{\partial}$. These conventions allow to compose paths as usually and at the same time to act from the left on fibres. Only conjugations by or some specific compositions of these virtual paths will appear. However, they are "responsible" for the combinatorics of the homotopy relations.

The edge relation. Let $e \in E_1$ be an edge. We define a virtual path $\vec{e} = \vec{\partial}_1(\vec{\partial}_0)^{-1} : \partial_1(e) \rightarrowtail \partial_0(e)$ and understand that its monodromy action on fibres is given by $e := m_{\partial_1} m_{\partial_0}^{-1} : M_{\partial_0(e)} \to M_{\partial_1(e)}$. The following picture shows virtual paths:



As the m_{∂_i} are $\mathscr{G}(e)$ equivariant we obtain for $g \in \mathscr{G}(e)$:

$$\mathscr{G}(\partial_1)(g)e = \mathscr{G}(\partial_1)(g)m_{\partial_1}m_{\partial_0}^{-1} = m_{\partial_1}gm_{\partial_0}^{-1} = m_{\partial_1}m_{\partial_0}^{-1}\mathscr{G}(\partial_0)(g) = e\mathscr{G}(\partial_0)(g) \ .$$

Hence for all $e \in E_1$ and $g \in \mathscr{G}(e)$ we have the edge relation of $\mathscr{G}$-doted paths from $\partial_1(e)$ to $\partial_0(e)$:

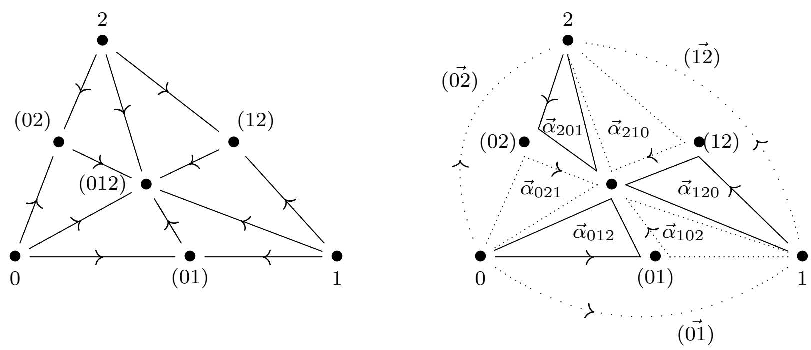
$$\boxed{\mathscr{G}(\partial_1)(g)\vec{e} \quad = \quad \vec{e}\mathscr{G}(\partial_0)(g)} \tag{2.2.6}$$

The cocycle relation. Let $f \in E_2$ be a face. Its closure $E(f) \subset E$ consists of all its boundaries $\partial(f)$ and consists of the image of the unique map of the standard 2-simplex $\Delta_2 = \mathrm{Hom}(-,[2])_{\leq 2}$ such that the unique 2-simplex of Δ_2 is mapped to f. More precisely

$$(\Delta_2)_0 = \{0,1,2\} \ , \ (\Delta_2)_1 = \{(01),(02),(12)\} \ , \ (\Delta_2)_2 = \{(012)\}$$

where the corresponding simplices are maps $\partial : [n] \to [2]$ in Δ which are described by their respective image in $\{0,1,2\}$. The map $\Delta_2 \twoheadrightarrow E(f)$ alluded above maps (012) to f.

We want to study $M \in \mathsf{lcs}\big(E,(\mathscr{G},\alpha)\big)$ locally at f. The above reduces this to the study of its pullback $M|_{\Delta_2}$ on the standard 2-simplex. We set $\alpha_{ijk} = \alpha_{i,(ij),(ijk)}$ for any permutation $\{i,j,k\} = \{0,1,2\}$ and also apply to it the interpretation of being a monodromy operator along the opposite direction of a virtual path $\vec{\alpha}_{ijk}$ this time being a loop. We obtain the following pictures of virtual paths for Δ_2



(2.2.7)

where the loops $\vec{\alpha}_{ijk}$ turn once around a face of the barycentric subdivision by passing through the simplices in ascending order according to their degrees.

There are six relations that M is required to satisfy on $E(f)$. For any permutation $\{i,j,k\} = \{0,1,2\}$ we have

$$\alpha_{ijk}m_{v_i} = \begin{cases} m_{\partial_0}m_{\partial_k} & i > j \\ m_{\partial_1}m_{\partial_k} & i < j \end{cases} \tag{2.2.8}$$

$$\text{with } \begin{cases} \partial_0 : (ij) \to i & i > j \\ \partial_1 : (ij) \to i & i < j \end{cases} \text{ and } \partial_k : (ijk) \to (ij) \ .$$

which holds as maps $M_f \to M_{v_i(f)}$ due to the compatibility condition (iii) that was asked for in the definition of a $(\mathscr{G},\alpha)$ system. According to the interpretation with paths relation (2.2.8) is due to composition of paths around the 2-simplex $i(ij)(ijk)$ of the barycentric subdivision, as can be seen in (2.2.7) . In accordance with the paragraph about the edge relation we abbreviate the monodromy operator $m_{\partial_1}m_{\partial_0}^{-1} : M_{\partial_0\partial_k(f)} \to M_{\partial_1\partial_k(f)}$ by $\partial_k f$

which will be understood as the monodromy action along (the opposite direction of) a virtual path $\overrightarrow{\partial_k f}: \partial_1(e) \nrightarrow \partial_0(e)$ for $e = \partial_k(f)$. By turning once around the barycenter of diagram (2.2.7) — thus incorporating all the six of the relations once — an easy calculation yields the cocycle relation (2.2.9) which is a relation of $\mathscr{G}$-doted loops based at $v_0(f)$ for any $f \in E_2$:

$$\boxed{1 \;=\; \big(\overrightarrow{\partial_2 f}\big)\vec{\alpha}_{102}(\vec{\alpha}_{120})^{-1}\big(\overrightarrow{\partial_0 f}\big)\vec{\alpha}_{210}(\vec{\alpha}_{201})^{-1}\big((\overrightarrow{\partial_1 f})^{-1}\big)\vec{\alpha}_{021}(\vec{\alpha}_{012})^{-1}} \tag{2.2.9}$$

The homotopy relations consist of the normal subgroup

$$H \;\lhd\; \mathop{\ast}_{v\in E_0} \mathscr{G}(v) * \widehat{Fr}(E_1)$$

generated by the cocycle relation (2.2.9) and the edge relation (2.2.6) for all possible parameters.

$$H = \left\langle\!\!\left\langle \begin{array}{c} \mathscr{G}(\partial_1)(g)\vec{e} = \vec{e}\,\mathscr{G}(\partial_0)(g) \\ (\overrightarrow{\partial_2 f})\alpha_{102}{\alpha_{120}}^{-1}(\overrightarrow{\partial_0 f})\alpha_{210}{\alpha_{201}}^{-1}\big((\overrightarrow{\partial_1 f})\big)^{-1}\alpha_{021}{\alpha_{012}}^{-1} = 1 \end{array} \right\rangle\!\!\right\rangle$$

The appearing virtual edge paths in these relations are interpreted as the respective elements of $\widehat{Fr}(E_1)$. By intersecting with the group $\Omega(E, \mathscr{G}; v)$ we obtain the normal subgroup $H_{\Omega,v}$ of homotopy relations for loops based at v.

2.2.5 The theorem

Theorem 2.2.9 (abstract Van Kampen theorem). *Let E be a connected 2-complex with group data $(\mathscr{G}, \alpha)$. With the above notation we have a commutative diagram that yields two descriptions of the fundamental group of* $\mathsf{lcs}\big(E, (\mathscr{G}, \alpha)\big)$.

$$\begin{array}{ccc} \Omega(E,\mathscr{G};v)/H_{\Omega,v} & \subset & \mathop{\ast}_{v\in E_0}\mathscr{G}(v) * \widehat{Fr}(E_1)/H \\ \downarrow r & & \downarrow q \\ \pi_1(E,\mathscr{G};v) & \xleftarrow{\;c\;} & \pi_1(E,\mathscr{G};T) \end{array} \tag{2.2.10}$$

The morphisms c, q, r have the following properties:

(1) c is the isomorphism between the two π_1 induced by the natural isomorphism $F_v \to F_T$ derived from $v \in T$,

(2) q is induced by the functor Q of Proposition 2.2.7, it is surjective and its kernel is generated as a normal subgroup by $\{\vec{e} \mid e \in T\}$,

(3) r is an isomorphism which is induced by the canonical operation of v-based $\mathscr{G}$-doted loops on the fibre $F_v(\cdot)$.

Proof: (1) The assertion on c is trivial by the theory of Galois categories. (2) Next we deal with q. As the kernel of $\widehat{Fr}(E_1) \twoheadrightarrow \widehat{\pi}_1(E_{\le 1}, T)$ is generated by the edges from T we need to show that the functor Q yields an isomorphism

$$\Big(\mathop{\ast}_{v\in E_0}\mathscr{G}(v) * \widehat{\pi}_1(E_{\le 1}, T)\Big)/\overline{H} \;\xrightarrow{\sim}\; \pi_1(E, \mathscr{G}; T)$$

where $\overline{H}$ is the image of H. Equivalently we need to show that Q induces an equivalence of $\mathsf{lcs}\big(E,(\mathscr{G},\alpha)\big)$ with the full subcategory of those $\ast_{v\in E_0}\mathscr{G}(v)\ast\widehat{\pi}_1(E_{\leq 1},T)$-sets on which $\overline{H}$ act trivially. By Propositon 2.2.7 Q is fully faithful. Furthermore, the interpretation that leads to the edge and cocycle relations clearly is mirrored by the actions of $\mathscr{G}(v)$ and $\vec{e}$ for $e\in E_1$ (acting as $e=m_{\partial_1}m_{\partial_0}^{-1}$) on the various $F_v(\cdot)$ for $v\in E_0$ which are identified along T by $v\in T$ with $F_T(\cdot)$. Thus $\overline{H}$ acts trivial on the essential image of Q.

Let $\tilde{M}$ be a $\Big(\ast_{v\in E_0}\mathscr{G}(v)\ast\widehat{\pi}_1(E_{\leq 1},T)\Big)$-set on which $\overline{H}$ acts trivial. We need to show that it is isomorphic to a $Q(M)$ for an $M\in\mathsf{lcs}\big(E,(\mathscr{G},\alpha)\big)$. We build M degree by degree. For $v\in E_0$ we set $M_v=\tilde{M}$ with the induced $\mathscr{G}(v)$ action. For $e\in E_1$ we set $M_e=\tilde{M}$ with the $\mathscr{G}(e)$ action induced by $\mathscr{G}(\partial_0):\mathscr{G}(e)\to\mathscr{G}\big(\partial_0(e)\big)$. For $\partial_i:e\to v$ from degree 1 to 0 we thus set $m_{\partial_0}=id$ and $m_{\partial_1}=\vec{e}$ as element of $\widehat{\pi}_1(E_{\leq 1},T)$ acting on $\tilde{M}$. The edge relations (2.2.6) imply that these m_∂'s are $\mathscr{G}(e)$ equivariant.

To extend this construction up to the 2-simplices $f\in E_2$ it suffices to work in the respective closures $E(f)$, or even over the standard 2-simplex Δ_2, which we will do in the sequel. We set $M_{(012)}=\tilde{M}$ with $\mathscr{G}(012)$ action induced by $\mathscr{G}(v_0):\mathscr{G}(012)\to\mathscr{G}(0)$. By the six relations (2.2.8) discussed above the maps m_∂ for $\partial:(012)\to s$ for any simplex s are fixed uniquely by turning around the barycenter of the picture (2.2.7) of virtual paths of Δ_2. This assignement being well defined is precisely equivalent to the cocycle condition to hold.

(3) Finally we treat r. It is defined by the action on M_v via the following two operations. For an edge $e\in E_1$ the path $\vec{e}\in\widehat{\pi}_1(E_{\leq 1},T)$ acts by "continuation" with $e=m_{\partial_1}m_{\partial_0}^{-1}$: $M_{\partial_0(e)}\to M_{\partial_1(e)}$. For a vertex v' and $g\in\mathscr{G}(v')$ the path g acts by the $\mathscr{G}(v')$ action on $M_{v'}$. Note that this is possible due to the condition of letters in $\mathscr{G}$-doted paths to be geometrically adjacent. It yields a left action on M_v as we start from the right by continuating in the direction opposite to the one of the path. This definition of r obviously is compatible with q proving that $H_{\Omega,v}$ acts trivial. This defines r in a compatible way such that the diagram (2.2.10) commutes.

For $v_1,v_2\in E_0$, let γ_{v_1,v_2} denote the finite geodesic from v_1 to v_2 in the tree T, i.e., the unique minimal connecting path in T (this is a finite combinatorial path and not a pro-path). We treat it as an element of $\widehat{Fr}(E_1)$ such that $\vec{e}=\gamma_{\partial_1(e),\partial_0(e)}$ and $(\vec{e})^{-1}=\gamma_{\partial_0(e),\partial_1(e)}$ and so on by composition.

Let $\rho:\Omega(E,\mathscr{G};v)/H_{\Omega,v}\to\mathrm{Aut}(M)$ be a continuous action on a finite discrete set M. For injectivity of r it is sufficient to extend any ρ to an action of $\ast_{v\in E_0}\mathscr{G}(v)\ast\widehat{Fr}(E_1)$ on M such that H and $e\in T$ act trivially. We define for $g\in\mathscr{G}(v')$ and $e\in E_1$

$$\begin{aligned} g &\mapsto \rho(\gamma_{v,v'}g\gamma_{v,v'}^{-1}):M\to M\\ \vec{e} &\mapsto \rho(\gamma_{v,\partial_1(e)}\vec{e}\gamma_{v,\partial_0(e)}^{-1}):M\to M \end{aligned}$$

which is possible as ρ is only applied to $\mathscr{G}$-doted paths. This obviously extends the action to $\rho:\ast_{v\in E_0}\mathscr{G}(v)\ast\widehat{Fr}(E_1)\to\mathrm{Aut}(M)$ such that H (homotopy relations are $\mathscr{G}$-doted paths) and $e\in T$ act trivially (due to the properties of geodesics in trees).

The elements $q(\gamma_{v,v'}g\gamma_{v,v'}^{-1})=q(g)$ for $g\in\mathscr{G}(v')$ and $q(\gamma_{v,\partial_1(e)}\vec{e}\gamma_{v,\partial_0(e)}^{-1})=q(\vec{e})$ for $e\in E_1$ generate $\pi_1(E,\mathscr{G};T)$ as a pro-finite group. They are in the image of $c^{-1}r$ whence r is surjective. $\square$

2.3 Discretising descent data

A Van Kampen formula for the fundamental group of a space in terms of its parts and their configuration is achieved in three steps. First, the category of coverings is equivalent by an effective descent result to a category of descent data. The second step discretises the situation. By discretising descent data we mean to construct a 2-complex with group data and an equivalence of the category of descent data with the associated category of locally constant systems. As a third step we plug in the abstract Van Kampen Theorem and provide an amalgamation formula for the fundamental groups of the equivalent categories

$$\mathsf{Rev}(S) \xrightarrow{h^*} \mathsf{DD}(h, \mathsf{Rev}) \xrightarrow{\text{disc}} \mathsf{lcs}\big(E, (\mathscr{G}, \alpha)\big) \ .$$

In this section we proceed with the second step.

The setup. Let $T_{\leq 2}$ be a simplicial scheme in degrees up to 2 consisting of locally noetherian schemes T_n. Let $E = \pi_0(T_{\leq 2})$ be its 2-complex of sets of connected components. For the category $\mathscr{F}$ fibred above the category of schemes (compare with Section 2.1) we take our favorite choice Rev, i.e., finite étale covers above variable bases. Actually, the only requirement on $\mathscr{F}$ comprises in the property that over connected bases the sections form a Galois category. Once we have defined logarithmic étale covers such that $\mathsf{Rev}^{\log}$ is fibred in Galois categories over fs-log schemes, cf. Section 3.3.1, one easily derives log versions of the following.

For a simplex $s \in E$ we fix a base point $t(s) \in s$ (the second s has to be considered as a connected component of a T_n). This base point corresponds to a fibre functor $F_{t(s)}$ of $\mathsf{Rev}(s)$. To a boundary map $\partial : s \to s'$ we associate a path (in the sense of V [SGA 1]) $\vec{\partial} : \ t(s') \rightarrowtail T(\partial)t(s)$ corresponding to a transformation $\vec{\partial} : F_{t(s)} \circ T(\partial)^* \to F_{t(s')}$. For a 2-simplex (vef) of the barycentric subdivision with $\partial' : f \to e$ and $\partial : e \to v$ we obtain a loop

$$\alpha_{vef} := \vec{\partial}\vec{\partial'}(\overrightarrow{\partial\partial'})^{-1} \in \pi_1\big(v, t(v)\big) \tag{2.3.11}$$

that is in the fundamental group of the component $v \subset T_0$ (actually $T(\partial)\vec{\partial'}$ instead of $\vec{\partial'}$ is correct). We define the group data $(\mathscr{G}, \alpha)$ on E as follows: $\mathscr{G}(s) = \pi_1\big(s, t(s)\big)$ for any simplex $s \in E$ and $\partial : s \to s'$ is associated

$$\mathscr{G}(\partial) : \pi_1\big(s, t(s)\big) \xrightarrow{T(\partial)_\#} \pi_1\big(s', T(\partial)t(s)\big) \xrightarrow{\vec{\partial}()\vec{\partial}^{-1}} \pi_1\big(s', t(s')\big) \ .$$

The element α is defined as in (2.3.11) above.

Proposition 2.3.1. *The fibre functors $F_{t(s)}$ and paths $\vec{\partial}$ merge together forming a functor*

$$\text{disc} : \mathsf{DD}(T_{\leq 2}, \mathsf{Rev}) \to \mathsf{lcs}\big(E, (\mathscr{G}, \alpha)\big)$$

which is an equivalence of Galois categories.

Proof: Let (X', φ) be a descent datum relative $T_{\leq 2}$. We define $\text{disc}(X', \varphi) = M$ as follows.

$$\begin{array}{lccl} v \in E_0 & \rightsquigarrow & M_v = F_{t(v)}\big(X'|_v\big) & \text{as } \pi_1(v, t(v))\text{-}\mathsf{sets} \ , \\ e \in E_1 & \rightsquigarrow & M_e = F_{t(e)}\big(T(\partial_0)^* X'|_e\big) & \text{as } \pi_1(e, t(e))\text{-}\mathsf{sets} \ , \\ f \in E_2 & \rightsquigarrow & M_f = F_{t(f)}\big(T(\partial_0\partial_0)^* X'|_f\big) & \text{as } \pi_1(f, t(f))\text{-}\mathsf{sets} \ . \end{array}$$

We define boundary maps m_∂ for $\partial : e \to v$ with $v \in E_1$ as

$$\begin{aligned} m_{\partial_1} &: F_{t(e)}\big(T(\partial_0)^*X'|_e\big) \xrightarrow{\varphi|_e} F_{t(e)}\big(T(\partial_1)^*X'|_e\big) \xrightarrow{\vec{\partial}_1} F_{t(\partial_1(e))}\big(X'|_{\partial_1(e)}\big) \\ m_{\partial_0} &: F_{t(e)}\big(T(\partial_0)^*X'|_e\big) \xrightarrow{\vec{\partial}_0} F_{t(\partial_0(e))}\big(X'|_{\partial_0(e)}\big) \ . \end{aligned}$$

Recall that the map $[0] \to [2]$ with image i was called v_i, e.g., $v_2 = \partial_0\partial_0$. We define boundary maps m_∂ starting from the face $f \in E_2$ as:

$$\begin{aligned} m_{v_0} &: F_{t(f)}\big(T(v_2)^*X'|_f\big) \xrightarrow{\partial_1^*\varphi|_f} F_{t(f)}\big(T(v_0)^*X'|_f\big) \xrightarrow{\vec{v}_0} F_{t(v_0(f))}\big(X'|_{v_0(f)}\big) \\ m_{\partial_1} &: F_{t(f)}\big(T(v_2)^*X'|_f\big) \xrightarrow{\vec{\partial}_1} F_{t(\partial_1(f))}\big(T(\partial_0)^*X'|_{\partial_1(f)}\big) \\ m_{v_2} &: F_{t(f)}\big(T(v_2)^*X'|_f\big) \xrightarrow{\vec{v}_2} F_{t(v_2(f))}\big(X'|_{v_2(f)}\big) \\ m_{\partial_0} &: F_{t(f)}\big(T(v_2)^*X'|_f\big) \xrightarrow{\vec{\partial}_0} F_{t(\partial_0(f))}\big(T(\partial_0)^*X'|_{\partial_0(f)}\big) \\ m_{v_1} &: F_{t(f)}\big(T(v_2)^*X'|_f\big) \xrightarrow{\partial_0^*\varphi|_f} F_{t(f)}\big(T(v_1)^*X'|_f\big) \xrightarrow{\vec{v}_1} F_{t(v_1(f))}\big(X'|_{v_1(f)}\big) \\ m_{\partial_2} &: F_{t(f)}\big(T(v_2)^*X'|_f\big) \xrightarrow{\partial_0^*\varphi|_f} F_{t(f)}\big(T(v_1)^*X'|_f\big) \xrightarrow{\vec{\partial}_2} F_{t(\partial_2(f))}\big(T(\partial_0)^*X'|_{\partial_2(f)}\big) \end{aligned}$$

This was obtained so far in accordance with the principle of turning once around the barycenter in Δ_2 and obeying the relations with α's (for each 2-simplex of the barycentric subdivision). Note that we have used the fact that for their effect on fibres the following identity holds:

$$\partial_i^*\varphi := T(\partial_i)^*\varphi = (\vec{\partial}_i)^{-1}\varphi\vec{\partial}_i \ . \tag{2.3.12}$$

However, there remains one such relation, namely for the 2-simplex $\big(v_0(f), \partial_2(f), f\big)$. This amounts by the definition of the various m_∂ to

$$\big(\vec{\partial}_1\varphi\big)\big(\vec{\partial}_2\partial_0^*\varphi\big) = m_{\partial_1}m_{\partial_2} = \alpha_{(v_0(f),\partial_2(f),f)}m_{v_0} = \big(\vec{\partial}_1\vec{\partial}_2(\vec{v}_0)^{-1}\big)\big(\vec{v}_0\partial_1^*\varphi\big) \ . \tag{2.3.13}$$

Using (2.3.12) the relation (2.3.13) is equivalent to

$$\partial_2^*\varphi\partial_0^*\varphi = \partial_1^*\varphi$$

which agrees with the cocycle condition (2.1.1) on φ from descent theory.

The assignment $(X', \varphi) \mapsto \mathrm{disc}(X', \varphi)$ is clearly functorial. Hence we have defined a functor. To define an inverse let $M \in \mathsf{lcs}\big(E, (\mathscr{G}, \alpha)\big)$ and determine $X' \in \mathsf{Rev}(T_0)$ as $X'|_v$ corresponding to M_v for all $v \in E_0$. The boundary maps from edges to vertices define a $\varphi : \mathrm{T}(\partial_0)^*X' \to T(\partial_1)^*X'$ and the cocycle condition holds for this (X', φ) just by reversing the above argument that proved that disc yields locally constant systems. □

Corollary 2.3.2. *Let $h : S' \to S$ be an effective descent morphism for* Rev*. Assume that S is connected and S, S' are locally noetherian. Let $S' = \coprod S'_v$ be the decomposition into connected components. Let $\overline{s}$ be a geometric point of S, let $\overline{s}(t)$ be a geometric point of the simplex $t \in \pi_0\big(S_{\leq 2}(h)\big)$, and let T be a maximal tree in the graph $\Gamma = \pi_0\big(S_{\leq 2}(h)\big)_{\leq 1}$. For every boundary map $\partial : t \to t'$ let $\gamma_{t',t} :\ \overline{s}(t') \to S_{\leq 2}(h)\big(\partial\big)\overline{s}(t)$ be a fixed path in the sense of algebraic paths between base points. Then canonically with respect to all these choices*

$$\pi_1(S, \overline{s}) \cong \Big(\mathop{\ast}_{v \in \pi_0(S')} \pi_1\big(S'_v, \overline{s}(v)\big) \ast \widehat{\pi}_1\big(\Gamma, T\big)\Big) / H$$

where H is the normal subgroup generated by the cocycle and edge relations

$$\boxed{\begin{array}{c}\pi_1(\partial_1)(g)\vec{e} = \vec{e}\pi_1(\partial_0)(g) \\ (\overrightarrow{\partial_2 f})\alpha^{(f)}_{102}(\alpha^{(f)}_{120})^{-1}(\overrightarrow{\partial_0 f})\alpha^{(f)}_{210}(\alpha^{(f)}_{201})^{-1}(\overrightarrow{\partial_1 f})^{-1}\alpha^{(f)}_{021}(\alpha^{(f)}_{012})^{-1} = 1\end{array}}$$

for all parameter values $e \in S_1(h)$, $g \in \pi_1\big(e, \overline{s}(e)\big)$, and $f \in S_2(h)$. The map $\pi_1(\partial_i)$ uses the fixed path $\gamma_{\partial_i(e),e}$ and finally $\alpha^{(f)}_{ijk}$ is defined using $v \in S_0(h)$ and $e \in S_1(h)$ determined by $v_i(f) = v$, $\{\partial_0(e), \partial_1(e)\} = \{v_i(f), v_j(f)\}$ as

$$\alpha^{(f)}_{ijk} = \gamma_{v,e}\gamma_{e,f}\big(\gamma_{v,f}\big)^{-1} \in \pi_1\big(v, \overline{s}(v)\big) \ .$$

Proof: Definition 2.1.2, Proposition 2.3.1, Theorem 2.2.9. Compare also with IX Theorem 5.1 [SGA 1]. □

Corollary 2.3.3. *Let $h : S' \to S$ be an effective descent morphism for* Rev. *Assume that S is connected and S, S' are locally noetherian. Let $S' = \coprod_v S'_v$ be the decomposition into connected components and assume that $h|_{S'_v} : S'_v \to S$ is a monomorphism.*

With the analogous choices of base points and paths connecting them as above and with a maximal tree T in the graph $\Gamma = \pi_0\big(S^{\preceq}_{\leq 2}(h)\big)_{\leq 1}$ we have canonically with respect to all these choices an isomorphism

$$\pi_1(S, \overline{s}) \cong \Big(\mathop{\ast}_{v \in \pi_0(S')} \pi_1\big(S'_v, \overline{s}(v)\big) \ast \widehat{\pi}_1\big(\Gamma, T\big) \Big) / H$$

where H is the normal subgroup generated by the cocycle and edge relations for the respective group data on $S^{\preceq}_{\leq 2}(h)$.

Proof: In addition to the above we involve also Proposition 2.1.5. □

Examples: (1) The original Van Kampen Theorem requires a refinement of the above for discrete groups acting on arbitrary (not necessarily finite) sets. But then with $\mathscr{F} = \mathsf{Rev}^{\mathrm{top}}$ being the category of topological covering maps fibred above the category of topological spaces the Corollary 2.3.3 yields the following. Let $X = U_1 \cup U_2$ be an open covering of a topological space such that U_1, U_2 and $U = U_1 \cap U_2$ are path-connected. The map $U_1 \coprod U_2 \to X$ is an effective descent morphism for $\mathsf{Rev}^{\mathrm{top}}$ composed of monomorphisms. When we choose a base point $\ast \in U$ simultaneously for X, U_1, U_2, U all paths $\gamma_{\cdot,\cdot}$ and α vanish. The simplicial set for this descent situation has the interval as its geometric realisation. Hence

$$\pi_1^{\mathrm{top}}(X, \ast) \cong \pi_1^{\mathrm{top}}(U_1, \ast) \ast_{\pi_1^{\mathrm{top}}(U,\ast)} \pi_1^{\mathrm{top}}(U_2, \ast) \ .$$

(2) Let $\coprod h_v : \coprod S'_v \to S$ be an effective descent morphism for Rev composed of monomorphisms h_i. Assume moreover that every 1-simplex is simply connected and $\big(S^{\preceq}_{\leq 2}(\coprod h_v)\big)_2$ is empty. This holds, for example, in the case of a semistable curve over an algebraically closed field with smooth irreducible components with respect to the covering by the disjoint union of its irreducible components. Let Γ be the graph $\big(\pi_0(S^{\preceq}_{\leq 2}(\coprod h_v))\big)_{\leq 1}$ as above which in the case of a semistable curve is just its dual graph. Then

$$\pi_1(S, \overline{s}) \cong \mathop{\ast}_{v \in \pi_0(S')} \pi_1\big(S'_v, \overline{s}(v)\big) \ast \widehat{\pi}_1\big(\Gamma, T\big) \ .$$

(3) Let $h : S' \to S$ be an effective descent morphism for Rev such that S' is connected and simply connected. Let $E = \pi_0\big(S_{\leq 2}(h)\big)$ be the corresponding 2-complex. The edge relation holds for trivial reasons and the cocycle relation becomes the path homotopy relation for the combinatorial fundamental group of the simplicial space E. Let $T \subset E$ be a maximal subtree. Then

$$\pi_1(S, \bar{s}) \cong \widehat{\pi}_1\big(E, T\big) \ .$$

These assumptions are satisfied for a reduced, irreducible, proper, but singular curve over an algebraically closed field k which is birational to $\mathbb{P}^1_k$ with respect to its normalisation map. If X is semistable with a single component whose normalisation has genus 0 then

$$\pi_1(X) \cong \widehat{Fr}\big(\operatorname{Sing}(X)(k)\big) \ .$$

(4) This may also be applied to number theoretic questions. Let L/K be a Galois extension of algebraic number fields with Galois group G and $\mathfrak{o}_L, \mathfrak{o}_K$ the respective rings of integers. Then the natural map $\operatorname{Spec}(\mathfrak{o}_L) \to \operatorname{Spec}(\mathfrak{o}_K)$ is an effective descent morphism for Rev. Let $S'' = \operatorname{Spec}(\mathfrak{o}_L \otimes_{\mathfrak{o}_K} \mathfrak{o}_L)$ which has $\#G$ irreducible components:

$$\coprod_G \operatorname{Spec}(\mathfrak{o}_L) \xrightarrow{(id,g)} S'' \ .$$

Connectedness of S'' is equivalent to the inertia subgroups generate G which we will assume for the sequel, e.g., $\pi_1(\mathfrak{o}_K) = 1$ or even $K = \mathbb{Q}$. Then $\operatorname{Spec}(\mathfrak{o}_L \otimes_{\mathfrak{o}_K} \mathfrak{o}_L \otimes_{\mathfrak{o}_K} \mathfrak{o}_L)$ is connected as well and the topological part of the Van Kampen formula vanishes. If we select the base points on the diagonals then we may choose all $\alpha = 1$. Thus

$$\pi_1(\mathfrak{o}_L \otimes_{\mathfrak{o}_K} \mathfrak{o}_L) \ \rightrightarrows\ \pi_1(\mathfrak{o}_L) \longrightarrow \pi_1(\mathfrak{o}_K) \tag{2.3.14}$$

is exact for pro-finite groups.

For example, let $L = \mathbb{Q}(\zeta_{p^n})$ and $K = \mathbb{Q}$. Then there is a unique ramified point $\mathfrak{p}|p$ whose inertia coincides with the Galois group $G = (\mathbb{Z}/p^n)^*$. We choose $p, \mathfrak{p}, (\mathfrak{p}, \mathfrak{p})$ as base points for $\operatorname{Spec}(\mathbb{Z}), \operatorname{Spec}(\mathbb{Z}[\zeta_{p^n}]), S''$. For the covering of S'' by its connected components the Van Kampen Theorem gives

$$\pi_1(S'') = \Big(\mathop{\ast}_{g \in G} \pi_1\big(\operatorname{Spec}(\mathbb{Z}[\zeta_{p^n}]), \mathfrak{p}\big) \Big) / \text{identify all } Frob_{\mathfrak{p}} \ .$$

and consequently the exact sequence (2.3.14) yields a statement about the coinvariants:

$$\big(\pi_1(\mathbb{Z}[\zeta_{p^n}])\big)_{G(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})} = \pi_1(\mathbb{Z}) = 1 \ .$$

Of course, thus also the coinvariants of the abelianised group which is the Galois group of the big Hilbert class field or also the coinvariants of the class group vanish.

The above statement about the coinvariants may be obtained directly by field theory and even group theory. For that purpose let L/K be a finite G-Galois extension of number fields such that there is a unique totally ramified prime $\mathfrak{q}|\mathfrak{p}$ and all the other primes are unramified. Here one has to choose how to treat the infinite primes. But either choice – asking real places to stay real or completely ignoring the infinite primes – does work with the following. Let L^{nr} (resp. K^{nr}) be the maximal unramified extension of L (resp.

K) and let $\Pi = G(L^{\mathrm{nr}}/L)$, $\pi = G(K^{\mathrm{nr}}/K)$ and $\Gamma = G(L^{\mathrm{nr}}/K)$ be the respective Galois groups. The extension

$$1 \to \Pi \to \Gamma \to G \to 1 \tag{2.3.15}$$

can be split using inertia groups of primes above $\mathfrak{q}$ in L^{nr}. Let $\Pi_G = \Pi/\langle\langle (g.\sigma)\sigma^{-1}\rangle\rangle$ be the coinvariants of the induced action of G on Π by a fixed splitting s. Let L_G^{nr}/L denote the corresponding Galois extension. It is even Galois over K with Galois group $\Pi_G \times G$. As all inertia groups of L_G^{nr}/K equal $1 \times G$, the fixed field corresponding to G is unramified over K. Hence, it coincides with K^{nr} as the restriction $\Pi \twoheadrightarrow \pi$ is surjective L/K not having unramified subextensions. Therefore $\Pi_G = \pi$.

Now we take only a split extension Γ as in (2.3.15) and claim that the natural map $\Pi_G \to \Gamma/\langle\langle s(G)\rangle\rangle$ is an isomorphism. This clearly yields a group-theoretical proof of our statement on coinvariants. Indeed, one checks easily that Π_G is the quotient of Π by the subgroup

$$N = \langle \sigma g \sigma^{-1} \tau g^{-1} \tau^{-1}; \sigma, \tau \in \Pi, g \in s(G)\rangle \ .$$

However, the quotient of Γ by the conjugates of $s(G)$ is isomorphic to the quotient of Π by

$$M = \langle \prod \tau_i g_i \tau_i^{-1}; \tau_i \in \Pi, g_i \in s(G), \prod g_i = 1\rangle \ .$$

But N being also normal in Γ and being contained in M we may consider M/N in the extension

$$1 \to \Pi_G \to \Gamma/N \to G \to 1$$

where $\tau g \tau^{-1} \equiv g$ and thus $M/N = 1$. This proves the assertion.

The true topological component. Finally, maps between 2-complexes with group data yield functors between categories of locally constant systems in the reverse direction by pullback. The forgetful map $\big(E, (\mathscr{G}, \alpha)\big) \to \big(E, 1\big)$ where 1 is the trivial group data such that all $\mathscr{G}(s) = 1$ and all $\alpha = 1$ yields an exact functor

$$\mathsf{cov}(|E|) \cong \mathsf{lcs}(E, 1) \to \mathsf{lcs}\left(E, (\mathscr{G}, \alpha)\right)$$

between Galois categories. It is easily verified to be fully faithful. Consequently, we obtain a surjection $\pi_1(E, \mathscr{G}) \twoheadrightarrow \widehat{\pi}_1^{\mathrm{top}}(|E|)$.

Chapter 3

Logarithmic Geometry

3.1 The basics of logarithmic geometry

Logarithmic geometry was invented by Fontaine-Illusie [Ka89] and developped basically by Kato [Il94]. Its purpose was to treat certain mild singularities as smooth objects of another geometric world. The additional structure works with monoids and their "geometry". Most of the material in this chapter is not new, e.g., the introduction of logarithmic structures, logarithmic blow-ups, the logarithmic fundamental group, topological invariance of the Kummer étale site and the logarithmic specialisation map for logarithmic fundamental groups. But some aspects appear to be new, or at least are not well documented in the literature, e.g., descent theory for Kummer étale covers, the logarithmic Van Kampen Theorem and logarithmic smooth reduction of curves.

From the results obtained in this chapter we need mainly the following. The effective descent result combines with the abstract Van Kampen Theorem of Chapter 2 to a logarithmic Van Kampen formula for the log fundamental group of a stable curve in (6.2.5). Furthermore, the theory of logarithmic good reduction will be used for controlling group-theoretically the logarithmic specialisation map.

To learn about logarithmic geometry, instead of the following, the reader is advised to read the wonderful accounts [Ka89] and [Il02] on the subject. The author wants to emphasize that he learnt an enormous amount of the material while studying [Vi01] and Chapter I [Vi02].

3.1.1 The very basics

Monoids. All monoids P are assumed to be commutative and possess units which are preserved by morphisms. The forgetful functor from abelian groups to monoids has a left adjoint $P \mapsto P^{\mathrm{gp}}$ (sometimes called the Grothendieck group of P) and a right adjoint $P \mapsto P^{\times}$ = the set of invertible elements. A monoid is called integral if the cancellation law holds, i.e., $ac = bc \Rightarrow a = b$, or equivalently if the canonical map $P \to P^{\mathrm{gp}}$ is injective. The image of the latter is denoted by P^{int} and yields a left adjoint for the inclusion of integral monoids into monoids. An integral monoid P is called saturated if $a \in P^{\mathrm{gp}}$ belongs to P if and only if a power a^n for some $n \geq 1$ belongs to P. There is a saturation functor $P^{\mathrm{sat}} = \{a \in P^{\mathrm{gp}} \mid \exists n \geq 1 : a^n \in P^{\mathrm{int}}\}$ which is again a left adjoint of a forgetful functor. Furthermore the category of monoids (resp. integral or saturated monoids) possesses pushouts and fibre products.

Let P be a monoid. Its monoidring $R[P]$ over a ring R consists of a free R module with basis P and inherits a ring structure by R-linear continuation of monoid composition. For example $R[\mathbb{N}] = R[t]$ is the polynomial ring in one variable.

We call a finitely generated and integral monoid fine and abbreviate fine and saturated by fs. If P is finitely generated as a monoid the same holds for P^{int} and P^{gp}.

Lemma 3.1.1 (Gordon's Lemma). *Let P be a finitely generated monoid. Then P^{sat} is an fs monoid.*

Proof: We may assume P is fine and study $\mathrm{Spec}(k[P])$ for a field of characteristic 0. It is reduced as it contains a schematically dense subscheme $\mathrm{Spec}(k[P^{\mathrm{gp}}])$ which is smooth over $\mathrm{Spec}(k[P^{\mathrm{gp}}_{\mathrm{tors}}])$ which itself is étale over k.

We set $Q = P^{\mathrm{sat}}$ and study $i : k[P] \subset k[Q]$. As $P^{\mathrm{gp}} = Q^{\mathrm{gp}}$ the map i is birational and integral and thus dominated by the normalisation $k[P]^{\mathrm{norm}}$ (in fact *is* the normalisation). By finiteness of integral closure for varieties i is even finite and hence $k[Q]$ is a noetherian $k[P]$ module. A finite set of generators from Q plus the generators of P hence generate Q. □

Log structures. A *pre-log structure* on a scheme X is a pair (M_X, α_X) consisting of an étale sheaf of monoids M_X and a map of sheaves of monoids $\alpha_X : M_X \to \mathcal{O}_X$. Here $\mathcal{O}_X$ carries the multiplicative monoid structure. If no confusion arises the indices are omitted. A pre-log structure is called a log structure if α_X induces an isomorphism $\alpha_X^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$. We identify $\mathcal{O}_X^*$ with its pre-image. Maps of (pre-)log structures on X are just maps of étale sheaves of monoids over $\mathcal{O}_X$. Finally, a log scheme (X, M_X, α_X) is a scheme X endowed with a log structure (M_X, α_X). For convenience, a log scheme is denoted by X. To specify its underlying scheme we use the notation $\mathring{X}$. A morphism $f : X \to Y$ of (pre-)log schemes consists of a map $\mathring{f} : \mathring{X} \to \mathring{Y}$ of schemes and a map $u_f : (\mathring{f})^{-1} M_Y \to M_X$ of monoid sheaves such that u and pullback by $\mathring{f}$ are compatible with the α's. The trivial log structure $\mathcal{O}_X^* \to \mathcal{O}_X$ is final (over the identity) for log structures on X and embeds the category of schemes into the category of log schemes. There is a natural map $\epsilon : X \to (\mathring{X}, \mathcal{O}_X^*)$.

To a pre-log structure on a scheme we can form an associated log structure which is universal with respect to maps to log structures, cf. 1.3 [Ka89]. This allows to pullback log structures as follows, cf. 1.4 [Ka89]. Let X be a scheme and $f : X \to \mathring{Y}$ be a map to a log scheme Y. The log structure $f^* M_Y$ on X is the log structure associated to $f^{-1} M_Y \to f^{-1} \mathcal{O}_Y \to \mathcal{O}_X$. It enhances X to a log scheme and f to a map of log schemes which is universal for maps from X endowed with any log structure to Y such that the scheme component is the original f. A map $f : X \to Y$ of log schemes is called strict if $u_f : f^* M_Y \to M_X$ is an isomorphism.

The procedures of pulling back and taking the associated log-structures commute, cf. 1.4.2 [Ka89]. Furthermore, if the quotient $M_X/\mathcal{O}_X^*$ is denoted by $\overline{M}_X$ then canonically $f^{-1}\overline{M}_Y = \overline{f^* M_Y}$, cf. 1.4.1 [Ka89]. The sheaf $\overline{M}_X$ carries the additional information of "local phases" whereas M_X encodes how the phases extend the classical geometry.

Examples. (1) The basic example is the following. Let P be a monoid and R a ring. On $X = \mathrm{Spec}(R[P])$ we have a natural map from the constant sheaf of monoids P_X to $\mathcal{O}_X$ which by adjointness is given as the inclusion $P \subset R[P]$ on global sections. We consider the associated log structure and agree that $\mathrm{Spec}(R[P])$ will always mean the respective associated log scheme. The log-scheme $\mathrm{Spec}(\mathbb{Z}[P])$ represents the following contravariant functor on log schemes: $T \mapsto \mathrm{Hom}\big(P, M_T(T)\big)$.

(2) Let X be a normal scheme with a divisor $D \subset X$. Let $j : X \setminus D \to X$ be the open immersion of the "trivial" part. Then the log structure $M(\log D)$ on X is defined as the canonical map $M_X = (j_* \mathcal{O}^*_{X\setminus D} \cap \mathcal{O}_X) \hookrightarrow \mathcal{O}_X$.

Fs-log schemes. A log scheme X is said to have a coherent log structure if étale locally there exist charts, i.e., strict maps $U_i \to \mathrm{Spec}(\mathbb{Z}[P_i])$ for an étale covering $\{U_i \to X\}_{i\in I}$, such that the P_i are finitely generated monoids. It is furthermore called a fine log scheme if the monoids P_i can be chosen to be integral. Finally, X is a fs-log scheme if, moreover, the charts are built from saturated monoids, cf. 1.7 [Na97]. We will work mainly in the category of fs-log schemes. In the second example above X with log structure $M(\log D)$ is fs if for example X is regular, cf. Lemma 3.4.2.

The saturation of a fine log scheme X is étale locally given in the presence of a chart $X \to \mathrm{Spec}(\mathbb{Z}[P])$ as the schematic fibre product $X^{\mathrm{sat}} = X \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[P^{\mathrm{sat}}])$ with log structure induced by the second projection. These local data glue together as they describe a universal object for maps of fs-log schemes into X. The resulting scheme X^{sat} yields a right adjoint for the inclusion of fs-log schemes into log schemes, cf. 1.8 [Na97]. The natural map $X^{\mathrm{sat}} \to X$ is finite and of finite presentation by Lemma 3.1.1.

For a (quasi-compact) fs-log scheme X the sheaf $\overline{M}_X$ is a constructible sheaf of finitely generated saturated monoids without invertible elements. In particular, it does not contain torsion. The same is valid for its associated sheaf of groups $\overline{M}^{\mathrm{gp}}_X$. Moreover, for any geometric point $\overline{x} \in X$ of an fs-log scheme there is always (noncanonically) a chart in an étale neighbourhood of $\overline{x}$ with the stalk monoid $\overline{M}_{\overline{x}}$ as the building block, cf. 2.10 [Ka89].

A chart for a map $f : X \to Y$ of fs-log schemes is a map of monoids $u : P \to Q$ together with charts $Y \to \mathrm{Spec}(\mathbb{Z}[P])$ and $X \to \mathrm{Spec}(\mathbb{Z}[Q])$ such that the following commutes.

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Spec}(\mathbb{Z}[Q]) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{Spec}(\mathbb{Z}[u])} \\ Y & \longrightarrow & \mathrm{Spec}(\mathbb{Z}[P]) \end{array}$$

By the proof of 2.6 [Ka89] charts always exist étale locally for maps between fs-log schemes.

Without fibre products we definitely were not satisfied. But fibre products exist in the category of fs-log schemes by 1.6, 2.6, 2.7 [Ka89] and 1.8 [Na97]. However, taking the underlying scheme does not commute with fibre products as the latter are constructed as follows. Let X, Y be fs-log schemes over S. Then $X \times^{\mathrm{fs}}_S Y$ is the saturation of the fine log scheme

$$\left(\overset{\circ}{X} \times_{\overset{\circ}{S}} \overset{\circ}{Y}, pr_X^{-1} M_X \oplus_{pr_S^{-1} M_S} pr_Y^{-1} M_Y \right) .$$

In particular there is a finite map $(X \times^{\mathrm{fs}}_S Y)^{\circ} \to \overset{\circ}{X} \times_{\overset{\circ}{S}} \overset{\circ}{Y}$ of finite presentation which in general is not an isomorphism and may even be not surjective. In the special case where $X \to S$ is strict the underlying scheme of the fs fibre product is the schematic fibre product and the projection to the factor Y is again strict (strict base extension commutes with taking the underlying scheme and strictness is stable under fs base change).

A map $u : P \to Q$ of integral monoids is called exact if $P = (u^{\mathrm{gp}})^{-1}(Q)$ in P^{gp}. Furthermore a map $u : P \to Q$ of fs monoids is called Kummer if it is injective and

$Q^{\mathrm{gp}}/P^{\mathrm{gp}}$ is torsion. For a Kummer map the order of $Q^{\mathrm{gp}}/P^{\mathrm{gp}}$ is called its index. A map $f : X \to Y$ of fs-log schems is called exact (resp. Kummer) if the map $f^{-1}\overline{M}_Y \to \overline{M}_X$ is exact (resp. Kummer) in all stalks. Kummer maps are exact.

3.1.2 The Kummer étale site

Log-smooth, log-étale. There is a notion for morphisms between fs-log schemes of being log-étale or log-smooth following the classical analogues by imposing certain (unique) lifting properties, cf. 3 [Ka89], and being locally of finite presentation. By Theorem 3.5 [Ka89] a map $f : X \to Y$ is log-smooth (resp. log-étale) if and only if étale locally on X there are charts $u : P \to Q$ such that

(i) the kernel and the torsion part of the cokernel (resp. the kernel and the cokernel) of $u^{\mathrm{gp}} : P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$ are finite groups of orders invertible on X, and

(ii) the induced morphism $X \to Y \times^{\mathrm{fs}}_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[Q])$ is smooth (resp. étale) in the classical sense.

The epitome of a log-étale map is $\mathbb{A}^1_k \to \mathbb{A}^1_k; t \mapsto t^n$, where n is prime to the exponential characteristic of k and $\mathbb{A}^1_k = \mathrm{Spec}(k[\mathbb{N}])$ carries its natural log structure. Thus the notion of a log-étale morphism generalises tame ramification and turns it into a "smooth property".

By Proposition 3.5 [Ka89] a strict map of fs-log schemes is log-étale (resp. log-smooth) if and only if the underlying map of schemes is étale (resp. smooth). A map which is Kummer and log-étale is called Kummer log-étale or két for short. For a log-étale map being Kummer is equivalent to being exact. Being log-smooth, log-étale or két is stable under composition and fs base extension.

Example. Let $u : P \hookrightarrow Q$ be an injective Kummer map of fs monoids, $\Gamma = Q^{\mathrm{gp}}/P^{\mathrm{gp}}$, $N = \#\Gamma$, and $R = \mathbb{Z}[\frac{1}{N}]$. Let $\varphi : X \to \mathrm{Spec}(R[P])$ be a chart for the fs-log scheme X. Let X_Q denote the fs base change $X \times^{\mathrm{fs}}_{\mathrm{Spec}(R[P])} \mathrm{Spec}(R[Q])$. Then $\mathrm{pr}_1 : X_Q \to X$ is két. Being also finite we call it a standard két cover (associated to the chart φ and u). The X-fs-log scheme X_Q represents the following contravariant functor on X-fs-log schemes: $T \mapsto \mathrm{Hom}_P\big(Q, M_T(T)\big)$.

The két site. One defines the (small) Kummer étale site $X_{\text{két}}$ for a fs-log scheme X as the category of all X-fs-log schemes which are két over X endowed with the Grothendieck topology of settheoretically covering families. Surprisingly, the crucial point with this definition is the stability under fs base extension of being a settheoretically covering family which is not obvious but guaranteed by Nakayama's 4 Point Lemma, cf. 2.2.2 [Na97] 2.2 [Il02].

Lemma 3.1.2 (4 Point Lemma). *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian square of fs-log schemes, and let $\overline{x} \in X, \overline{y}' \in Y'$ be geometric points such that $g(\overline{y}') = \overline{y} = f(\overline{x})$. Assume that the inverse image of $\overline{M}_{Y',\overline{y}'} \oplus \overline{M}_{X,\overline{x}}$ under

$$\overline{M}^{\mathrm{gp}}_{Y,\overline{y}} \to \overline{M}^{\mathrm{gp}}_{Y',\overline{y}'} \oplus \overline{M}^{\mathrm{gp}}_{X,\overline{x}}; \quad a \mapsto (u_g(a), u_f(a)^{-1}) \tag{3.1.1}$$

is trivial (this holds, for example, when either f or g is exact over $\overline{y}$). Then there is a geometric point $\overline{x}' \in X'$ which maps to $\overline{y}'$ and $\overline{x}$ under f' and g'. □

All X-maps between Kummer log-étale X-fs-log schemes are automatically két morphisms themselves by 1.3 [Vi01]; in particular they are exact. Therefore the condition (3.1.1) is always satisfied. This proves that surjectivity of két coverings is stable under fs base extension.

A map $f : X \to Y$ between fs-log schemes determins by pullback a functor $f^{-1} : Y_{\text{két}} \to X_{\text{két}}$ which commutes with fibre products and is continuous by Lemma 3.1.2. Hence it induces a map of sites $f_{\text{két}} : X_{\text{két}} \to Y_{\text{két}}$. Kummer étale cohomology is the cohomology of sheaves of abelian groups on this site $X_{\text{két}}$, cf. [Na97]. It behaves as the cohomology of a constructible torus fibration over the classical étale site X_{et}, cf. 5 [Il02].

3.1.3 The local structure of a két morphism

The proof of Proposition 4.1 [Vi01] allows to formulate a structure theorem for két morphisms, cf. 1.6 [Il02]. But first we recall the key property which holds for fs monoids but not for fine monoids.

Lemma 3.1.3. *Let $u : P \to Q$ be an injective Kummer map of fs monoids, and set $\Gamma = Q^{\mathrm{gp}}/P^{\mathrm{gp}}$. Then the natural map*

$$(Q \oplus_P Q)^{\mathrm{sat}} \to Q \oplus \Gamma; \quad (a, b) \mapsto (a + b, b)$$

is an isomorphism of fs monoids (we do not distinguish between b and its class in Γ).

Proof: Lemma 3.3 [Il02]. □

Proposition 3.1.4. *Let $f : Y \to X$ be két. Then f is étale locally over the image of f in X a standard két morphism. More precisely, let $\overline{y} \in Y, f(\overline{y}) = \overline{x} \in X$ be geometric points and $P = \overline{M}_{\overline{x}}$. Then there is an étale neighbourhood $\overline{x} \in U \to X$ and a Zariski neighborhood $\overline{y} \in V \subset f^{-1}(U)$ such that $f|_V : V \to U$ is isomorphic to some $U_Q = U \times^{\mathrm{fs}}_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[Q]) \xrightarrow{\mathrm{pr}_1} U$ for a chart $\varphi : U \to \mathrm{Spec}(\mathbb{Z}[P])$ and an injective Kummer map $u : P \to Q$ of fs monoids without invertible elements and with $\#(Q^{\mathrm{gp}}/P^{\mathrm{gp}})$ invertible on U.*

Proof: Using $\varinjlim$-arguments of Proposition 4.3 [Vi01] we may assume that $X = \mathrm{Spec}(R)$ for some noetherian strict henselian local ring R such that its closed point localises $\overline{x}$. Let $P = \overline{M}_{\overline{x}}$ and choose a chart $X \to \mathrm{Spec}(\mathbb{Z}[P])$. By Lemma 1.2 [Vi01] the induced map $P = \overline{M}_{\overline{x}} \to \overline{M}_{\overline{y}} = Q$ is an injective Kummer map of the required type. The scheme X_Q is finite over X and local (P maps to the maximal ideal of R), hence the spectrum of a noetherian strict henselian local ring with closed point x_Q. Let $\overline{x}_Q$ be a lift of $\overline{x}$ to X_Q.

We consider $W = Y \times^{\mathrm{fs}}_X X_Q$ and its projections $\mathrm{pr}_1, \mathrm{pr}_2$. By the 4 Point Lemma 3.1.2 there is a geometric point $\overline{w} \in W$ that projects to $\overline{y}$ and $\overline{x}_Q$. By 2.1.1 [Na97] and Lemma 3.1.3 both projections are strict in $\overline{w}$ and thus étale in the classical sense. Therefore pr_2 has a section s as a morphism of schemes mapping $\overline{x}_Q$ to $\overline{w}$. But that implies that $s(X_Q)$ is an open and closed neighborhood of $\overline{w}$ in W, where pr_2 maps strict isomorphically to X_Q. Hence s is a section for maps of fs-log schemes.

Let us consider the factorisation $X_Q \xrightarrow{\mathrm{pr}_1 s} Y \xrightarrow{f} X$ of the canonical map which is finite. Therefore $\mathrm{pr}_1 s$ is finite étale and $\mathrm{pr}_1 s(X_Q) = V$ is an open, closed and affine (because local) neighbourhood of $\overline{y}$. Clearly for global sections $R \to \mathcal{O}(V) \subset \mathcal{O}(X_Q)$ which proves that V is finite over X and hence also the spectrum of a noetherian strict henselian local ring. Thus $X_Q \to V$ is an isomorphism. □

Analogously, the proof of Proposition 4.2 [Vi01] yields the following.

Proposition 3.1.5. *Let $f : Y \to X$ be log-étale. Then f is étale locally over the image of f in X of the following type: let $\overline{y} \in Y, f(\overline{y}) = \overline{x} \in X$ be geometric points and $P = \overline{M}_{\overline{x}}$. There is an étale neighbourhood $\overline{x} \in U \to X$ and a Zariski neighborhood $\overline{y} \in V \subset f^{-1}(U)$ such that $f|_V : V \to U$ is isomorphic to some $U_Q = U \times^{\mathrm{fs}}_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[Q]) \xrightarrow{\mathrm{pr}_1} U$ for a chart $\varphi : U \to \mathrm{Spec}(\mathbb{Z}[P])$ and an injective map $u : P \to Q$ of fs monoids without invertible elements and with $\#(Q^{\mathrm{gp}}/P^{\mathrm{gp}})_{\mathrm{tors}}$ invertible on U.*

Proposition 3.1.6. *Let $Y = X_Q \to X$ be a standard két morphism for some chart $\varphi : X \to \mathrm{Spec}(\mathbb{Z}[P])$ and Kummer map $u : P \hookrightarrow Q$, such that $\Gamma = Q^{\mathrm{gp}}/P^{\mathrm{gp}}$ yields a finite (classically) étale diagonalizable group scheme $D(\Gamma)_X = \mathrm{Spec}(\mathcal{O}_X[\Gamma])$ over X. Then $Y \to X$ is a két torsor under $D(\Gamma)_X$, i.e., the natural (right) action $\mu : Y \times^{\mathrm{fs}}_X D(\Gamma)_X \to Y$ induced by*

$$Q \to Q \oplus_P \Gamma; \quad b \mapsto (b, b)$$

yields an isomorphism $(\mathrm{pr}_1, \mu) : Y \times^{\mathrm{fs}}_X D(\Gamma)_X \to Y \times^{\mathrm{fs}}_X Y$, cf. a két variant of XI 4.2 [SGA 1].

Furthermore, standard két morphisms are open, finite and surjective on the underlying schemes, and preserve these properties under arbitrary fs base change. Finally, general két morphisms are open.

Proof: This is Proposition 3.2 [Il02]. By fs base change one reduces to $X = \mathrm{Spec}(\mathbb{Z}[P])$. With

$$\mathrm{Spec}(\mathbb{Z}[Q]) \times^{\mathrm{fs}}_{\mathrm{Spec}(\mathbb{Z}[P])} D(\Gamma)_{\mathrm{Spec}(\mathbb{Z}[P])} = \mathrm{Spec}(\mathbb{Z}[Q \oplus \Gamma])$$

$$\mathrm{Spec}(\mathbb{Z}[Q]) \times^{\mathrm{fs}}_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[Q]) = \mathrm{Spec}(\mathbb{Z}[(Q \oplus_P Q)^{\mathrm{sat}}])$$

the map (pr_1, μ) is induced by $(a, b) \mapsto (a + b, b)$ which is an isomorphism by Lemma 3.1.3.

The asserted properties of standard két morphisms follow by an argument of Kato, cf. Proposition 3.2 [Il02]. Being an open morphism is an étale local property, hence Proposition 3.1.4 implies that also general két morphisms are open. □

Corollary 3.1.7. *Quasi-compact Kummer log-étale maps are fs-universally open and quasi-finite.* □

Being quasi-finite distinguishes the két morphisms among all log-étale maps by a result of Kisin.

Proposition 3.1.8 (Kisin). *Let $f : Y \to X$ be a log-étale map. Then f being quasi-finite is equivalent to f being Kummer.*

Proof: Proposition 1.7 [Ki00]. □

Finite két covers. Let X be a fs-log scheme. Let $\mathsf{Rev}^{\log}(X)$ be the category of all finite két X-fs-log schemes. This will be the category that is governd by the log fundamental group of X, cf. Section 3.3. As an example we discuss the case of a fs-log scheme X such that $\mathring{X}$ is the spectrum of a strict henselian local noetherian ring R with residue field k of exponential characteristic p. Set $P = \overline{M}_X$ which is an fs monoid without invertible elements, hence $P^{\mathrm{gp}} \cong \mathbb{Z}^r$ for some $r \in \mathbb{N}$. By 2.10 [Ka89] there is a chart $X \to \mathrm{Spec}(\mathbb{Z}[P])$ induced by a map $P \to R$ of multiplicative monoids factoring through the maximal ideal of R. With this choice of a chart $M_X = R^* \oplus P$. Similarly $X_Q = \mathrm{Spec}(R_Q)$ is strict étale local and has $M_Q = R_Q^* \oplus Q$. We define $\tilde{P} = \varinjlim_{(n,p)=1} \frac{1}{n} P \subset P^{\mathrm{gp}} \otimes \mathbb{Q}$ which yields by fibre product in log schemes an integral log scheme $\tilde{X} = X_{\tilde{P}} = X \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[\tilde{P}])$ playing the role of a universal két cover.

Lemma 3.1.9. *All endomorphisms of $\tilde{X}$ as an X-log scheme are automorphisms and*

$$\mathrm{Aut}_X(\tilde{X}) = \mathrm{Hom}\Big(P^{\mathrm{gp}}, \hat{\mathbb{Z}}(1)\big(k\big)\Big) .$$

In particular $\mathrm{Aut}_X(\tilde{X})$ is abelian. Here $\hat{\mathbb{Z}}(1)\big(k\big)$ means $\varprojlim_{n\in\mathbb{N}} \mu_n(k)$ and $\mu_n(k)$ is the group of n-th roots of unity in k.

Proof: $\tilde{X}$ is the spectrum of a strict étale local ring $\tilde{R}$ with log structure $M_{\tilde{X}}(\tilde{X}) = \tilde{R}^* \oplus \tilde{P}$.

$$\mathrm{End}_X(\tilde{X}) = \mathrm{Hom}_P(\tilde{P}, \tilde{R}^* \oplus \tilde{P}) = \left\{ \begin{pmatrix} \mathrm{id} & h \\ & \mathrm{id} \end{pmatrix} \in \mathrm{End}(\tilde{R}^* \oplus \tilde{P}) \mid h(P) = 1 \right\} .$$

The id in the lower row follows from $\tilde{P}^{\mathrm{gp}}$ being uniquely divisible by all n prime to p. Hence all endomorphisms are automorphisms and

$$\mathrm{Aut}_X(\tilde{X}) = \mathrm{Hom}_P\big(\tilde{P}, \tilde{R}^*\big) = \mathrm{Hom}\big(\tilde{P}^{\mathrm{gp}}/P^{\mathrm{gp}}, \mu(\tilde{R})\big)$$

$$= \varprojlim_{(n,p)=1} \mathrm{Hom}\big(P^{\mathrm{gp}} \otimes (\frac{1}{n}\mathbb{Z}/\mathbb{Z}), \mu_n(k)\big) = \mathrm{Hom}\Big(P^{\mathrm{gp}}, \hat{\mathbb{Z}}(1)\big(k\big)\Big)$$

where $\mu(\tilde{R}) = \mu(R) = \mu(k) = \bigcup_n \mu_n(k)$ is the set of all roots of unity. □

Proposition 3.1.10. *The functor*

$$F_{\tilde{X}} : \mathsf{Rev}^{\log}(X) \to \mathrm{Hom}\Big(P^{\mathrm{gp}}, \hat{\mathbb{Z}}(1)\big(k\big)\Big)\text{-sets}$$

which maps $Y \mapsto \mathrm{Hom}_X(\tilde{X}, Y)$ is an equivalence of categories.

Proof: Of course $\mathrm{Hom}\Big(P^{\mathrm{gp}}, \hat{\mathbb{Z}}(1)\big(k\big)\Big)$ acts as $\mathrm{Aut}_X(\tilde{X})$ on $F_{\tilde{X}}(Y)$. Any object of $\mathsf{Rev}^{\log}(X)$ is by Proposition 3.1.4 a finite disjoint union of schemes X_Q for suitable Kummer extensions $P \hookrightarrow Q$ of fs monoids without invertible elements such that $Q^{\mathrm{gp}}/P^{\mathrm{gp}}$ is prime to p. Any such Q is uniquely imbedded in $\tilde{P}$ as monoid over P. Call this $i_Q : Q \hookrightarrow \tilde{P}$. Consequently, there is a canonical element $\pi_Q := (1, i_Q)$ in $F_{\tilde{X}}(X_Q) = \mathrm{Hom}_P(Q, \tilde{R}^* \oplus \tilde{P})$. In fact $\tilde{X} = \varprojlim X_Q$ where the transfer maps $\pi_{Q_1,Q_2} : X_{Q_1} \to X_{Q_2}$ are induced in the same vain in case $Q_2 \subset Q_1$ as subsets of $\tilde{P}$. The natural projection $\tilde{X} \to X_Q$ is π_Q.

For a prime to p torsion group A let $\mathrm{Hom}(A, \mu(k))$ be denoted by $A^\vee$ and call it the Pontrjagin dual. Indeed, it is uncanonically isomorphic to the latter. Now

$$F_{\tilde{X}}(X_Q) = \mathrm{Hom}_P(Q, \tilde{R}^* \oplus \tilde{P}) = \mathrm{Hom}\big(Q^{\mathrm{gp}}/P^{\mathrm{gp}}, \mu(k)\big) = (Q^{\mathrm{gp}}/P^{\mathrm{gp}})^\vee \ .$$

Therein π_Q is the zero element and $\mathrm{Aut}_X(\tilde{X}) = (\tilde{P}^{\mathrm{gp}}/P^{\mathrm{gp}})^\vee$ acts via translation after restriction by means of i_Q. Consequently $F_{\tilde{X}}$ takes values in finite sets (with group action) so $F_{\tilde{X}}$ is well defined. Moreover the action on $F_{\tilde{X}}(X_Q)$ is transitive.

We prove essential surjectivity. A transitive action of $(\tilde{P}^{\mathrm{gp}}/P^{\mathrm{gp}})^\vee$ on a finite set M amounts to a finite quotient which by Pontrjagin duality corresponds to a finite subgroup $A \subset \tilde{P}^{\mathrm{gp}}/P^{\mathrm{gp}}$. Let $Q = \{a \in \tilde{P} \mid (a \mod P^{\mathrm{gp}}) \in A\}$ which is contained in $\frac{1}{n}P$ for an appropriate n. Hence Q is an fs monoid, $Q^{\mathrm{gp}}/P^{\mathrm{gp}} = A$ and clearly $F_{\tilde{X}}(X_Q) = M$.

As $\tilde{X}$ is local it is sufficient to consider only connected covers to prove fully faithfulness, i.e., X_Q's. Now in

$$F_{\tilde{X}} : \mathrm{Hom}_X(X_{Q_1}, X_{Q_2}) \to \mathrm{Hom}_{(\tilde{P}^{\mathrm{gp}}/P^{\mathrm{gp}})^\vee\text{-}\mathsf{sets}}\big((Q_1^{\mathrm{gp}}/P^{\mathrm{gp}})^\vee, (Q_2^{\mathrm{gp}}/P^{\mathrm{gp}})^\vee\big)$$

for both sides being empty corresponds to $Q_2 \not\subset Q_1$ in $\tilde{P}$. But if $Q_2 \subset Q_1$ then $F_{\tilde{X}}(\pi_{Q_1,Q_2})$ is the group homomorphism obtained by restriction with $i_{Q_2} : Q_2 \subset Q_1$ and both sides are principally homogeneous under

$$\mathrm{Aut}_X(X_{Q_2}) = \mathrm{Hom}\big(Q_2^{\mathrm{gp}}/P^{\mathrm{gp}}, \mu(k)\big) = (Q_2^{\mathrm{gp}}/P^{\mathrm{gp}})^\vee$$

respectively. □

Corollary 3.1.11. *Let X be a strict henselian local noetherian fs-log scheme with closed point $\overline{x}$ and residue field k. Then with $\tilde{X}$ as above $\mathsf{Rev}^{\log}(X)$ is a Galois category with fibre functor $F_{\tilde{X}}$. Its fundamental group is canonically*

$$\pi_1\big(\mathsf{Rev}^{\log}(X), F_{\tilde{X}}\big) = \mathrm{Hom}\Big(\overline{M}_{\overline{x}}^{\mathrm{gp}}, \hat{\mathbb{Z}}(1)\big(k\big)\Big)$$

which is a free $\hat{\mathbb{Z}}(1)\big(k\big)$ module of finite rank. □

3.1.4 Topological invariance

This section discusses Kummer universal homeomorphisms by giving a résumé of the results of [Vi01]. This will finally lead to a slight generalisation with the notion of a log-radicial map in Section 3.2.2. Log-radicial maps will be exploited in our treatment of descent theory as in [SGA 1].

Definition 3.1.12 (2.1 [Vi01]). *Let $q : X \to Y$ be a morphism of fs-log schemes. We call q a* **Kummer universal homeomorphism** *(abbreviated as "kuh") if q is Kummer and for all fs base changes of q the underlying map of schemes is a homeomorphism.*

Remark 3.1.13 (2.2 [Vi01]).
(1) If q is kuh then its underlying map of schemes is a universal homeomorphism of ordinary schemes as strict base change commutes with taking the underlying scheme. Hence q is integral, radicial and surjective, cf. 18.12.11 [EGA$_{IV}$].
(2) If q is strict and $\mathring{q}$ is a universal homeomorphism then q is kuh.
(3) Being a kuh morphism is stable under composition and fs base extension.

Theorem 3.1.14 (2.4 [Vi01]). *A morphism $q : X \to Y$ of fs-log schemes is kuh if and only if it satisfies the following conditions:*

(i) $\mathring{q}$ *is a universal homeomorphism of schemes, and*

(ii) for any geometric point $\overline{x}$ *with image* $q(\overline{x}) = \overline{y}$ *the morphism* $u_q : \overline{M}_{\overline{y}} \to \overline{M}_{\overline{x}}$ *is* p *inseparable (cf. 4.9 [Ka89]) where* p *is the exponential residue characteristic at* $\overline{x}$*.* □

The second condition just requires $\overline{M}_{\overline{y}} \to \overline{M}_{\overline{x}}$ to be injective and Kummer of p primary index.

Corollary 3.1.15 (2.7 [Vi01]).
(1) Kuh morphisms satisfy the property "two from three", i.e., if $h = g \circ f$ *is a composition of morphisms of fs-log schemes then* f, g, h *are kuh if at least two of them are kuh.*
(2) A morphism that is log-étale and kuh is necessarily an isomorphism. □

The main theorem of [Vi01] is the following effective descent statement. It constitutes the fs-log analogue of the topological invariance of the étale site, cf. VIII 1.1 [SGA 4].

Theorem 3.1.16 (0.1 [Vi01]). *Let* $q : X \to Y$ *be a kuh morphism of fs-log schemes. Then fs base change with* q *induces an equivalence* $q^{-1} : Y_{\text{két}} \to X_{\text{két}}$ *of Kummer étale sites.* □

3.2 Descent theory for log-étale covers

This section follows IX [SGA 1]. It merely contains logarithmic versions of the respective assertions which therefore will be named by the respective numbering from there with a *log* as a prefix.

We present descent theory for the category $\mathsf{Rev}^{\log}$ fibred over fs-log schemes. Its category of sections over X is $\mathsf{Rev}^{\log}(X)$. As we can pull back finite két covers this indeed yields a fibred category and descent theory deals with the properties of

$$f^* : \mathsf{Rev}^{\log}(X) \to \mathsf{DD}(f, \mathsf{Rev}^{\log})$$

for a map $f : X' \to X$ of fs-log schemes, cf. Section 2.1. We recall Definition 2.1.2: f is called a faithful (resp. descent, or even effective descent) morphism for $\mathsf{Rev}^{\log}$ if f^* is a faithful functor (resp. fully faithful functor, or even an equivalence of categories) between $\mathsf{Rev}^{\log}(X)$ and $\mathsf{DD}(f, \mathsf{Rev}^{\log})$. The adjective universal may be added as usual in case the respective property persists to hold after arbitrary base extension.

Our treatment of descent theory will provide the necessary effective descent assertion that goes into the application of the abstract Van Kampen Theorem in (6.2.5). What is more, descent theory admits a more local study of finite két covers that leads to a construction of the logarithmic fundamental group. This group is not new, but has already been constructed by different people.

This section contains new material and answers a question in [Vi02] where she asks for a good descent theory for Kummer étale covers and a logarithmic Van Kampen Theorem. For completeness, we mention that in [Il02] there is a hint that Fujiwara might have already developed a log Van Kampen Theorem for the case of curves (which would suffice for the application in this thesis). The given reference is in japanese so that it is out of reach for the author.

3.2.1 General facts

The comprehensive source for descent theory is [Gi64] which constitutes the "unpublished" Exposé VII of [SGA 1]. For the convenience of the reader we present the required results from that source. These results will later on allow formal reduction steps in proving effective descent statements.

However, [Gi64] uses a different terminology: For a fibred category $\varphi : \mathsf{F} \to \mathsf{E}$ and a map f in E it is defined in [Gi64] for f to be a morphism of F-i-descent (suppressed: with respect to φ) for $0 \leq i \leq 2$. By Theorem 9.3 and Theorem 9.10 of [Gi64] being F-0-decente (resp. F-1-descent or even F-2-descent) is equivalent to being a faithful (resp. descent or even an effective descent) morphism for F.

Lemma 3.2.1 (section-lemma). *Let $\varphi : \mathsf{F} \to \mathsf{E}$ be a fibred category such that E has fibre products. Let $r : S' \to S$ be a morphism in E that possesses a section g, i.e., a right inverse $rg = \mathrm{id}_S$. Then r is universally F-2-descent.*

Proof: Having a section is preserved under base extension. So it suffices to prove that morphisms with sections are F-2-descent. We use Theorem 10.4 [Gi64] with $f = \mathrm{id}, u = \mathrm{id}, r = r, g = g$. As g, g' are sections they are faithful and by the rows 4 and 5 of loc. cit. r is fully faithful. But then g being a right inverse is also fully faithful and thus row 6 applies. □

The property F-i-descent ($0 \leq i \leq 2$) may be checked locally with respect to universal F-2-descent morphisms:

Lemma 3.2.2. *Let $\varphi : \mathsf{F} \to \mathsf{E}$ be a fibred category such that E has fibre products. Let $f : S' \to S$ and $r : T \to S$ be morphisms in E such that r is a universal F-2-descent morphism. Let $f_T, f_{T \times_S T}$ and $f_{T \times_S T \times_S T}$ be the base extension of f with respect to r and the natural maps $T \times_S T \to S$ and $T \times_S T \times_S T \to S$. Then the following holds.*

(0) If f_T is F-0-descent then f is F-0-descent.

(1) If f_T is F-1-descent and $f_{T \times_S T}$ is F-0-descent then f is F-1-descent.

(2) If f_T is F-2-descent, $f_{T \times_S T}$ an F-1-descent morphism and $f_{T \times_S T \times_S T}$ an F-0-descent morphism then f is F-2-descent.

Proof: Theorem 10.8 [Gi64] □

Corollary 3.2.3. *With the notation from above if r and f_T are universally F-2-descent then the same holds for f.* □

Lemma 3.2.4 (domination-composition). *Let $\varphi : \mathsf{F} \to \mathsf{E}$ be a fibred category such that E has fibre products. Let f, g be composable morphisms in E, i.e., $f \circ g$ exists. Then the following holds.*

(d) Let $0 \leq i \leq 2$. If $f \circ g$ is F-i-descent and universally F-$(i-1)$-descent (which is the empty condition for $i = 0$) then f is F-i-descent.

(c) Let $0 \leq i \leq 2$. If f, g are F-i-descent and moreover g universally F-$(i-1)$-descent (which is the empty condition for $i = 0$) then $f \circ g$ is F-i-descent.

Proof: (d) Theorem 10.10 [Gi64], (c) Theorem 10.11 [Gi64]. □

3.2.2 Log-radicial maps

In [SGA 1] the property radicial plays an important role while checking that a morphism satisfies descent for morphisms between étale covers. Here we proceed in a log-analogous way, cf. Theorem 3.2.11. We use essentially Vidal's work on topological invariance.

Definition 3.2.5. *Let $f : X \to Y$ be a morphism of fs-log schemes. We call f* **log-radicial** *if $\mathring{f}$ is a radicial map of schemes, and for any geometric point $\overline{x}$ with image $f(\overline{x}) = \overline{y}$ the morphism $u_f : \overline{M}_{\overline{y}} \to \overline{M}_{\overline{x}}$ is p inseparable (cf. 4.9 [Ka89]) where p is the exponential residue characteristic at $\overline{x}$.*

The second condition just requires $\overline{M}_{\overline{y}} \to \overline{M}_{\overline{x}}$ to be injective and Kummer of p primary index.

Remark 3.2.6.
(1) A log-radicial map is automatically Kummer of p primary index, where p is the respective exponential residue characteristic.
(2) In characteristic 0 a log-radicial map is strict radicial.
(3) Being log-radicial is stable under composition.

Lemma 3.2.7. *Let $f : X \to S$ be a map of fs-log schemes. Then the following are equivalent:*

(a) f is log-radicial,

(b) for all strict geometric points $\overline{s} \to S$ the base change $f_{\overline{s}} : X_{\overline{s}} \to \overline{s}$ is kuh or $X_{\overline{s}}$ is empty.

Proof: Only $(a) \Rightarrow (b)$ requires proof. As strict fs base change preserves radicial maps and p-inseparability we may assume that $\mathring{S}$ is the spectrum of a separably closed field. Then X is either empty or $\mathring{f}$ is a universal homeomorphism. The criterion of Theorem 3.1.14 proves that f is kuh. □

Corollary 3.2.8. *f is log-radicial $\iff$ f is fs-universally injective.* □

Corollary 3.2.9. *f is log-radicial $\iff$ f is fs-universally log-radicial.* □

Proposition 3.2.10 (log SGA1 VIII 3.1). *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

be a cartesian square of fs-log schemes. Assume that g is surjective and f is Kummer. Then the following holds:

(1) f is surjective $\iff$ f' is surjective,

(2) f is log-radicial $\iff$ f' is log-radicial.

Proof: (1) follows from Nakayama's 4 Point Lemma, Lemma 3.1.2, and (2) follows from the preceeding lemma and its corollaries together with the 4 Point Lemma. □

Theorem 3.2.11 (log SGA1 I 5.1). *Let $f : X \to Y$ be a morphism of fs-log schemes of finite presentation. For f to be an open immersion (resp. an isomorphism) it is necessary and sufficient to be log-étale and log-radicial (resp. log-étale, log-radicial and surjective).*

Proof: This comprises in a slight generalisation of 2.7 (2) [Vi01]. A log-radicial and log-étale map is strict, thus classically étale. By I 5.1 [SGA 1] étale and radicial implies being an open immersion. □

3.2.3 Descent morphisms

As in [SGA 1] descent of morphisms between két covers succeeds due to a topological description of the respective graphs, cf. Corollary 3.2.13. The essential notion in this section is that of a log-radicial map. Thereby we can derive for a large class of maps that they are descent morphisms for $\mathsf{Rev}^{\log}$, cf. Proposition 3.2.18. Hence, we establish a first step towards an effective descent result that is required for the log Van Kampen formula.

Proposition 3.2.12 (log SGA1 I 5.3). *Let $f : X \to S$ be a log-étale map. There is a bijection*

$$\left\{ \begin{array}{l} \textit{sections of} \\ X \to S \end{array} \right\} = \left\{ \Gamma \subset X \;\middle|\; \begin{array}{l} \Gamma \textit{ is open and } f|_\Gamma : \Gamma \to S \\ \textit{is log-radicial and surjective} \end{array} \right\}$$

which maps a section s to its image $s(S)$.

Proof: By Proposition 1.1 [Vi01] such sections are strict open immersions. Now we use Theorem 3.2.11. □

Corollary 3.2.13 (log SGA1 IX 1.6). *Let X, Y be fs-log schemes over S such that Y/S is log-étale. There is a bijection*

$$\mathrm{Hom}_S(X,Y) = \left\{ \Gamma \subset X \times_S Y \;\middle|\; \begin{array}{l} \Gamma \textit{ is open and } \mathrm{pr}_1|_\Gamma : \Gamma \to X \\ \textit{is log-radicial and surjective} \end{array} \right\}$$

which maps a map f to its graph. □

Now we possess with Corollary 3.2.13 a topological description of graphs for maps between két covers. Thus, we may proceed by essentially topological considerations for the proof of the following descent statement for maps.

Proposition 3.2.14 (log SGA1 IX 3.1). *Let $g : S' \to S$ be a surjective map of fs-log schemes. Then g is faithful for maps between Kummer log-étale S-schemes, i.e., for Kummer log-étale S-schemes X, Y and their base extensions X', Y' with respect to g the natural map $\mathrm{Hom}_S(X,Y) \to \mathrm{Hom}_{S'}(X',Y')$ is injective.*

Proof: Identifying maps with their graphs as in Corollary 3.2.13 the natural map becomes $\Gamma \mapsto \Gamma' = \Gamma \times_S^{\mathrm{fs}} S'$. By Nakayama's 4 Point Lemma, Lemma 3.1.2, the projection $\Gamma' \to \Gamma$ is surjective. □

Let $\mathsf{Of}(X)$ denote the set of simultaneously open (**o**uvert) and closed (**f**ermé) subsets of an fs-log scheme X.

Proposition 3.2.15 (log SGA1 IX 3.2). *Let $g : S' \to S$ be a surjective map of fs-log schemes such that for all base extensions $T' \to T$ by finite két morphisms $T \to S$ the sequence*

$$\mathsf{Of}(T) \longrightarrow \mathsf{Of}(T') \rightrightarrows \mathsf{Of}(T' \times^{\mathrm{fs}}_T T')$$

is exact in sets. Then g is a descent morphism for $\mathsf{Rev}^{\log}$.

Proof: Let $X, Y \in \mathsf{Rev}^{\log}(S)$ and denote their base extensions with g (resp. the natural map $S'' = S' \times^{\mathrm{fs}}_S S' \to S$) by X', Y' (resp. by X'', Y''). Graphs of morphisms are open and closed as két covers are separated. Hence the sequence in sets

$$\mathrm{Hom}_S(X, Y) \longrightarrow \mathrm{Hom}_{S'}(X', Y') \rightrightarrows \mathrm{Hom}_{S''}(X'', Y'')$$

is the subsequence of

$$\mathsf{Of}(X \times^{\mathrm{fs}}_S Y) \longrightarrow \mathsf{Of}(X' \times^{\mathrm{fs}}_{S'} Y') \rightrightarrows \mathsf{Of}(X'' \times^{\mathrm{fs}}_{S''} Y'')$$

consisting of those Γ (resp. Γ' or Γ'') such that the restriction of the first projection is log-radicial and surjective. By the assumption applied to the két map $X \times^{\mathrm{fs}}_S Y \to S$ the latter is exact. Now, let Γ' be a graph of a S'-map $X' \to Y'$ such that $\mathrm{pr}_1^*\Gamma' = \mathrm{pr}_2^*\Gamma'$ in $X'' \times^{\mathrm{fs}}_{S''} Y'' = (X \times^{\mathrm{fs}}_S Y) \times^{\mathrm{fs}}_S S''$ via $\mathrm{pr}_i : S'' \to S'$ for $i = 1, 2$. Then Γ' is the preimage of a unique $\Gamma \in \mathsf{Of}(X \times^{\mathrm{fs}}_S Y)$. By 3.2.10 applied to the base extension $X' \to X$ the restriction of pr_X to Γ is log-radicial and surjective as the same holds for $\mathrm{pr}_{X'} : \Gamma' \to X'$. Indeed, $X' \to X$ is surjective being the base extension of the surjective g with respect to a Kummer map. Hence Γ is uniquely a graph of a map by Corollary 3.2.13. ☐

A map $h : X \to Y$ of topological spaces is called submersive if it is surjective and Y carries the quotient topology with respect to h.

Proposition 3.2.16 (log SGA1 IX 2.3). *Let $g : S' \to S$ be a submersive map of fs-log schemes such that the natural map $(S' \times^{\mathrm{fs}}_S S')^\circ \to \mathring{S}\,' \times_{\mathring{S}} \mathring{S}\,'$ is surjective. Then the sequence*

$$\mathsf{Of}(S) \longrightarrow \mathsf{Of}(S') \rightrightarrows \mathsf{Of}(S' \times^{\mathrm{fs}}_S S')$$

of sets is exact.

Proof: By assumption the natural map $\mathsf{Of}(\mathring{S}\,' \times_{\mathring{S}} \mathring{S}\,') \to \mathsf{Of}(S' \times^{\mathrm{fs}}_S S')$ is injective. Hence if $X' \in \mathsf{Of}(S')$ has two identical pullbacks to $S' \times^{\mathrm{fs}}_S S'$ then it is the preimage of a unique subset $X \subset S$ which is open and closed as g is submersive. ☐

Corollary 3.2.17 (log SGA1 IX 3.3). *Let g be a map of fs-log schemes that is exact and fs-universally submersive. Then g is a universal descent morphism for $\mathsf{Rev}^{\log}$.*

Proof: By Nakayama's 4 Point Lemma the map g fs-universally satisfies the assumptions for Proposition 3.2.16 which in turn yields the prerequisits for Proposition 3.2.15. ☐

Proposition 3.2.18. *A map of fs-log schemes that is exact and either (1) fpqc, or (2) fppf, or (3) proper and surjective, or (4) két and surjective satisfies fs-universally descent for $\mathsf{Rev}^{\log}$.*

Proof: The fs base extension of a map is the composition of the schematic base extension with a finite map. Here the latter is also surjective due to exactness of the map and Nakayama's 4 Point Lemma, hence a map of type (3). As composition of submersive maps are submersive again it suffices that conditions (1)-(4) each imply that the map is scheme-universally submersive. This follows from classical fpqc descent, cf. VIII 4.3 [SGA 1], or from the maps being either universally open or closed. Now we may apply Proposition 3.2.17. □

3.2.4 Criteria for effectivity

In this section we finally prove effective descent results for Kummer étale covers. We proceed by reduction to the strict local case. By Proposition 3.1.10 we further reduce to dimension 0. Then we use the explicit description of $\mathsf{Rev}^{\log}$ in group-theoretical terms in the strict local case by the very same proposition.

The later chapters only use Theorem 3.2.20 through (6.2.5). But the additional effective descent results easily follow by formal devissage arguments and the classical case. So they are included to give a more complete treatment of logarithmic descent theory along the lines of [SGA 1].

Proposition 3.2.19. *Let f be a strict, étale and surjective morphism of fs-log schemes. Then f is a universal effective descent morphism for* $\mathsf{Rev}^{\log}$.

Proof: This is Proposition 4.4 [Vi01]. We sketch a proof. By Corrollary 3.2.17 f is universally a descent morphism. The effective descent as a scheme follows from VIII 7.9 [SGA 1]. The descent of the log structure is by glueing étale sheaves in the étale topology and the resulting log structure is again fs as this is an étale local question. □

Theorem 3.2.20 (log SGA1 IX 4.7). *Let $f : S' \to S$ be a finite, surjective and exact morphism of finite presentation between fs-log schemes. Then f satisfies universally effective descent for* $\mathsf{Rev}^{\log}$.

Proof: The proof follows the classical analogue but requires additional reasoning for the zero dimensional case. By Lemma 3.2.2 and a Zariski cover by affines we may reduce to the case S affine, hence quasi-compact. For quasi-compact S we may use the usual $\varinjlim$-arguments due to Proposition 4.3 [Vi01] reducing to the case of affine and noetherian S. Again by Lemma 3.2.2 and $\varinjlim$-arguments we may assume that S is the spectrum of a strict henselian local noetherian ring.

Let $S_0 \hookrightarrow S$ be the inclusion of the closed point with induced log structure. By 18.5.10 [$\mathrm{EGA_{IV}}$] $S', S'' = S' \times_S^{\mathrm{fs}} S'$ and $S''' = S' \times_S^{\mathrm{fs}} S' \times_S^{\mathrm{fs}} S'$ are finite disjoint unions of strict henselian local noetherian spectra which are finite over S. By 18.5.5 [$\mathrm{EGA_{IV}}$] their set of connected components bijects by restriction to the set of connected components of the fibres over S_0 which are denoted S_0', S_0'' and S_0''' respectively. In the following diagram

$$\begin{array}{ccccccc} S_0 & \longleftarrow & S_0' & \leftleftarrows & S_0'' & \Lleftarrow & S_0''' \\ {\scriptstyle i}\downarrow & & {\scriptstyle i'}\downarrow & & {\scriptstyle i''}\downarrow & & {\scriptstyle i'''}\downarrow \\ S & \longleftarrow & S' & \leftleftarrows & S'' & \Lleftarrow & S''' \end{array}$$

the vertical arrows are monomorphisms which induce equivalences of the respective categories of sections for $\mathsf{Rev}^{\log}$ by Proposition 3.1.10. Hence

$$\begin{array}{ccc} \mathsf{Rev}^{\log}(S) & \xrightarrow{f^*} & \mathsf{DD}(f, \mathsf{Rev}^{\log}) \\ {\scriptstyle i^*}\downarrow\cong & & \downarrow\cong \\ \mathsf{Rev}^{\log}(S_0) & \xrightarrow{f_0^*} & \mathsf{DD}(f_0, \mathsf{Rev}^{\log}) \end{array}$$

and we may replace f by f_0 and assume that S is the spectrum of a separably closed field. Moreover, by considering a single component of S' and Lemma 3.2.4 we may assume that $f : S' \to S$ is a finite purely inseparable field extension k'/k of exponential characteristic p. We factorise f as $S' \to S_1 \to S$ by setting $S_1 = (\overset{\circ}{S}, M_1)$ and

$$M_1 = \{m \in M_{S'} \mid m \text{ is } p'\text{-torsion in } M_{S'}^{\mathrm{gp}}/M_S^{\mathrm{gp}}\} .$$

Then we have to discuss the following cases.

(a) The map $S_1 \to S$ is up to taking the strict reduced subscheme a standard két cover of S. By Proposition 3.1.10 we may as well deal with standard két covers.

(b) The map $S' \to S_1$ has the property that the cokernel of $\overline{M}_{S_1}^{\mathrm{gp}} \to \overline{M}_{S'}^{\mathrm{gp}}$ has at most p-primary torsion.

As by Proposition 3.2.18 both parts are universal descent morphisms it suffices to deal with (a) and (b) separately invoking Lemma 3.2.4.

(a) This is a special case of *internal Galois descent* within a Galois category. Let X be a Galois object in a Galois category $\mathscr{C}$ with final object $*$ then the map $X \to *$ is a universal effective descent morphism for objects of $\mathscr{C}$. The reason is that descent data consist in group actions of $\mathrm{Aut}(X/*)$ on Y/X. Taking the quotient in $\mathscr{C}$ yields the effective descent.

(b) For notational convenience we replace S_1 by S. First of all, by an easy exercise $S'' = S' \times_S^{\mathrm{fs}} S'$ and $S''' = S' \times_S^{\mathrm{fs}} S' \times_S^{\mathrm{fs}} S'$ have only one point so that by 3.1.10 the respective categories of finite két fs-log schemes are equivalent to finite sets with continuous action by

$$\begin{aligned} A &= \mathrm{Hom}\Big(\overline{M}_S^{\mathrm{gp}}, \hat{\mathbb{Z}}(1)\big(k\big)\Big), \\ A' &= \mathrm{Hom}\Big(\overline{M}_{S'}^{\mathrm{gp}}, \hat{\mathbb{Z}}(1)\big(k\big)\Big), \\ A'' &= \mathrm{Hom}\Big(\overline{M}_{S''}^{\mathrm{gp}}, \hat{\mathbb{Z}}(1)\big(k\big)\Big), \\ A''' &= \mathrm{Hom}\Big(\overline{M}_{S'''}^{\mathrm{gp}}, \hat{\mathbb{Z}}(1)\big(k\big)\Big). \end{aligned}$$

In this way the problem becomes purely elementary group theory. By Proposition 2.1.1 [Na97] $\overline{M}_{S''}^{\mathrm{gp}} = \overline{M}_{S'}^{\mathrm{gp}} \oplus_{\overline{M}_S^{\mathrm{gp}}} \overline{M}_{S'}^{\mathrm{gp}}$ and thus $A'' = A' \times_A A'$. Let $\mathrm{pr}_i : A'' \to A'$ for $i = 1, 2$ be the natural projections. Then

$$\mathsf{DD}(S' \to S, \mathsf{Rev}^{\log}) = \left\{ (M', \varphi) \;\middle|\; \begin{array}{l} M' \in A'\text{-}\mathsf{sets} \\ \varphi : \mathrm{pr}_1^* M' \xrightarrow{\sim} \mathrm{pr}_2^* M' + \text{cocycle condition} \end{array} \right\}$$

Exploiting the cocycle condition yields $\varphi^2 = \varphi$ as maps of sets, hence $\varphi = \mathrm{id}$. Thus the actions of A'' on M' via both projections to A' coincide. Consequently, the natural functor induces an equivalence

$$\big(A'/\ker(A' \to A)\big)\text{-}\mathsf{sets} \xrightarrow{\cong} \mathsf{DD}(f, \mathsf{Rev}^{\log})$$

and it remains to prove that $A' \to A$ is surjective. Setting $D = \hat{\mathbb{Z}}(1)\big(k\big) = \hat{\mathbb{Z}}(1)\big(k'\big)$ the sequence

$$\mathrm{Hom}\big(\overline{M}^{\mathrm{gp}}_{S'}, D\big) \to \mathrm{Hom}\big(\overline{M}^{\mathrm{gp}}_{S}, D\big) \to \mathrm{Ext}^1\left(\overline{M}^{\mathrm{gp}}_{S'}/\overline{M}^{\mathrm{gp}}_{S}, D\right) \to 0$$

is exact as $\overline{M}^{\mathrm{gp}}_{S'}$ is torsionfree and $S' \to S$ exact. However, the Ext^1-term vanishes by the assumptions on torsion in $\overline{M}^{\mathrm{gp}}_{S'}/\overline{M}^{\mathrm{gp}}_{S}$ and D being uniquely p-divisible. Hence $A' \twoheadrightarrow A$ is surjective. □

In case (b) $S' \to S_1$ is kuh. Thus one could also argue by Theorem 3.1.16 and prove that $A' \to A$ is an isomorphism.

Corollary 3.2.21 (log SGA1 IX 4.9). *Let $f : S' \to S$ be a surjective and exact morphism of finite presentation between fs-log schemes such that the underlying morphism of schemes is universally open for base extensions of schemes. Then f satisfies universally effective descent for* $\mathsf{Rev}^{\log}$.

Proof: We factorise f as $S' \to (\mathring{S}\,', f^*M_S) \to S$, i.e., in a finite exact surjective morphism of finite presentation followed by a strict universally open morphism of finite presentation. They both satisfy universal descent by Proposition 3.2.18. Thus by Lemma 3.2.4 and Theorem 3.2.20 it suffices to treat the case of f being strict. As above by Zariski descent and $\varinjlim$-arguments we may assume that S is affine and noetherian. We now invoke the following lemma from the theory of schemes.

Lemma 3.2.22 (Mumford, 14.5.10 [EGA$_{\mathbf{IV}}$]). *Let $f : X \to S$ be a universally open, surjective morphism which is locally of finite type and S noetherian. Then there exists a finite surjective $T \to S$ and a Zariski covering $\coprod U \to T$ such that $X \times_S U \to U$ has a section for all parts U of the Zariski covering.* □

We choose $\coprod U \to T \to \mathring{S}$ appropriate for $\mathring{f} : \mathring{S}\,' \to \mathring{S}$ according to Lemma 3.2.22 and regard them as fs-log schemes with the induced fs-log structure from M_S.

$$\begin{array}{ccccc}
\coprod S' \times_S U & \longrightarrow & S' \times_S T & \longrightarrow & S' \\
\uparrow\!\downarrow & & \downarrow & & \downarrow \\
\coprod U & \longrightarrow & T & \longrightarrow & S
\end{array}$$

By Lemma 3.2.4 it suffices to treat the individual steps $\coprod S' \times_S U \to \coprod U, \coprod U \to T$ and $T \to S$ for which universal effective descent holds by Lemma 3.2.1, Proposition 3.2.19 and Theorem 3.2.20. □

Corollary 3.2.23. *Let f be an exact morphism between fs-log schemes such that the underlying map of schemes is fppf. Then f satisfies universal effective descent for* $\mathsf{Rev}^{\log}$.

Proof: fppf implies universally open as a map in the category of schemes. □

Corollary 3.2.24. *Let $f : S' \to S$ be a két covering and S quasi-compact. Then f is a universal effective descent morphism for* $\mathsf{Rev}^{\log}$.

Proof: By Lemma 3.2.4 we may assume that S' is itself quasi-compact. Then f is exact, surjective and of finite presentation. By Corollary 3.1.7 a két morphism is universally open. Thus Corollary 3.2.21 applies. □

Theorem 3.2.25 (log SGA1 IX 4.12). *Let $f : S' \to S$ be a proper, surjective and exact morphism of finite presentation between fs-log schemes. Then f satisfies universally effective descent for* $\mathsf{Rev}^{\log}$.

Proof: By a reduction as in the proof of Corollary 3.2.21 we may assume that f is strict and by those of Theorem 3.2.20 we may assume that S is strict henselian local noetherian. We consider the strict inclusion of the closed point $S_0 \hookrightarrow S$. The cartesian square

$$\begin{array}{ccc} S'_0 & \xrightarrow{f_0} & S_0 \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ S' & \xrightarrow{f} & S \end{array}$$

induces a commutative diagram of functors

$$\begin{array}{ccc} \mathsf{Rev}^{\log}(S) & \overset{f^*}{\hookrightarrow} & \mathsf{DD}(f, \mathsf{Rev}^{\log}) \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle \mathsf{DD}(i)} \\ \mathsf{Rev}^{\log}(S_0) & \xrightarrow{f_0^*} & \mathsf{DD}(f_0, \mathsf{Rev}^{\log}) \end{array}$$

where i^* is an equivalence by Proposition 3.1.10, f_0^* is an equivalence by Corollary 3.2.23 and f^* is fully faithful as a consequence of Proposition 3.2.18. To ensure that f^* is even an equivalence it suffices to verify that $\mathsf{DD}(i)$ is full which in turn follows from the claim that i'^* is fully faithful and i''^* (for $i'' : S'_0 \times^{\mathrm{fs}}_{S_0} S'_0 \to S' \times^{\mathrm{fs}}_S S'$) is faithful for $\mathsf{Rev}^{\log}$. Finally this claim follows from the following lemma. □

Lemma 3.2.26. *Let S be a strict henselian, local, noetherian fs-log scheme with (strict) closed point $i : S_0 \hookrightarrow S$. Let $X \to S$ be a proper map of fs-log schemes and let $i_X : X_0 \hookrightarrow X$ be the base change of i. Then $i_X^* : \mathsf{Rev}^{\log}(X_0) \to \mathsf{Rev}^{\log}(X)$ is fully faithful.*

Proof: By 18.5.19 [EGA$_{\mathrm{IV}}$] the natural map $\pi_0(X_0) \to \pi_0(X)$ is bijective. Hence we may assume that X and X_0 are connected. Note: the lemma follows from that if we had verified already that $\mathsf{Rev}^{\log}(T)$ is a Galois category for arbitrary T, cf. Theorem 3.3.6.

By Corollary 3.2.13 and again 18.5.19 [EGA$_{\mathrm{IV}}$] (applied to $X \times^{\mathrm{fs}}_S Y$) it suffices to prove that a finite két map $\Gamma \to X$ is log-radicial and surjective if this holds for its fibre $\Gamma_0 \to X_0$ over S_0. So let us assume this, in particular $\Gamma_0 \to X_0$ is strict.

As két implies open and the strict locus of a map is open in the domain, we conclude that $\Gamma \to X$ is strict étale and surjective. Indeed, X is the only open neighbourhood of X_0. Then we may apply the classical counterpart XII Theorem 5.9 bis [SGA 4]. □

Remark 3.2.27. *The morphisms treated in Theorem 3.2.20 and Corollary 3.2.21 are even universally effective decent morphisms for relative fs-log schemes which are separated, két and of finite presentation. This only uses a refinement of the reasoning as in the classical counterparts the basic ingredient being 18.5.11 [EGA$_{IV}$].*

3.3 The logarithmic fundamental group

The log fundamental group has already been constructed and defined by different people, see for example Chapter 4 [Il02]. Here we give a construction for matters of completeness. Moreover, using descent theory as developed in Section 3.2, we may provide proofs that are skipped by most references.

An algebraic fundamental group is determined by its Galois category of finite covers. Therefore we need to construct a Galois category. The good properties of a Galois category are easily verified for locally constant finite sheaves in case there are locally only finitely many connected components. But covers should be actual geometric objects and not merely sheaves of sections. An effective descent result remedies the discrepancy.

3.3.1 Locally constant finite sheaves

Let X be a fs-log scheme, $X_{\text{két}}$ its Kummer étale site, and $\widetilde{X}_{\text{két}}$ its category of sheaves of sets. In fact, any sufficiently good site works for the following. The global section functor has as left adjoint the constant sheaf functor.

Definition 3.3.1. *Let $\mathsf{cf}(X)$ denote the full subcategory of $\widetilde{X}_{\text{két}}$ of all constant sheaves A_X on X associated to finite sets A. The category $\mathsf{lcf}(X)$ of* **locally constant finite sheaves** *of sets on X is defined as the full subcategory of all $\mathscr{F} \in \widetilde{X}_{\text{két}}$ such that there is a két covering $\{U \to X\}$ such that $\mathscr{F}|_U \in \mathsf{cf}(U)$.*

Lemma 3.3.2. *Let X be quasi-compact and A a finite set. Then the global sections of A_X are naturally isomorphic to $A^{\pi_0(X)}$.*

Proof: The constant presheaf $Y/X \mapsto A$ is separated, hence sheafification is realised by applying $\check{\mathrm{H}}^0$ only once. Thus

$$A_X(X) = \varinjlim_{\{U \to X\}} \ker\left(\Gamma(U, A) \rightrightarrows \Gamma(U \times_X U, A) \right) = \varinjlim_{\{U \to X\}} A^{\pi_0|\{U \to X\}|} = A^{\pi_0(X)}$$

where $\{U \to X\}$ runs through all két coverings of X and $|\{U \to X\}|$ stands for the nerve of the associated simplicial set of components. Note that the last equality uses that két maps are open by Proposition 3.1.6 and Nakayamas 4 Point Lemma, Lemma 3.1.2. □

Lemma 3.3.3. *The category $\mathsf{lcf}(X)$ has (1) a final object, (2) an initial object, (3) finite coproducts, (4) quotients by actions of finite groups, (5) fibre products.*

Proof: We may construct sheaves satisfying a universal property locally as one can glue sheaves. Then: (1) the one point valued sheaf, (2) the empty set sheaf, (3) and (4) locally as the respective operation on sets due to the constant sheaf functor being a left adjoint. Note that over a connected fs-log scheme $\mathsf{cf}(X)$ is equivalent to sets by the preceeding Lemma. Hence the group action locally originates from an action on the underlying set of the constant sheaf.

The solution for (5), let's say $\mathscr{F}' \times_{\mathscr{F}} \mathscr{F}''$, is constructed in the same way. We may assume that all three sheaves are constant A_X, A'_X, A''_X and the maps originate from maps $A' \to A, A'' \to A$. Then we claim that $A'_X \times_{A_X} A''_X = (A' \times_A A'')_X$. To prove this we need to check that for $\mathscr{H} \in \mathsf{lcf}(X)$ the natural map

$$\mathrm{Hom}(\mathscr{H}, (A' \times_A A'')_X) \to \mathrm{Hom}(\mathscr{H}, A'_X) \times_{\mathrm{Hom}(\mathscr{H}, A_X)} \mathrm{Hom}(\mathscr{H}, A''_X)$$

is bijective. This being a local question again we may assume that $\mathscr{H}$ is constant and even the one point sheaf. This reduces us to the bijectivity of

$$\left(A' \times_A A''\right)_X(X) \to A'_X(X) \times_{A_X(X)} A''_X(X)$$

which follows from Lemma 3.3.2. □

Lemma 3.3.4. *Let $\varphi : \mathscr{F} \to \mathscr{H}$ be a map in $\mathsf{lcf}(X)$. Then there is an image sheaf $\varphi(\mathscr{F}) \in \mathsf{lcf}(X)$ and $\mathscr{F} \to \varphi(\mathscr{F}) \subset \mathscr{H}$ yields a factorisation as strict epimorphism followed by a monomorphism which is an isomorphism onto a direct summand.*

Proof: This is again a local matter. We therefore may assume that X is connected and $\mathscr{F}, \mathscr{H}, \varphi$ are constant. Then the respective factorisation in sets does the job. □

Consequently $\mathsf{lcf}(X)$ satisfies the axioms (G1)-(G3) of V §4 [SGA 1] for Galois categories. Furthermore any log geometric point $\tilde{x} \to X$, cf. 4.4 [Il02], yields a stalk functor $F_{\tilde{x}} : \mathsf{lcf}(X) \to \mathsf{sets}$ which satisfies the axioms (G4) - (G5) of V §4 [SGA 1] for fibre functors.

Proposition 3.3.5. *Let X be a connected fs-log scheme. Then $\mathsf{lcf}(X)$ is a Galois category with fibre functors associated to any log geometric point.*

Proof: As X is connected and két morphisms are open by Proposition 3.1.6 the above fibre functor associated to a log geometric point is also conservative, i.e., it satisfies (G6) of V §4 [SGA 1]. □

Theorem 3.3.6. *Let X be a fs-log scheme. Then the following holds:*

(1) Any $Y \in \mathsf{Rev}^{\log}(X)$ defines a representable sheaf $\mathrm{Hom}_X(-, Y) \in \widetilde{X}_{\text{két}}$ which is locally constant and finite.

(2) The functor $\mathsf{Rev}^{\log}(X) \to \mathsf{lcf}(X)$ is fully faithful.

(3) $\mathsf{Rev}^{\log}$ satisfies effective descent relative két coverings. In particular, the above functor $\mathsf{Rev}^{\log}(X) \to \mathsf{lcf}(X)$ is an equivalence.

(4) Moreover, if X is connected then $\mathsf{Rev}^{\log}(X)$ is a Galois category.

Proof: (1) By 2.6 [Il02] or 3 [Ka91] representable presheaves on $X_{\text{két}}$ are indeed sheaves. We sketch the proof for $Y \in \mathsf{Rev}^{\log}(X)$: by Proposition 3.1.4 the topology of $X_{\text{két}}$ is generated by strict étale coverings and standard két morphisms. For a strict étale covering $\mathrm{Hom}_X(-, Y)$ satisfies the sheaf property as the classical étale topology is coarser than the canonical one and — concerning the log structure — étale maps of sheaves can be uniquely glued from local data. Again by Proposition 3.1.4 and the usual $\varinjlim$-arguments we may therefore assume that X is strict henselian local noetherian and that we need to check the sheaf property for a covering $X_Q \to X$ which is of standard két type. By Proposition 3.1.10 this now becomes a question for the standard Grothendieck topology on the site of finite sets with action of a pro-finite group. Here representable presheaves are indeed sheaves.

We come back to the proof of the theorem. By Proposition 3.1.4 and Proposition 3.1.6 any $Y \in \mathsf{Rev}^{\log}(X)$ is két locally on X a disjoint union of copies of X, hence represents a finite constant sheaf. Part (2) follows from Yoneda. Part (3) is a consequence of Corollary 3.2.24. Indeed, $\mathscr{F} \in \mathsf{lcf}(X)$ is locally representable as $\mathrm{Hom}_U(-, Y_U)$ and yields a descent datum for $Y_U \in \mathsf{Rev}^{\log}(U)$ relative a két covering $\{U \to X\}$. This being effective it glues to some $Y \in \mathsf{Rev}^{\log}(X)$. The local identifications $\mathscr{F}|_U = \mathrm{Hom}_U(-, Y_U)$ glue to a global isomorphism $\mathscr{F} = \mathrm{Hom}_X(-, Y)$. Finally, Part (4) follows from the above in conjunction with Proposition 3.3.5. □

Definition 3.3.7. *Let X be a connected fs-log scheme and $\tilde{x}$ a log geometric point. We define* **logarithmic fundamental group** $\pi_1^{\log}(X, \tilde{x})$ *as the fundamental group of the Galois category* $\mathsf{Rev}^{\log}(X)$ *equipped with the fibre functor* $F_{\tilde{x}}$.

Examples of log fundamental groups. (0) The natural map $\varepsilon : X \to (\overset{\circ}{X}, \mathcal{O}_X^*)$ induces a natural surjective map $\pi_1^{\log}(X) \twoheadrightarrow \pi_1(\overset{\circ}{X})$ of the log fundamental group to the classical étale fundamental group of the underlying scheme.

(1) Let X be a strict henselian fs-log scheme with log geometric point $\tilde{x}$ over the closed point $\overline{x}$. Then by Proposition 3.1.10

$$\pi_1^{\log}(X, \tilde{x}) = \mathrm{Hom}\Big(\overline{M}_{X,\overline{x}}^{\mathrm{gp}}, \hat{\mathbb{Z}}(1)\big(k\big)\Big)$$

where k is the residue field at $\overline{x}$.

(2) Let S be the spectrum of a discrete valuation ring R with closed point s, residue field k, and $R \subset K$ its field of fractions. We endow S with the fs-log structure $M(\log s)$, i.e., $R' \setminus \{0\} \hookrightarrow R'$ for an étale R-algebra R'. Let S^{sh} be a strict henselisation in a geometric point $\overline{s}$ over the closed point and endow it with the analogous fs-log structure. With the choice of a log-geometric point $\tilde{s}$ there is a canonical short exact sequence

$$1 \to \pi_1^{\log}(S^{\mathrm{sh}}, \tilde{s}) \to \pi_1^{\log}(S, \tilde{s}) \to G_k \to 1$$

that is isomorphic to the following:

$$1 \to I^{\mathrm{tame}} \to G_K^{\mathrm{tame}} \to G_k \to 1 \ .$$

Here G_K^{tame} is the tame quotient of the absolute Galois group of K, G_k is the absolute Galois group of k, and $I^{\mathrm{tame}} \cong \hat{\mathbb{Z}}(1)\big(k\big)$ is the tame inertia subgroup. Hence, the isomorphism of (1) induces the "correct" G_k action on $\pi_1^{\log}(S^{\mathrm{sh}})$. By Proposition 3.1.10 these sequences are furthermore isomorphic to

$$1 \to \pi_1^{\log}(\overline{s}, \tilde{s}) \to \pi_1^{\log}(s, \tilde{s}) \to G_k \to 1 \ .$$

Here s and $\overline{s}$ carry the log structure of the *standard log point* induced by $\mathbb{N} \to k$ (resp. k^{sep}); $(n > 0) \mapsto 0$ and $\tilde{s}$ is a log geometric point above s.

(3) According to Kato [Ka94], cf. 7.3 [Il02], a locally noetherian fs-log scheme is called log-regular if the following is satisfied.

(i) $R_{\overline{x}} = \mathcal{O}_{X,\overline{x}} / \big(\alpha_X(M_{X,\overline{x}} \setminus M_{X,\overline{x}}^{\times})\big)$ is regular for all geometric points $\overline{x}$ of X.

(ii) $\dim R_{\overline{x}} + rk_{\mathbb{Z}}\big(\overline{M}_{X,\overline{x}}^{\mathrm{gp}}\big) = \dim \mathcal{O}_{X,\overline{x}}$

Kato proves that such schemes are automatically normal and Cohen-Macaulay, cf. 4.1 [Ka94]. Furthermore if $j : X^{\mathrm{triv}} = \{x \in X | \overline{M}_{X,\overline{x}} = 1\} \hookrightarrow X$ then $\alpha_X : M_X = j_* \mathbb{G}_m \cap \mathcal{O}_X \hookrightarrow \mathcal{O}_X$ is the natural map, cf. 11.6 [Ka94]. We now assume in addition that X is regular, $D \subset X$ is a normal crossing divisor and the fs-log structure is $M(\log D)$, i.e., induced as above by the open immersion $j : X \setminus D \hookrightarrow X$. One checks that indeed $(X, M(\log D))$ is log-regular. By 3.1 [FK95] the identity map induces an equivalence $\mathsf{Rev}^{\log}(X) \to \mathsf{Rev}^{\mathrm{tame}}(X, D)$ where the latter describes tamely ramified covers of the pair (X, D) in the sense of [GM71]. This implies a natural isomorphims $\pi_1^{\log}(X) = \pi_1^{\mathrm{t}}(X, D)$. We sketch a proof:

Proof: At points of codimension 1 being a két cover for the log structure $M(\log D)$ easily implies being at most tamely ramified along D. Hence there is a functor "identity" $\mathsf{Rev}^{\log}(X) \to \mathsf{Rev}^{\mathrm{tame}}(X, D)$. This functor is fully faithful. The question on essential surjectivity is by étale effective descent and $\varinjlim$-arguments reduced to the strict henselian case. By 2.3.4 [GM71] a tame Y/X is a generalised Kummer cover. A cofinal system of those consist in the pullbacks of the coordinatewise n^{th}-power map on $(\mathbb{A}^1)^{\times r}$ via a map $(a_i) : X \to (\mathbb{A}^1)^{\times r}$ such that the a_i are local parameters whose vanishing locus is the divisor D. But this map clearly enhances to a strict fs-log map. Hence Y/X is also the fs base extension of a standard két cover. □

(4) The special case of a regular 1 dimensional scheme with reduced divisor $D \subset X$, i.e., pointed curves, yields $\pi_1^{\log}(X, M(\log D)) = \pi_1^{\mathrm{t}}(X, D)$.

3.3.2 Logarithmic blow-up

Logarithmic blow-ups are special log-étale maps unlike the classical counterpart. They will appear in the study of the logarithmic fundamental group of a semistable curve. Their properties and in particular their existence are well known. However, we use logarithmic descent theory as developed in Section 3.2 to derive an alternative proof for the result that log blow-ups induce isomorphisms on logarithmic fundamental groups.

A subset $I \subset P$ of a monoid is called an ideal if $IP \subset I$. A monoid possesses a unique maximal Ideal $I_{max} = P \setminus P^{\times}$. If I is finitely generated and P is fine then any set of generators for I contains a set of representatives of $(I \setminus I \cdot I_{max})/P^{\times}$. Therefore we call the cardinality of the latter the essential size of I and denote it by $\mathrm{ess}(I)$.

Let X be a fs-log scheme and $\mathscr{I} \lhd M_X$ a coherent ideal, i.e., étale locally for suitable charts $X \to \mathrm{Spec}(\mathbb{Z}[P])$ it is induced from finitely generated ideals $I \lhd P$. The logarithmic blow-up of X along $\mathscr{I}$ is a fs-log scheme $X_{\mathscr{I}}$ over X that represents the following functor:

$$\Big(f : Y \to X\Big) \mapsto \begin{cases} * & \text{if } \mathrm{im}\big(f^{-1}\mathscr{I} \subset f^{-1}M_X \xrightarrow{u_f} M_Y\big)M_Y \\ & \text{is locally monogeneous} \\ \emptyset & \text{else} \end{cases}$$

where $*$ means a set with one element and monogenous means generated as an ideal by a single element.

Proposition 3.3.8. *Logarithmic blow-ups exist.*

Proof: This can be found in [Ni99], or 6.1 [Il02]. We recall it here for the convenience of the reader.

Claim (1): If $\sigma : X_{\mathscr{I}} \to X$ is a log blow-up of X along $\mathscr{I}$ and $h : U \to X$ arbitrary then the log blow-up on U along the image sheaf of ideals

$$\mathscr{I}|_U = \mathrm{im}\big(h^{-1}\mathscr{I} \subset h^{-1}M_X \xrightarrow{u_h} M_U\big)M_U$$

exists as well. Furthermore, it can be represented by the second projection $X_{\mathscr{I}} \times_X^{\mathrm{fs}} U \to U$. This is trivial.

Claim (2): Let $\{U \to X\}$ be a covering of X in the classical étale topology and let us assume that the log blow-ups of U along $\mathscr{I}|_U$ exist for all U in the covering. Then X has a log blow-up along $\mathscr{I}$ which furthermore is obtained by glueing from the local solutions with the help of certain relatively ample linebundles.

Of course claim (1) yields glueing data for the log blow-ups $U_{\mathscr{I}|_U} \to U$ relative the covering $\{U \to X\}$ as fs-log schemes over X. This actually glues by VIII 7.8 [SGA 1] as schemes over $\mathring{X}$ thanks to a natural relatively ample linebundle: a local generator of $\mathscr{I}$ generates within M_U a torsor under $\mathcal{O}_U^*$ that corresponds to a linebundle which is unique and independent from the choice of a generator. Being relatively ample over U will become clear from the local construction. The fs-log structures also glue being étale local by definition, hence we obtain a $X_{\mathscr{I}} \to X$ that has the required property: the pullback of $\mathscr{I}$ along $Y \to X$ is locally monogenous if and only if it is so for all $Y_U = Y \times_X^{\mathrm{fs}} U$ and the resulting maps $Y_U \to U_{\mathscr{I}|_U}$ glue together yielding a map $Y \to X_{\mathscr{I}}$.

(3) By (2) we may assume that X and $\mathscr{I}$ possess a global chart and by (1) that actually $X = \mathrm{Spec}(\mathbb{Z}[P])$ for some fs monoid P and $\mathscr{I}$ is induced by $I \lhd P$. Let $a \in I$. We set $P_a = \langle P, \frac{b}{a} : b \in I \rangle \subset P^{\mathrm{gp}}$. The natural map $\mathrm{Spec}(\mathbb{Z}[P_a^{\mathrm{sat}}]) \to \mathrm{Spec}(\mathbb{Z}[P])$ represents the open subfunctor

$$\Big(f : Y \to X\Big) \mapsto \begin{cases} * & \text{if im}\left(f^{-1}\mathscr{I} \subset f^{-1}M_X \xrightarrow{u_f} M_Y\right)M_Y \\ & \text{is generated by } a \\ \emptyset & \text{else} \end{cases} .$$

Varying a over a set of generators of I yields an open covering of the functor we would like to represent. It is easily verified that thus $X_{\mathscr{I}}$ is the saturation of the schematic blow-up of X along the ideal generated by $\alpha_X(\mathscr{I})$ which is endowed with a log structure by taking the standard one on its standard open cover by $\mathrm{Spec}(\mathbb{Z}[P_a])$, cf. 6.1 [Il02]. The pullback of the relatively ample linebundle of the (classical) blow-up gives the one required in claim (2) above. ☐

Corollary 3.3.9. *Let X be a fs-log scheme and $\mathscr{I} \lhd M_X$ a coherent sheaf of ideals. The log blow-up $\sigma : X_{\mathscr{I}} \to X$ satisfies the following properties.*

(1) $\overline{u}_\sigma^{\mathrm{gp}} : \sigma^* \overline{M}_X^{\mathrm{gp}} \to \overline{M}_{X_{\mathscr{I}}}^{\mathrm{gp}}$ is an isomorphism,

(2) σ is log-étale,

(3) σ is a monomorphism, in particular $X_{\mathscr{I}} \times_X^{\mathrm{fs}} X_{\mathscr{I}} = X_{\mathscr{I}}$,

(4) a base change of σ is the logarithmic blow-up of the ideal generated by the pullback of $\mathscr{I}$,

(5) σ is proper with geometrically connected fibres, in particular, the log blow-up of a connected fs-log scheme stays connected,

(6) σ is universally surjective.

Proof: (1)-(4) and being proper are obvious from the construction. The rest of (5) and (6) may be checked étale locally and fibrewise and are consequences of the following. The log blow-up of $X = \mathrm{Spec}(k[P])$ along I is dominant birational projective, hence surjective, and satisfies $\mathcal{O}_X = \sigma_* \mathcal{O}_{X_{\mathscr{I}}}$ as $k[P]$ is normal, cf. 2.1 [Fu93]. We conclude by Zariski's Connectedness Theorem. □

Example. Let $X \to S$ be a generically smooth, semistable curve over a discrete valuation ring R with uniformising element π and closed point s. We endow X and S with the log structure $M(\log X_s)$, resp. $M(\log s)$, and need to check that X thus becomes a fs-log scheme. The crucial locus lies in the double points. But the problem is étale local and therefore may be verified for the standard model $X_0 = \mathrm{Spec}\big(R[u,v]/(uv - \pi^e)\big)$ of a double point of thickness e. Let $Q(e)$ be the monoid generated by $(1,0), (1,1)$ and $(e-1, e)$ in $\mathbb{Z}^2$. It is saturated and isomorphic to the pushout $(\mathbb{N}^2 \oplus_{\mathbb{N}} \frac{1}{e}\mathbb{N})^{\mathrm{sat}}$ with respect to the diagonal and inclusion. There is a map of monoids $Q(e) \to R[u,v]/(uv - \pi^e)$ sending $(1,0) \mapsto u, (1,1) \mapsto \pi$ and $(e-1,e) \mapsto v$ (being normal $\big(R[u,v]/(uv-\pi^e)\big)^\times$ is saturated). The induced fs-log structure yields a log-regular fs-log scheme structure on X_0 with the generic fibre as the open set of triviality. Hence, by 11.6 [Ka94] it coincides with $M(\log X_s)$ and detects the latter as a fs-log structure. Now, $X \to S$ has been enhanced to a map of fs-log schemes that furthermore is log-smooth. Let $\sigma : X' \to X$ be a classical blow-up in a double point of thickness $e > 1$. Then also X' is generically smooth and semistable over S and σ enhances to a map of fs-log schemes. By the local construction it coincides with a log blow-up. If the double point has thickness $e = 1$ then the exceptional fibre has multiplicity 2. Hence, X'/S is not semistable. However, the blow-up $\sigma : X' \to X$ as schemes still enhances to a log blow-up of the respective fs-log schemes where the fs-log structure is induced by the special fibre. Thus X'/S is still log-smooth (even in characteristic 2).

The original proof from [FK95] of the following uses a log purity result. We are going to give a proof based on logarithmic descent theory.

Theorem 3.3.10. *A logarithmic blow-up is a universal effective descent morphism for* $\mathsf{Rev}^{\log}$.

Corollary 3.3.11 (2.4 [FK95], 6.10 [Il02]). *Let X be a connected fs-log scheme. Let $\sigma : X_{\mathscr{I}} \to X$ be a log blow-up along the coherent ideal $\mathscr{I}$. Then $X_{\mathscr{I}}$ is connected and*

$$\pi_1^{\log}(\sigma) : \pi_1^{\log}(X_{\mathscr{I}}) \xrightarrow{\sim} \pi_1^{\log}(X)$$

is an isomorphism.

Proof: The corollary follows from σ being a monomorphism and the following consequence of this property for descent theory:

Lemma 3.3.12. *Let $i : T_0 \hookrightarrow T$ be a monomorphism in a category E with fibre products and $\mathsf{F} \to \mathsf{E}$ be a fibred category. Then the forgetful functor induces an equivalence $\mathsf{DD}(i, \mathsf{F}) \to \mathsf{F}_{T_0}$ where F_{T_0} is the category of sections of F over T_0.*

Proof: $T_0 \times_T T_0 = T_0$ □

In fact, the assertion of the corollary is equivalent to the assertion of the theorem. Furthermore, the universality of the descent properties is automatic by (4) of Corollary 3.3.9.

To establish surjectivity of $\pi_1^{\log}(\sigma)$ or, equivalently, that σ is a descent morphism for $\mathsf{Rev}^{\log}$ we take a connected $Y \in \mathsf{Rev}^{\log}(X)$ and show that $Y' = X_{\mathscr{I}} \times_X^{\mathrm{fs}} Y$ is still connected. But the projection $Y' \to Y$ being a log blow-up is proper with geometrically connected fibres, hence Y' is connected.

Now we prove effectivity or equivalently that $\pi_1^{\log}(\sigma)$ is bijective. By the usual reasoning with Lemma 3.2.2 and Proposition 3.2.19 we may assume that X and $\mathscr{I}$ have a global chart and X is quasi-compact. We proceed by induction on the essential size of I due to the following devissage argument. Let $J \subset I$ be an ideal of P generated by $\mathrm{ess}(I) - 1$ of the essential generators of I and consider the cartesian square

$$\begin{array}{ccc} X_{\mathscr{I},\mathscr{J}} & \longrightarrow & X_{\mathscr{I}} \\ \downarrow & & \downarrow \\ X_{\mathscr{J}} & \longrightarrow & X \end{array} \tag{3.3.2}$$

where $\mathscr{J}$ is associated to J and all maps are log blow-ups. If interpreted as the maximum about the local values, obviously $\mathrm{ess}(\mathscr{I}|_{X_{\mathscr{J}}}) \leq 2$ and $\mathrm{ess}(\mathscr{J}|_{X_{\mathscr{I}}}) \leq \mathrm{ess}(\mathscr{J})$ holds. But if three of the log blow-ups in the diagram (3.3.2) induce isomorphisms on $\pi_1^{\log}$ then so does the fourth. We may therefore assume that $\mathrm{ess}(\mathscr{I}) \leq 2$.

Claim: If $\mathrm{ess}(\mathscr{I}) \leq 2$ then all geometric fibres are simply connected (for the classical π_1). This follows by direct calculation. Let a, b generate the stalk $I_{\overline{x}}$ at some geometric point $\overline{x}$ such that $\mathrm{ess}(I_{\overline{x}}) = 2$. Then up to nilpotent elements the fibre above $\overline{x}$ is obtained by glueing $\mathrm{Spec}(k[\frac{a}{b}])$ with $\mathrm{Spec}(k[\frac{b}{a}])$ in the obvious way. Hence the relevant reduced fibre is a projective line which is simply connected. Furthermore, the fibres above points where $\mathscr{I}$ is already monogeneous consist of one point.

We conclude from IX 6.11 [SGA 1] that $\pi_1(\overset{\circ}{\sigma}) : \pi_1(\overset{\circ}{X}_{\mathscr{I}}) \to \pi_1(\overset{\circ}{X})$ is an isomorphism. Now we again reduce by Lemma 3.2.2 and Proposition 3.2.19 to the strict local case of X strict henselian local noetherian with chart modelled on $P = \overline{M}(X)$. By the above $\overset{\circ}{X}_{\mathscr{I}}$ is simply connected. Let $Y' \in \mathsf{Rev}^{\log}(X_{\mathscr{I}})$ be a connected Kummer log-étale cover. As $X_{\mathscr{I}}$ is quasi-compact and $\overline{u}_{\sigma}^{\mathrm{gp}}$ an isomorphism we may find a Kummer extension $P \hookrightarrow Q$ of fs monoids of index prime to the residue characteristics such that the cover belonging to $(Q^{\mathrm{gp}}/P^{\mathrm{gp}})^{\vee}$ by Proposition 3.1.10 dominates $Y'/X_{\mathscr{I}}$ locally. Thus the second projection

$$\mathrm{pr}_2 : Y' \times^{\mathrm{fs}}_{X_{\mathscr{I}}} \left(X_{\mathscr{I}} \times^{\mathrm{fs}}_X X_Q \right) \to X_{\mathscr{I}} \times^{\mathrm{fs}}_X X_Q$$

is strict étale. Because $X_{\mathscr{I}} \times^{\mathrm{fs}}_X X_Q$ is again a log blow-up of the above type it is simply connected and thus pr_2 has a section. Consequently Y' is dominated by a két cover which was pulled back from X. Hence $\pi_1^{\log}(\sigma)$ is also injective. □

Corollary 3.3.13. *Let X be a connected fs-log scheme. Let $\sigma : X_{\mathscr{I}} \to X$ be a log blow-up along the coherent ideal $\mathscr{I}$. Then $X_{\mathscr{I}}$ is connected and also*

$$\pi_1(\overset{\circ}{\sigma}) : \pi_1(\overset{\circ}{X}_{\mathscr{I}}) \xrightarrow{\sim} \pi_1(\overset{\circ}{X})$$

is an isomorphism. □

3.4 Logarithmic good reduction

Throughout Section 3.4 let S be the spectrum of an excellent henselian discrete valuation ring R with perfect residue field $k = R/(\pi)$ of exponential characteristic $p \geq 0$ and field of fractions K. Let η (resp. s) denote its generic (resp. closed) point. We fix a geometric point $\overline{\eta} = \mathrm{Spec}(\overline{K})$ (resp. $\overline{s} = \mathrm{Spec}(\overline{k})$) over η (resp. s). We endow S with the standard fs-log structure $M(\log s)$ induced by $\mathbb{N} \to R$, $1 \mapsto \pi$. We fix furthermore log geometric points $\tilde{\eta}$ (resp. $\tilde{s}$) over $\overline{\eta}$ (resp. $\overline{s}$).

The theory of good reduction seeks to find log-smooth models over S for varieties over η. The theory of specialisation compares the $\pi_1^{\log}$ of the log geometric fibres of proper, log-smooth models and hints at a criterion for their existence. With the help of M. Artin's results about rational singularities of surfaces and a theorem of T. Saito we establish a new criterion for log-smooth reduction of curves. Finally, we treat good reduction of két covers aiming at a group-theoretical control of the logarithmic specialisation map.

Sections 3.4.1 and 3.4.2 are independent. But both are used in the subsequent sections 3.4.3 and 3.4.4. Unfortunately the author didn't find an interaction in parallel to Tamagawa's case between the logarithmic view on Saito's theorem, Section 3.4.3, and the result about logarithmic good reduction of covers, Section 3.4.4. Only the latter is used in later chapters.

3.4.1 Semistability

Here we show the well known fact that semistability implies logarithmic smoothness. This is well known, but it is used in the proof of Theorem 3.4.8 and therefore appears here for illustration.

Definition 3.4.1. *Let $D \subset X$ be a closed subscheme on the scheme X. Let $j : U \hookrightarrow X$ be the inclusion of the complement $U = X \setminus D$. The log-structure $M(\log D)$ on X is the natural map $M_X = j_*\mathbb{G}_{m,U} \times_{j_*\mathcal{O}_U} \mathcal{O}_X \to \mathcal{O}_X$.*

Lemma 3.4.2. *Let $D \subset X$ be a divisor on a regular scheme. Let $j : U \hookrightarrow X$ be the inclusion of the complement $U = X \setminus D$. Then $M(\log D)$ equals the natural map $M_X = j_*\mathbb{G}_{m,U} \cap \mathcal{O}_X \hookrightarrow \mathcal{O}_X$ and is a fs-log structure on X.*

Proof: Let $\mathrm{Div}(D) = \bigoplus \underline{\mathbb{Z}}_{\eta_C}$ with $\underline{\mathbb{Z}}_{\eta_C} = i_{\eta_C,*}\mathbb{Z}$ be the sheaf of divisors with support in D; for a component C let $i_{\eta_C} : \eta_C \hookrightarrow X$ be the inclusion of its generic point. Then the short exact sequence

$$0 \to \mathbb{G}_{m,X} \to j_*\mathbb{G}_{m,U} \xrightarrow{\nu} \mathrm{Div}(D) \to 0$$

involving the valuations along the components of the divisor induces an isomorphism $\overline{M}_X \cong \bigoplus \underline{\mathbb{N}}_C$. Local sections yield charts for $M(\log D)$ by fs monoids. □

Definition 3.4.3. *Let X/S be flat, separated and of finite type with X reduced. The scheme X is* **relavely semistable over** *S if étale locally on X we have: (i) X_η is smooth over η, (ii) X_s is reduced, (iii) the irreducible components X_i of X_s are divisors on X and (iv) for each nonempty $J \subset \{$irreducible components of $X_s\}$ the intersection $X_J = \bigcap_{j \in J} X_j$ is smooth over s of codimension $\#J$ in X.*

Lemma 3.4.4. *Let X/S be relatively semistable. Then X is regular and étale locally X is smooth over $\mathrm{Spec}\, R[t_1, \ldots, t_r]/(t_1 \cdot \ldots \cdot t_r - \pi)$. In particular $(X, M(\log X_s))$ is (in a unique way) log-smooth over S with its fs-log structure $M(\log s)$.*

Proof: Let $x \in X_s$ be a point and $J = \{j \,|\, x \in X_j\}$. Renumber to get $J = \{1, \ldots, r\}$. Let $t_j \in \mathcal{O}_{X,x}$ be an equation for X_j near x. Then $\mathcal{O}_{X,x}$ is regular as $\mathcal{O}_{X,x}/(t_1, \ldots, t_r)$ is regular of codimension r by assumption.

By changing one of the t_j by a unit we may assume that $\pi = t_1 \cdot \ldots \cdot t_r$. The map $R[[t_1, \ldots, t_r]]/(t_1 \cdot \ldots \cdot t_r - \pi) \to \hat{\mathcal{O}}_{X,x}$ is seen to be formally smooth be general arguments (reduce mod t_j).

Thus $(X, M(\log X_s))$ is strict (classically) smooth over $\operatorname{Spec}\big(R[\mathbb{N}^r]/\pi - (1, \ldots, 1)\big)$ which is itself log-smooth over S being the fibre of morphism modelled over R on the diagonal embedding of $\mathbb{N}$ in $\mathbb{N}^r$, cf. 3.5 [Ka89]. $\square$

Lemma 3.4.5. *Let X/S be flat, of relative dimension 1, such that X is regular and the reduced special fibre $X_{s,\mathrm{red}}$ is a divisor with normal crossings on X. If p does not divide the multiplicities of the components of X_s then $(X, M(\log X_s))$ is log-smooth over S in a natural way.*

Proof: By Theorem 3.3 [Ba95] we may work két local on S and X. In particular we allow finite tamely ramified extensions of R. Let $S' \to S$ be such a két cover on the base. The fs base change X' is normal being log-étale over the log-regular X. Thus X' is the normalisation of the schematic base change its fs-log structure being again $M(\log X'_s)$. A (complete) local analysis shows that X' has again a normal crossing divisor as reduced special fibre, cf. [Ab00].

By Abhyankar's Lemma and the assumptions on the multiplicities of the components of X_s we reduce to the case of a reduced special fibre. But we have lost that X is regular. Nevertheless X/S is smooth outside the double points of X_s while the canonical enriched log-morphism is strict there. Hence X/S may violate being log-smooth only in the double points. These are étale locally of the form $Y = \operatorname{Spec}(R[u,v]/uv - \pi^e)$. But here we have a chart modelled on $Q(e) = \langle (1,0), (e-1, e)\rangle^{\mathrm{sat}} \subseteq \mathbb{Z}^2$ such that $Y = \operatorname{Spec}\big(R[Q(e)]/\pi - (1,1)\big)$ which is easily seen to be log-smooth over S. $\square$

3.4.2 Logarithmic specialisation

Generalising the classical Theorem X 3.8 [SGA 1] about the specialisation of étale fundamental groups, Vidal has proven in chapter I [Vi02] the following theorem about log specialisation.

Theorem 3.4.6 (I 2.2 [Vi02]). *Let S_t be the normalisation of S in the maximal tamely ramified extension of K in $\overline{K}$ endowed with the integral log structure $M_{S_t} = \mathcal{O}_{S_t} \setminus \{0\}$ and fix an isomorphism of $\tilde{s}$ with the closed point of S_t with its induced log structure.*

Let $f : X \to S$ be a proper, log-smooth morphism of fs-log schemes with connected log geometric special fibre $X_{\tilde{s}}$, e.g., geometrically connected generic fibre. Then the following holds:

(1) The natural map $\pi_1^{\log}(X_{\tilde{s}}) \to \pi_1^{\log}(X_{S_t})$ is an isomorphism.

(2) The specialisation map $\mathrm{sp}_{\log} : \pi_1^{\log}(X_{\overline{\eta}}) \to \pi_1^{\log}(X_{S_t}) \cong \pi_1^{\log}(X_{\tilde{s}})$ is surjective.

(3) The specialisation map induces an isomorphism on the prime-to-p quotients:

$$\mathrm{sp}_{\log}{}^{p'} : \pi_1^{\log}(X_{\overline{\eta}})^{p'} \xrightarrow{\sim} \pi_1^{\log}(X_{\tilde{s}})^{p'} \ . \qquad \square$$

Remark. The log fundamental group of the integral log schemes X_{S_t} and $X_{\tilde{s}}$ are defined as follows. These log schemes are pro fs-log schemes induced by fs base changes from the normalisations (and closed points) of the normalisations S' (and $s' \in S'$) in finite tamely ramified extensions of S. We may define $\mathsf{Rev}^{\log}(X_{S_t})$ (resp. $\mathsf{Rev}^{\log}(X_{\bar{s}})$) as the two-category limit of the direct system of all $\mathsf{Rev}^{\log}(X_{S'})$ (resp. $\mathsf{Rev}^{\log}(X_{s'})$) which again are Galois categories by abstract nonsense and whose fundamental group happens to be the projective limit of the fundamental groups of the Galois categories from the system.

Corollary 3.4.7 (I 3.2 [Vi02]). *Let P be the p-Sylow subgroup of G_K, i.e., its wild ramification subgroup. Let $X \to S$ be proper, log-smooth morphism of fs-log schemes with log geometric connected fibres. Then P acts trivial on $\pi_1^{\log}(X_{\overline{\eta}})^{p'}$.* □

This in particular applies to proper, generically smooth semistable curves over S which are endowed with the fs-log structure $M(\log X_s)$ where X_s is the special fibre – or more generally – to schemes X proper over S such that the special fibre is a normal crossing divisor D and X is endowed with the fs-log structure $M(\log D)$.

3.4.3 Criteria for logarithmic good reduction

The known criteria describe an equivalence between good reduction and the vanishing of the monodromy action by the inertia group. Here we give an interpretation in terms of logarithmic geometry of the consequences of the vanishing of the monodromy action by solely the wild inertia in the case of curves. This gives a conceptual explanation of T. Saito's description (c) in the subsequent theorem of the reduction behaviour. Only the purely geometric step (c) ⇒ (d) is new. It is derived by a close examination of a rational singularity. The following theorem supplies a first step towards a group-theoretical control of the logarithmic specialisation map.

Theorem 3.4.8 (criterion for log-smooth reduction). *Let X_K be a smooth proper curve over $\mathrm{Spec}(K)$ of genus $g \geq 2$. Let ℓ be a prime number different from the residue characteristic of S. Then the following are equivalent:*

(a) The wild inertia group $P < G_K$ acts trivially on $\pi_1(X_{\overline{K}})^{\ell}$, i.e., the restriction of $\rho_{X_K} : G_K \to \mathrm{Out}\big(\pi_1(X_{\overline{K}})^{\ell}\big)$ to P is trivial.

(b) P acts trivially on $\pi_1(X_{\overline{K}})^{\mathrm{ab},\ell}$.

(c) The minimal regular model of X_K over S such that the reduced special fibre is a divisor with normal crossings satisfies the following property: any component of the special fibre with multiplicity divisible by p is isomorphic to $\mathbb{P}^1_k$ and meets the rest of the special fibre in exactly two points lying on components of multiplicity prime to p.

(d) X_K has log-smooth reduction over S, i.e., there is a model X/S such that $M(\log X_s)$ is a fs-log structure on X that turns it into a log-smooth fs-log scheme over S.

Proof: $(d) \Rightarrow (a)$: Corollary 3.4.7.

$(a) \Leftrightarrow (b)$: By the Theorem of Semistable Reduction, see [Ab00], there is a semistable regular model — thus in particular a log-smooth model, Lemma 3.4.4 — after eventually enlarging S by a finite generically étale but ramified extension. Hence by $(d) \Rightarrow (a)$ the wild inertia P acts through a finite quotient anyway. But the kernel of $\mathrm{Out}\big(\pi_1(X_{\overline{K}})^{\ell}\big) \to \mathrm{Aut}\big(\pi_1(X_{\overline{K}})^{ab,\ell}\big)$ contains no torsion by [As95]. Therefore the images of P in both groups of (outer-)automorphism are isomorphic.

$(b) \Leftrightarrow (c)$: Theorem 3 [Sai87], see also [Ab00].

$(c) \Rightarrow (d)$: Let X'/S be the model from (c). Lemma 3.4.5 tells us that X'/S is already log-smooth away from the components with multiplicity divisible by p. Hence we will work locally around such components. By §6.7 Theorem 1 and Lemma 4 [BLR90] we can blow down the components E of X'_s with multiplicity divisible by p and obtain a normal scheme X flat over S such that $\sigma : X' \to X$ is topologically the contraction of these E's and $\mathcal{O}_X \to \sigma_*\mathcal{O}_{X'}$ is an isomorphism. To study the properties of X/S we may work étale locally at a geometric point $\overline{x}$ localised in $\sigma(E)$ for a fixed component E of the above type.

Let $A = \mathcal{O}^{\mathrm{sh}}_{X,\overline{x}}$ and consider $\sigma : X' \to X$ as the pullback to Spec(A). Then $X'_{s,\mathrm{red}}$ decomposes into three components C', D', E as a divisor and

$$(C' \bullet E) = 1,\ (D' \bullet E) = 1,\ (E^2) < 0 \tag{3.4.3}$$

because E intersects C', D' transversally and due to [Mu61], the intersection pairing being partially defined as in [Ar66]. As the arithmetic genus of E vanishes and E is the fundamental cycle, X has a rational singularity in $\overline{x}$ by Theorem 3 [Ar66]. A suitable adaption of 1.7 [Ar62] yields that restriction induces an isomorphism $\mathrm{Pic}(X') \cong \mathrm{Pic}(E)$ which furthermore is isomorphic to $\mathbb{Z}$ by the degree.

Let $U = X_\eta$ and $j : U \hookrightarrow X$ (resp. X') the inclusion. Then the cohomology sequence of σ_* applied to

$$0 \to \mathbb{G}_{m,X'} \to j_*\mathbb{G}_{m,U} \xrightarrow{\nu} \mathrm{Div}(X'_{s,\mathrm{red}}) \to 0$$

yields

$$0 \to \mathbb{G}_{m,X} \to j_*\mathbb{G}_{m,U} \xrightarrow{\nu} \sigma_* \mathrm{Div}(X'_{s,\mathrm{red}}) \xrightarrow{\delta} \mathrm{R}^1\, \sigma_*\mathbb{G}_{m,X'} \to 0\ .$$

The map δ is surjective as it can canonically be identified with "minus the intersection with E":

$$-(\ \bullet E) : \mathbb{Z}_C \oplus \mathbb{Z}_D \oplus \mathbb{Z}_{\overline{x}} \to \mathbb{Z}_{\overline{x}}$$

where $C = \sigma(C'), D = \sigma(D')$. The image P of $M_X = j_*\mathbb{G}_m \cap \mathcal{O}$ in the kernel of $-(\ \bullet E)$ consists of the non-negative elements, i.e., by (3.4.3)

$$\begin{aligned} P &= \{(c, d, e) \in \mathbb{Z}^3 \mid c, d, e \geq 0,\ c + d + e(E^2) = 0\} && (3.4.4) \\ &= \{(d, e) \in \mathbb{Z}^2 \mid d, e \geq 0,\ d + e(E^2) \leq 0\}\ . && (3.4.5) \end{aligned}$$

Hence M_X is a fs-log structure on X a chart given locally by a section $P \to M_X$, cf. 2.10 [Ka89]. In the sequel we regard X as a fs-log scheme in this way.

The map $\mathrm{Spec}(R[P]) \to \mathrm{Spec}(R[\mathbb{N}])$ induced by

$$\mathbb{N} \to P;\ 1 \mapsto p_\pi = \big(\mathrm{mult}_D(\pi), \mathrm{mult}_E(\pi)\big)$$

is log-smooth by the assumption on the multiplicities of the components next to the bad $\mathbb{P}^1_k = E$. Thus it suffices to show that the strict map $\mathrm{Spec}(A) \to \mathrm{Spec}\big(R[P]/\pi = p_\pi\big)$ is (formally) étale in $\overline{x}$, i.e., induces an isomorphism on the completions of the strict henselisations. Because R is excellent with perfect residue field both completions are normal of dimension 2, hence surjectivity on cotangent spaces is enough.

Let $I_{max} = P \setminus \{0\}$ be the maximal ideal and let

$$W = I_{max} \setminus I^2_{max} = \{(c, d, 1) \mid c, d \geq 0,\ c + d = -(E^2)\}$$

be the essential set of generators. Then W forms a basis of the cotangent space of the strict henselisation of $\mathrm{Spec}(R[P]/\pi = p_\pi)$ as a k^{sep} vector space. Let $\mathfrak{m}$ be the maximal ideal of A. We need to identify the image of W in $\mathfrak{m}/\mathfrak{m}^2$.

Let $Z_{c,d}$ denote the divisor $-E - cC - dD$. The linebundle $\mathscr{L}(Z_{c,d})$ on X' such that $(c, d, 1) \in W$ is trivial as the degree of its restriction to E vanishes. Let $f_{c,d}$ be a trivializing section. It is uniquely defined up to multiplication by A^* and coincides with the image of $(c, d, 1)$ under a chart of the fs-log structure up to a factor from A^*. Next we consider the short exact sequence

$$0 \to \mathscr{L}(-E) \to \mathcal{O}(X') \to \mathcal{O}_E \to 0$$

and tensor with $\mathscr{L}(Z_{c,d}) \subset \mathscr{L}(-E)$. We obtain a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{L}(-2E) & \longrightarrow & \mathscr{L}(-E) & \longrightarrow & i_{E,*}\mathscr{N}_E & \longrightarrow & 0 \\ & & \cup & & \cup & & \cup & & \\ 0 & \longrightarrow & \mathscr{L}(Z_{c,d} - E) & \longrightarrow & \mathscr{L}(Z_{c,d}) & \longrightarrow & i_{E,*}\mathscr{N}_E\big(- c(C \cap E) - d(D \cap E)\big) & \longrightarrow & 0 \end{array}$$

with the normal bundle $\mathscr{N}_E$ of $i_E : E \subset X'$ and with exact rows and injective columns as $\mathrm{Tor}_1(\mathcal{O}_E, \mathcal{O}_{cC+dD})$ vanishes due to transversality. By Theorem 4 [Ar66] we may evaluate $\mathrm{H}^0(X', \cdot)$ of the diagram as

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{m}^2 & \longrightarrow & \mathfrak{m} & \longrightarrow & \mathfrak{m}/\mathfrak{m}^2 & \longrightarrow & 0 \\ & & \cup & & \cup & & \cup & & \\ 0 & \longrightarrow & \mathfrak{m}^2 \cap (A \cdot f_{c,d}) & \longrightarrow & A \cdot f_{c,d} & \longrightarrow & k^{\mathrm{sep}} \cdot f_{c,d} & \longrightarrow & 0 \end{array}$$

because the relavant H^1 vanish by Lemma 5 [Ar66]. In particular, $f_{c,d}$ is nontrivial in $\mathfrak{m}/\mathfrak{m}^2$. In fact, W forms a basis due to the following lemma and the theorem about the criterion for log-smooth reduction is proven. □

Lemma 3.4.9. *Let $P \neq Q \in \mathbb{P}^1_k$ be closed points. Then*

$$\bigcup_{p+q=n, p,q \geq 0} \mathrm{H}^0\big(\mathbb{P}^1_k, \mathcal{O}(n - pP - qQ)\big)$$

generates $\mathrm{H}^0\big(\mathbb{P}^1_k, \mathcal{O}(n)\big)$ as a vector space over k.

Proof: Let x_0, x_1 be homogeneous linear coordinates for $\mathbb{P}^1_k$ such that $P = 0$ and $Q = \infty$. Then

$$\mathrm{H}^0\big(\mathbb{P}^1_k, \mathcal{O}(n)\big) = gr_n k[x_0, x_1] \supseteq k \cdot x_0^p x_1^q = \mathrm{H}^0\big(\mathbb{P}^1_k, \mathcal{O}(n - p \cdot 0 - q \cdot \infty)\big) \ .$$

□

Corollary 3.4.10 (theorem of log-smooth reduction). *The equivalent criteria (a)-(d) of Theorem 3.4.8 are satisfied after possibly a finite (wildly ramified) extension of the discrete valuation ring.* □

Corollary 3.4.11. *Assume that the equivalent criteria of Theorem 3.4.8 are satisfied. Then the model in (c) is itself log-smooth over S in the canonical way.*

Proof: We use information and notation from the proof of (c) $\Rightarrow$ (d). We look at the contraction map $\sigma : X' \to X$ of one bad $E' = \mathbb{P}^1_k$ over the local scheme $X = \mathrm{Spec}(\mathcal{O}_{X,x})$ where $\{x\} = C \cap D$ is the (local) intersection of two components of the special fibre. Let $\tau : X_I \to X$ be the log blow-up of X along the ideal generated by $I = \langle (0,1),(1,1) \rangle \subset P \subset \mathbb{Z}^2$ where P is as in (3.4.5). The reduced fibre of τ is a $E_I = \mathbb{P}^1_k$. As log blow-ups are log-étale the fs-log scheme X_I is log-smooth over S and thus regular outside the locus $\{x \mid \mathrm{rk}_{\mathbb{Z}} \overline{M}^{\mathrm{gp}}_{X_I,x} \geq 2\}$. In the remaining two points where the strict transforms C_I, D_I of C, D meet E_I we have $\overline{M}_{X_I,x} = \langle P, (-1,0) \rangle^{\mathrm{sat}}$ or $\langle P, (1,0) \rangle^{\mathrm{sat}}$ which are both isomorphic to $\mathbb{N}^2$. Thus X_I is regular.

Both σ and τ yield a desingularisation of X to a regular surface. The birational map $\tau^{-1}\sigma : X' \to X_I$ is an isomorphism outside E' (resp. E_I) and due to the minimal model theory for regular surfaces must be an isomorphism everywhere. Hence σ may be identified with the log-étale τ showing that X' was already log-smooth over S. □

3.4.4 Logarithmic good reduction of covers

After having settled the question on log-smooth reduction for curves we now try to treat logarithmic good reduction of étale covers. By that we mean the reduction as a két cover of a log-smooth model, i.e., we will study the logarithmic specialisation map.

Lemma 3.4.12. *Let X_K be a smooth proper curve of genus $g \geq 2$ over K. Then all its semistable models (if any exists at all) considered as a fs-log scheme with the canonical fs-log structure $M(\log X_s)$ have canonically isomorphic $\pi_1^{\log}$, i.e., isomorphic as a quotient of $\pi_1(X_K)$.*

Proof: This follows as the map to the unique stable model is a composition of classical blow-ups $X' \to X$ along (1) double points of thickness > 1 or (2) smooth points of the special fibre.

Indeed, blow-ups of type (1) were discussed in the example on page 3.3.2 and shown to enrich to log blow-ups. Now we use Corollary 3.3.11.

Concerning blow-ups at points of type (2) we argue as follows. By Theorem 3.4.6 we may compare the $\pi_1^{\log}$ of the log geometric fibres instead. Any két cover of $X'_{\bar{s}}$ must be completely decomposed along the exceptional fibre which constitutes in a $\mathbb{P}^1$ connected with one point to $X_{\bar{s}}$ (a "hair"). Indeed, $\mathbb{A}^1$ has no nontrivial tame covers. Thus the natural map $\pi_1^{\log}(X'_{\bar{s}}) \to \pi_1^{\log}(X_{\bar{s}})$ is an isomorphism. (We could argue simultaneously for both types of blow-ups by applying the log Van Kampen theorem as in (6.2.5) to the special fibres). □

Proposition 3.4.13. *Let Y/S be a proper, generically smooth, semistable curve of genus $g \geq 2$. Let $G \subset \mathrm{Aut}(Y/S)$ be a finite group of S-automorphisms. Then the quotient Y/G exists and is a semistable model.*

Let $Y \to Y/G$ be étale in the generic fibre but Y not smooth over S. When equipped with the canonical fs-log structures $M(\log Y_s)$, resp. $M(\log(Y/G)_s)$, then moreover $Y \to Y/G$ is a Kummer étale cover if and only if G acts on the geometric set of double points with stabilizers that are prime-to-p groups.

Proof: The quotient $X = Y/G$ exists as Y is projective by an adaption of 2.8 [Li68]. Furthermore X is again semistable by the appendix of [Ra90]. We now use the details of the latter to proceed.

There is no wild ramification at generic points of components of the special fibre as otherwise there would be a double point that is fixed by a p-subgroup (the special fibre is geometrically connected and not smooth). Hence $Y \to X$ is at most tamely ramified outside a finite set of closed points. After base extension by a finite két cover of S, i.e., a finite tamely ramified cover, we may by Abhyankar's Lemma assume that $Y \to X$ is even étale over codimension 1. By Zariski-Nagata purity, cf. X 3.1 [SGA 1], $Y \to X$ is étale over the regular locus. Consequently, the set of double points of Y is the preimage of the set of double points of X and we need only study the map locally at a double point.

Let $\overline{y}$ be a geometric double point of Y and $\overline{y} \mapsto \overline{x}$. The degree of $\mathcal{O}^{\mathrm{sh}}_{Y,\overline{y}}/\mathcal{O}^{\mathrm{sh}}_{X,\overline{x}}$ is the cardinality of the stabilizer of $\overline{y}$ which therefore must be prime to p in case $Y \to X$ is két. On the other hand, let us assume the degree is e prime to p. Then we claim that there is a diagram of two cartesian squares of fs-log schemes

$$\begin{array}{ccccc} \mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{Y,\overline{y}}) & \xrightarrow{f_Y} & \mathrm{Spec}(R[u,v]/uv-\pi^n) & \longrightarrow & \mathrm{Spec}(R[Q(n)]) \\ \downarrow & & \downarrow & & \downarrow h \\ \mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{X,\overline{x}}) & \xrightarrow{f_X} & \mathrm{Spec}(R[U,V]/UV-\pi^{en}) & \longrightarrow & \mathrm{Spec}(R[Q(en)]) \end{array}$$

where f_Y, f_X are strict pro-étale, the horizontal arrows in the second cartesian square are the standard one and h is induced by the map

$$h : Q(en) \to Q(n);\ (1,0) \mapsto (e,0),\ (en-1,en) \mapsto (en-e,en),\ (1,1) \mapsto (1,1)$$

which is Kummer of index e. Consequently all vertical maps are két covers.

It remains to find suitable u, v, U, V. We first choose arbitrary parameters u_1, v_1 such that $\mathcal{O}^{\mathrm{sh}}_{Y,\overline{y}} = \big(R[u_1,v_1]/u_1v_1 - \pi^n\big)^{\mathrm{sh}}$. Their norms $U = N_{Y/X}(u_1), V = N_{Y/X}(v_1)$ are local parameters of X at $\overline{x}$ which satisfy $UV = \pi^{en}$, cf. Appendix [Ra90]. Note that X is normal and $Y \to X$ is flat over codimension 1 and hence the norm is well behaved. By examination of the associated divisors on Y near $\overline{y}$ we get $u_1^e = U\varepsilon$, $v_1^e = V\varepsilon'$ with $\varepsilon, \varepsilon'$ mutually inverse units in $\mathcal{O}^{\mathrm{sh}}_{Y,\overline{y}}$. Because e is prime to the residue characterisitic the equation $z^e = \varepsilon$ is étale and possesses a root in $\mathcal{O}^{\mathrm{sh}}_{Y,\overline{y}}$. Changing the local parameters with the help of this e^{th}-root and its inverse we find u, v such that $u^e = U, v^e = V, uv = \pi^n$. Then $f_X^{\mathrm{sh}} : \big(R[U,V]/UV - \pi^{en}\big)^{\mathrm{sh}} \to \mathcal{O}^{\mathrm{sh}}_{X,\overline{x}}$ is finite birational as $\mathcal{O}^{\mathrm{sh}}_{Y,\overline{y}}$ is finite of generic degree e over both rings. By normality f_X^{sh} is an isomorphism which proves that u, v, U, V as constructed above fulfill the properties of the claim. □

Theorem 3.4.14 (log good reduction of covers). *Let X/S be a proper, generically smooth, but not smooth, semistable curve of genus $g \geq 2$ equipped with the canonical fs-log structure $M(\log X_s)$. Let $\mathrm{pr} : \pi_1(X_K) \twoheadrightarrow G_K$ be the canonical map, and let H be an open normal subgroup of $\pi_1(X_K)$. Then the following are equivalent:*

(a) H contains the kernel of the log specialisation map

$$\mathrm{sp}_{\log} : \pi_1^{\log}(X_K) \twoheadrightarrow \pi_1^{\log}(X_s)\ .$$

(b) *There exists a finite tamely ramified extension K'/K (corresponding to an extension of spectra of valuation rings $S' \to S$) with inertia group $I' = I \cap G_{K'}$ such that if we set $H' = H \cap \mathrm{pr}^{-1}(G_{K'})$ and $\overline{H}' = H' \cap \pi_1(X_{\overline{K}})$ the following holds:*

   (i) $\mathrm{pr}(H') = G_{K'}$,

   (ii) *I' acts unipotently on $\left(\overline{H}'\right)^{\mathrm{ab},\ell}$ for some prime number $\ell \neq p$,*

   (iii) *For a $G = \mathrm{Aut}(Y'_{K'}/X_{K'})$-equivariant semistable model Y'/S' of the covering space $Y'_{K'}$ over $X_{K'}$ associated to H' the stabilizers of the G-action on the set of geometric double points of Y' have orders prime to p.*

Proof: $(a) \Rightarrow (b)$: $pr(H)$ contains the wild inertia and thus defines an at most tamely ramified extension. Furthermore, P acts trivial on $\overline{H}^{\ell}$ such that by the Monodromy Therorem there is a finite tame extension K'/K such that the inertia $I' \subset G_{K'}$ acts unipotently on $\left(\overline{H}'\right)^{ab,\ell}$. By the Theorem on Semistable Reduction in its precise form, cf. Theorem 1 [Sai87], there is a semistable model and by choosing a minimal one we obtain an equivariant Y'/S'. Its quotient by G as in (iii) exists and yields a semistable model of $X_{K'}$. As by Lemma 3.4.12 a specific cover of the generic fibre has simultaneously over all models good or bad reduction as a két cover the quotient $Y' \to Y'/G$ is Kummer étale. By Proposition 3.4.13 therefore (iii) is satisfied.

$(b) \Rightarrow (a)$: We may replace H by H' as a tamely ramified cover S'/S is just a két cover. By the Theorem of Semistable Reduction cited above (i)+(ii) guarantee the existence of a model Y'/S' as in (iii). But then again $Y' \to Y'/G$ is két by Proposition 3.4.13 and Lemma 3.4.12 shows that there is also log good reduction for the generic cover over $X_{S'}$. This implies that H' contains the kernel of $\mathrm{sp}_{\log}$. □

Unfortunately, due to condition (iii), the above criterion is not purely group-theoretic. Thus it does not allow a priori as one likes to control the kernel of $\mathrm{sp}_{\log}$ as the intersection of all H with log good reduction, and furthermore to control the map $\mathrm{sp}_{\log}$ itself. What is more, the result of Lemma 3.4.12 makes it impossible to reconstruct the set of double points together with a group action of just any semistable model. One should at least stick to a unique model as the stable or the minimal regular semistable one to have any hope. However, the assertion about the cardinality of the stabilizers is valid or false simultaneously (a priori for $p \neq 2$) for all models and therefore has a chance to be group-theoretically controlled.

Part II

Anabelian Geometry

Chapter 4

Fundamental Groups

This chapter supplies the framework for studying étale fundamental groups with respect to the Grothendieck Conjecture in Anabelian Geometry.

4.1 Preliminaries on fundamental groups

We recall without proofs the fundamental group as a functor π_1. We stress the well known interplay between the different viewpoints of π_1 of a variety over a non algebraically closed field as an extension or as an exterior representation. Then we formally invert geometric Frobenius maps since these have to be treated as isomorphisms from the point of view of Anabelian Geometry in positive characteristic. Finally, we study certain geometric properties in group-theoretical terms for fundamental groups, e.g., Isom-sheaves and G-torsors. Here it will be essential to apply the connection between extensions and representations also to a geometric monodromy action.

4.1.1 The functor π_1

The construction of a fundamental group depends on the choice of a base point. As a functor, it is only defined on pointed connected schemes. Neglecting the base point, forces us to regard π_1 in the category $\mathscr{G}$ of pro-finite groups with exterior (continuous) homomorphisms, i.e., equivalence classes of group homomorphisms modulo composition with inner automorphisms. This precisely ignores the ambiguity coming from a choice of a base point. Hence, π_1 becomes a functor from connected schemes to $\mathscr{G}$.

Exterior Galois representation. Let K be a field. We fix an algebraic closure $\overline{K}$. For geometrically connected varieties X over K, the fundamental group carries more structure. The absolute Galois group $G_K = \mathrm{Aut}(\overline{K}/K)$ acts from the right on $X_{\overline{K}}$ and thus, by functorial transport, also on $\pi_1(X_{\overline{K}})$ as an object in $\mathscr{G}$. Using inverses, we transform the right action to a left action which is denoted by $\rho_X : G_K \to \mathrm{Aut}_{\mathscr{G}}(\pi_1(X_{\overline{K}}))$.

Let $\mathscr{G}(G_K)$ be the category of G_K-representations in $\mathscr{G}$, i.e., the category of pairs (V, ρ) where $V \in \mathscr{G}$ and ρ is a continuous map $G_K \to \mathrm{Aut}_{\mathscr{G}}(V) = \mathrm{Out}(V)$. Clearly we obtain a functor

$$\begin{aligned} \pi_1 : \mathsf{Var}_K &\to \mathscr{G}(G_K) \\ X &\mapsto \rho_X \end{aligned}$$

on geometrically connected varieties over K with values in $\mathscr{G}(G_K)$.

$\mathbf{G}_K$-extensions. The fundamental group of a geometrically connected variety over K forms naturally an extension

$$1 \to \pi_1(X_{\overline{K}}) \to \pi_1(X) \to G_K \to 1\ . \tag{4.1.1}$$

We call this the homotopy short exact sequence, cf. IX 6.1 [SGA 1]. Maps of K-varieties produce morphisms between such short exact sequences with identity on the G_K part. The base points being neglected, only $\pi_1(X_{\overline{K}})$-conjugacy classes of maps are well defined.

Let $\mathrm{EXT}[G_K]$ be the category of G_K-extensions

$$1 \to \overline{\Pi} \to \Pi \to G_K \to 1\ .$$

Maps in $\mathrm{EXT}[G_K]$ are classes of maps of short exact sequences with identity on the G_K part up to conjugacy by elements of the kernel $\overline{\Pi}$. We obtain a functor

$$\begin{array}{rccc} \pi_1: & \mathsf{Var}_K & \to & \mathrm{EXT}[G_K] \\ & X & \mapsto & \big(1 \to \pi_1(X_{\overline{K}}) \to \pi_1(X) \to G_K \to 1\big)\ . \end{array}$$

Acting on the kernel $\pi_1(X_{\overline{K}})$ by conjugation passes to G_K if considered as an action in $\mathscr{G}$. Thus we obtain a functorial construction $R : \mathrm{EXT}[G_K] \to \mathscr{G}(G_K)$. We recover the representation ρ_X from the extension (4.1.1) by applying R. Moreover, R is an equivalence when restricted to isomorphisms, extensions with center free kernel, and G_K-representations on center free groups. The extension is recovered by pullback:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & V & \longrightarrow & \mathrm{Aut}(V) \times_{\mathrm{Out}(V)} G_k & \xrightarrow{\mathrm{pr}_2} & G_K & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow \rho & & \\ 1 & \longrightarrow & \mathrm{Inn}(V) & \longrightarrow & \mathrm{Aut}(V) & \longrightarrow & \mathrm{Out}(V) & \longrightarrow & 1\ . \end{array} \tag{4.1.2}$$

Monodromy. Let Y/X be an étale G-Galois cover of a geometrically connected variety X/K. Then $\pi_1(Y)$ is a normal subgroup N of $\pi_1(X)$. The homotopy sequence (4.1.1) induces for G the structure of an extension

$$1 \to \overline{G} \to G \to G(L/K) \to 1\ . \tag{4.1.3}$$

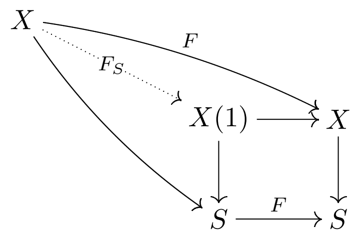
Here $G(L/K)$ is the Galois group of the field of constants L of Y over K and $\overline{G}$ is the geometric monodromy group $\pi_1(X_{\overline{K}})/\big(\pi_1(X_{\overline{K}}) \cap N\big)$. If we set $Y_{\overline{K}} = Y \times_L \overline{K}$, then $\overline{G}$ is naturally isomorphic to $\mathrm{Aut}^{\mathrm{opp}}\big(Y_{\overline{K}}/X_{\overline{K}}\big)$ which is the covering automorphism group of the associated geometric cover. The action of the covering automorphisms on $\pi_1(Y_{\overline{K}})$ corresponds — under this identification and up to taking inverses — to the conjugation by preimages in $\pi_1(X_{\overline{K}})$.

If a tame fundamental group π_1^{t} in the sense of [GM71] is defined, e.g., for the complement C of a relative étale divisor on a smooth curve over K, then the statements of the preceeding paragraphs hold for π_1^{t} as well. Let ρ_C^{t} denote the exterior G_K-representation attributed to C through π_1^{t}.

4.1.2 Topological invariance

It is known, that π_1 applied to universal homeomorphisms yields isomorphisms, cf. VIII 1.1 [SGA 4]. Theorem 3.1.16 ensures the same behaviour for $\pi_1^{\log}$, in particular for π_1^{t} of curves. Therefore the Grothendieck Conjecture of Anabelian Geometry cannot be true in positive characteristic for trivial reasons. The natural conclusion suggests to formally invert the class of universal homeomorphisms, a task already foreseen by Grothendieck in his "esquisse", footnote 3 [Gr84]. In what follows, we will formally invert geometric Frobenius maps in the category of varieties. Any anabelian statement in positive characteristic will be formulated with respect to this new geometric category.

Frobenius, cf. XV §1 [SGA 5]. Fix a prime number p. The Frobenius map F commutes with all maps between schemes of characteristic p, that is in $\mathrm{Sch}_{\mathbb{F}_p}$. If $S \in \mathrm{Sch}_{\mathbb{F}_p}$ then we have the following diagram in which the square is cartesian:



(4.1.4)

The diagram (4.1.4) defines a functor "Frobenius twist" $\cdot(1) : Sch_S \to Sch_S$ and a natural transformation $F_S : id_{Sch_S} \to \cdot(1)$, the "$S$-Frobenius" or geometric Frobenius. They behave well under base change: $\big(X \times_S T\big)(1) = X(1) \times_S T$ and $F_T = F_S \times_S T$. The m^{th} iterated twist will be denoted by $X(m)$.

In general X and its twist $X(1)$ are not isomorphic, e.g., the twist of $\mathbb{P}^1_k - \{0, 1, \lambda, \infty\}$ is still genus 0 but punctured in $0, 1, \lambda^p, \infty$. Or, the twist of an elliptic curve with invariant $\jmath$ has invariant $\jmath^p$.

Lemma 4.1.1. *Let S be a scheme in characteristic p. Then the Frobenius map $F : S \to S$ induces the identity on $\pi_1(S)$ in $\mathscr{G}$.*

Proof: For $T \in \mathsf{Rev}(S)$ the map $F_S : T \to T(1)$ is radicial, surjective and étale, hence an isomorphism by I 5.1 [SGA 1]. Therefore pullback by Frobenius is naturally isomorphic to the identity on $\mathsf{Rev}(S)$ and thus differs from the latter by an inner automorphism on $\pi_1(S)$. $\square$

Inverting Frobenius. Let K be a field of positive characteristic and consider the category Var_K of geometrically connected varieties over K. We would like to invert all universal homeomorphisms among varieties but find it technically easier just to invert all K-Frobenii. Let $\mathsf{Var}_{K,F_K^{-1}}$ denote the resulting localisation. One easily verifies that $\mathsf{Var}_{K,F_K^{-1}}$ possesses a calculus of left fractions for morphisms as explained in 10.3 [Wb94]. Therefore the set of morphisms in $\mathsf{Var}_{K,F_K^{-1}}$ is defined as

$$\mathrm{Hom}_{K,F_K^{-1}}(X,Y) = \varinjlim_n \mathrm{Hom}_K\big(X, Y(n)\big) \ .$$

The transfer maps of the directed system use the K-Frobenius. For a discussion of automorphism groups in the localisation see Appendix B. Lemma 4.1.2 shows that inverting K-Frobenii suffices if we restrict to normal varieties.

Lemma 4.1.2. *(1) Let $Y \to X$ be a universal homeomorphism between normal varieties. For $n \gg 0$, there is a factorisation $Y \to X \to Y(n)$ of the n^{th} power of the K-Frobenius.*

(2) If a functor Q on Var_K transforms K-Frobenii into isomorphisms then it also transforms all universal homeomorphisms between normal varieties into isomorphisms.

Proof: (1) We may work on affine pieces and need to show that for a finite, radicial inclusion $A \subset B$ of normal rings and n big enough we have $B^{p^n} \subset A$. This holds trivially for the fields of fractions $K \subset L$. By normality $B \cap K = A$, and part (1) follows.

(2) Let $f : Y \to X$ be a universal homeomorphism between normal varieties. By (1) we have for $n, m \gg 0$

$$Y \xrightarrow{f} X \xrightarrow{g} Y(n) \xrightarrow{h} X(m)$$

such that gf and hg are powers of K-Frobenii. Then $Q(g)$ has a left and a right inverse. Thus g and also f and h are transformed into isomorphisms. □

If we restrict further to the subcategory of $\mathsf{Var}_{K,F_K^{-1}}$ generated by smooth K-curves and dominant maps (the notion dominant still makes sense) then we get by Dedekind–Weber equivalence a category of function fields. More precisely, the localisation is realised by the perfection, i.e., the pure inseparable closure, of the function field together with the unique prolongation of the set of infinite places and maps respecting these places.

By topological invariance the functor π_1 (resp. π_1^{t}, if defined) restricted to geometrically connected varieties over K factorises through $\mathsf{Var}_{K,F_K^{-1}}$. We call the factorisation again π_1 (resp. π_1^{t}). To have a unified notation, we let $F_K = \mathrm{id}$ for fields K of characteristic 0, thus $\mathsf{Var}_{K,F_K^{-1}} = \mathsf{Var}_K$ in these cases.

4.1.3 Isom-sheaves and étale torsors

In this section we mimic certain geometric properties in group-theoretical terms. This transfer will facilitate some reduction steps for the proof of our main result Theorem 5.1.1.

Lemma 4.1.3. *Let X, X' be geometrically connected varieties over K, and let ρ (resp. ρ') be an exterior G_K-representation on the group V (resp. V'). Then*

(1) $\underline{\mathrm{Isom}}_{K,F_K^{-1}}(X, X') : L/K \mapsto \mathrm{Isom}_{L,F_L^{-1}}(X_L, X'_L)$

(2) $\underline{\mathrm{Isom}}_{G_K}(\rho, \rho') : L/K \mapsto \mathrm{Isom}_{G_L}(V, V')$

are étale sheaves of sets on $\mathrm{Spec}(K)_{et}$*. Moreover*

$$\pi_1 : \underline{\mathrm{Isom}}_{K,F_K^{-1}}(X, X') \to \underline{\mathrm{Isom}}_{G_K}\big(\rho_X, \rho_{X'}\big)$$

and, if π_1^{t} is defined, also

$$\pi_1^{\mathrm{t}} : \underline{\mathrm{Isom}}_{K,F_K^{-1}}(X, X') \to \underline{\mathrm{Isom}}_{G_K}\big(\rho_X^t, \rho_{X'}^t\big)$$

are morphisms of étale sheaves that behave natural with respect to composition.

Proof: (1) This is nothing but Galois descent, and having localised does not matter.

(2) This is obvious, as in the case of a Galois extension L/K' of finite seperable extensions of K the set $\underline{\mathrm{Isom}}_{G_{K'}}(\rho,\rho')(K')$ consists of the invariants of the $G(L/K')$-action on $\underline{\mathrm{Isom}}_{G_K}(\rho,\rho')(L)$ by conjugation.

The last statement is again obvious because π_1 (resp. π_1^{t}) is a functor and both Galois actions have identical geometric origin by conjugation with isomorphisms of schemes. □

Étale G-torsors. Let G be a finite group, and X be a connected scheme with geometric point $\overline{x}$. Then $\mathrm{Hom}(\pi_1(X,\overline{x}),G)$ is the set of pointed G-torsors (E,e) on $(X,\overline{x})$ up to isomorphism. Shifting the pointing $e\mapsto g.e$ within the fiber corresponds to composition with the inner automorphism $g(\cdot)g^{-1}$. Hence

$$\mathrm{Hom}_{\mathscr{G}}\big(\pi_1(X),G\big)=\left\{\begin{array}{c}\text{isomorphy classes of}\\ G\text{-torsors } E\to X\end{array}\right\}.$$

Surjectivity is equivalent to connectedness of the torsor.

Geometrically connected G-torsors on a geometrically connected variety X/K are described by homomorphisms $\psi:\pi_1(X)\to G$, such that $\overline{\psi}=\psi|_{\pi_1(X_{\overline{K}})}$ is surjective. An easy diagram chase shows that $\overline{H}=\ker(\overline{\psi})$ carries a commuting outer action of G and G_K. This motivates the following definition.

Definition 4.1.4. *Let $V\in\mathscr{G}(G_K)$ be center free. A* **center free, geometrically connected** G**-torsor** *on V is a center free $W\in\mathscr{G}(G\times G_K)$ together with an isomorphism $\mathrm{Aut}(W)\times_{\mathrm{Out}(W)}G\cong V$ in $\mathscr{G}(G_K)$, where the G_K-action on $\mathrm{Aut}(W)\times_{\mathrm{Out}(W)}G$ is by functoriality of the construction with respect to G-compatible isomorphisms.*

We have the following reason to impose the condition of vanishing center. Recall the construction in (4.1.2). This construction of an extension from a representation was previously only used for the extension (4.1.1). If it is applied to our definition of G-torsors, we get the following diagram of extensions:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & W & \longrightarrow & V=\mathrm{Aut}(W)\times_{\mathrm{Out}(W)}G & \xrightarrow{\mathrm{pr}_2} & G & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathrm{Aut}(W)\times_{\mathrm{Out}(W)}G_K & \longrightarrow & \mathrm{Aut}(V)\times_{\mathrm{Out}(V)}G_K & \xrightarrow{\mathrm{pr}_2} & G & \longrightarrow & 1\\
 & & \downarrow{\scriptstyle\mathrm{pr}_2} & & \downarrow{\scriptstyle\mathrm{pr}_2} & & & & \\
 & & G_K & = & G_K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}\tag{4.1.5}$$

Note that we have constructed extensions in both rows and columns. We could also have defined geometrically connected G-torsors for extensions by diagram (4.1.5).

Lemma 4.1.5. *Let X be a geometrically connected variety over K such that all open subgroups of $\pi_1(X_{\overline{K}})$ are center free. Then we have the following bijection:*

$$\left\{\begin{array}{c}\textit{isomorphy classes of}\\ \textit{geom. connected}\\ G\textit{-torsors on } X/K\end{array}\right\}\xleftrightarrow{1:1}\left\{\begin{array}{c}\textit{isomorphy classes of}\\ \textit{geom. connected, center free}\\ G\textit{-torsors on } \rho_X\end{array}\right\}$$

Proof: A geometric torsor $E \to X$ is mapped to $\pi_1(E)$ which has to be understood as a G-torsor given as an extension as in (4.1.5). A group-theoretical torsor $W \in \mathscr{G}(G \times G_K)$ is mapped to $\mathrm{pr}_2 : \pi_1(X) = \mathrm{Aut}(W) \times_{\mathrm{Out}(W)} (G \times G_K) \to G$. These maps are mutually inverse. $\square$

Lemma 4.1.6 (G-torsors). *Let X, X' be quasi-compact, geometrically connected varieties over K such that all open subgroups of $\pi_1(X_{\overline{K}}), \pi_1(X'_{\overline{K}})$ are center free. Let $F' \to X'$ be a G-torsor, $E' \to X'$ a geometrically connected G-torsor and $\bar{H}'$ the corresponding G-torsor for $\rho_{X'}$. Let W' be a center free geometrically connected G-torsor on $V' \in \mathscr{G}(G_K)$ and $V \in \mathscr{G}(G_K)$. Then the following holds.*

(1) We have a natural bijection

$$\Big(\coprod_{G\text{-torsors } F/X} \mathrm{Hom}_K^{G\text{-equiv}}(F, F') \Big)_{/\cong} \xrightarrow{\sim} \mathrm{Hom}_K(X, X') \ .$$

(2) We have a natural bijection

$$\Big(\coprod_{G\text{-torsors } W/V} \mathrm{Isom}_{G \times G_K}(W, W') \Big)_{/\cong} \xrightarrow{\sim} \mathrm{Isom}_{G_K}(V, V') \ .$$

(3) The following diagram commutes:

$$\begin{array}{ccc} \Big(\coprod_{E/X} \mathrm{Isom}_{K, F_K^{-1}}^{G\text{-equiv}}(E, E') \Big)_{/\cong} & \xrightarrow{\pi_1} & \Big(\coprod_{\overline{H}/\rho_X} \mathrm{Isom}_{G \times G_K}(\overline{H}, \overline{H}') \Big)_{/\cong} \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle =} \\ \mathrm{Isom}_{K, F_K^{-1}}(X, X') & \xrightarrow{\pi_1} & \mathrm{Isom}_{G_K}\big(\pi_1(X_{\overline{K}}), \pi_1(X'_{\overline{K}})\big) \ . \end{array}$$

Here E (resp. $\overline{H}$) ranges over G-torsors on X (resp. ρ_X), and $/\cong$ means up to equivalence induced by isomorphisms of the "variable" G-torsors.

Proof: (1) G-torsors are G-quotient maps and allow pullback construction. (2) There is a map, as the "base" V is naturally recovered by $V \cong \mathrm{Aut}(W) \times_{\mathrm{Out}(W)} G$. It is surjective by structure transport and obviously injective. Part (3) is trivial. $\square$

There are obvious analogues for π_1^{t} of hyperbolic curves and tame (geometrically connected) G-torsors.

4.2 Properties of fundamental groups

We describe a few well known properties shared by the (tame) fundamental group of smooth curves over an algebraically closed field. By GAGA, their isomorphism type is known in characteristic 0. However, in positive characteristic they lack a group-theoretical description.

We will focus on the properties implied by the fact that curves are algebraic and pro-ℓ $K(\pi, 1)$ spaces, cf. Appendix A. Let X be a variety over an algebraically closed field. For a continuous torsion $\pi_1(X)$-module A, we consider the associated locally constant torsion sheaf $\mathscr{A}$ on X. Being an algebraic $K(\pi,1)$ space means by definition that for any A the natural map

$$\mathrm{H}^*(\pi_1(X), A) \to \mathrm{H}^*_{et}(X, \mathscr{A})$$

is bijective. Being pro-ℓ $K(\pi,1)$ amounts to the same for ℓ-torsion modules of $\pi_1(X)^\ell$.

A particular feature of fundamental groups is the following locality property. Let $\mathscr{V}$ be a class of varieties that is stable under finite étale covers, e.g., smooth curves. Any group-theoretical statement, which is true for fundamental groups of varieties from $\mathscr{V}$, is immediately valid for all their open subgroups because these are the fundamental groups of étale covers. We will exploit this property frequently without further mentioning.

4.2.1 Torsion and trivial center results

We will first discuss fundamental groups of general varieties that are algebraic $K(\pi,1)$ spaces. Later on, we will specialise to the case of smooth curves which is the only relevant case for the subsequent chapters. In particular, we prove a trivial center result which is well known. However, here we derive it from conceptual assumptions from the viewpoint of Anabelian Geometry. This trivial center result allows to apply Section 4.1.3 to the fundamental groups of curves.

Proposition 4.2.1. *Let X/K be a variety over an algebraically closed field which is an algebraic $K(\pi,1)$ space. Then $\pi_1(X)$ is torsion free.*

Proof: The cohomological dimension can be bounded as $\mathrm{cd}\left(\pi_1(X)\right) = \mathrm{cd}(X) \leq 2\dim(X) < \infty$. Hence by I §3 Proposition 14 [Se97], all closed subgroups of $\pi_1(X)$ have finite cohomological dimension. The existence of torsion would imply the existence of a finite (necessarily closed) subgroup of infinite cohomological dimension. □

Proposition 4.2.2. *Let K be an algebraically closed field and ℓ a prime different from the characteristic. Let X/K be a smooth variety that is an algebraic $K(\pi,1)$ space.*

Suppose that the finite étale Galois covers Y/X with non-vanishing ℓ-adic Euler-characteristic $\chi(Y,\mathbb{Q}_\ell)$ form a cofinal system among all finite étale covers of X. Then $\pi_1(X)$ is center free.

Proof: The proof is stimulated by Lemma 1 in [Fa98]. Let σ belong to the center of $\pi_1(X)$. We identify $\pi_1(X)$ with the automorphism group of a universal pro-étale cover $\hat{X}/X$. The restriction $\sigma|_Y$ to a G-Galois subcover $Y \to X$ acts by inner automorphisms on $\pi_1(Y)$ as this action comes from the exact sequence

$$1 \to \pi_1(Y) \to \pi_1(X) \to G \to 1 \ . \tag{4.2.6}$$

By Proposition A.2.3 the cover Y is also an algebraic $K(\pi,1)$ space. Thus the action of $\sigma|_Y$ is trivial on $\mathrm{H}^*(\pi_1(Y), \mathbb{Z}/\ell^n) = \mathrm{H}^*(Y, \mathbb{Z}/\ell^n)$ and by Poincaré-duality on $\mathrm{H}^*_c(Y, \mathbb{Z}/\ell^n)$. Therefore the Lefschetz number of $\sigma|_Y$ in ℓ-adic cohomology is

$$\mathrm{tr}\left(\sigma|_Y^*, \mathrm{H}^*_c(Y, \mathbb{Q}_\ell)\right) = \chi(Y, \mathbb{Q}_\ell) \ .$$

By assumption, this does not vanish for Y from a cofinal system of all étale covers. Hence $\sigma|_Y$ has a fixed point for such Y as by the Lefschetz–Verdier fixed point formula of III

Corollary 4.10 [SGA 5] the Lefschetz number is a sum of local contributions at the fixed points. But covering automorphisms with fixed points are the identity. Hence also σ must be the identity. □

Strictly speaking, the action of $\sigma|_Y$ on $\pi_1(Y)$ from (4.2.6) is only determined up to an inner automorphism. However, those act trivially on group cohomology by VII §5 Proposition 3 [Se79]. The effect of $\sigma|_Y$ on $\mathrm{H}^*(\pi_1(Y), \mathbb{Z}/\ell^n)$ is well defined.

The assumption on the non-vanishing of the ℓ-adic Euler-characteristic for covers is equivalent to the non-vanishing of $\chi(X, \mathbb{Q}_\ell)$ in the following two cases: (1) if the characteristic is 0, or (2) when the covering $Y \to X$ has a tame compactification. By Corollary 2.8 [Il81] we have

$$\chi(Y, \mathbb{Q}_\ell) = \deg(Y/X)\chi(X, \mathbb{Q}_\ell)$$

in both cases.

Corollary 4.2.3. *Let X/K be a proper, smooth variety over an algebraically closed field. Suppose that X is an algebraic $K(\pi, 1)$ space with non-vanishing Euler-characteristic. Then $\pi_1(X)$ has trivial center.* □

For more trivial center results (also in the discrete setting of algebraic topology) see [Na92], [An74], [Go65], [St65].

Corollary 4.2.4. *Let K be an algebraically closed field and ℓ a prime different from the characteristic. Let X/K be a smooth variety. Let $\mathscr{C}$ be the full subcategory of* $\mathsf{Rev}(X)$ *associated to a quotient $\pi_1(X) \twoheadrightarrow Q$ such that $\mathscr{C}$ is stable under taking ℓ-primary étale Galois covers and its connected objects are pro-ℓ $K(\pi, 1)$ spaces.*

If the connected Galois objects $Y \in \mathscr{C}$ with non-vanishing ℓ-adic Euler-characteristic $\chi(Y, \mathbb{Q}_\ell)$ form a cofinal system in $\mathscr{C}$ then Q is center free.

Proof: We argue as in the proof of Proposition 4.2.2, but control cohomology of a cover $Y \in \mathscr{C}$ by $\mathrm{H}^*(\pi_1(Y)^\ell, \mathbb{Z}_\ell) = \mathrm{H}^*(Y, \mathbb{Z}_\ell)$. Again, an element of the center acts trivially and in case of non-vanishing $\chi(Y, \mathbb{Q}_\ell)$ must have a fixed point. □

Corollary 4.2.5. *Let X/K be a smooth hyperbolic curve over an algebraically closed field. Then $\pi_1(X)$, $\pi_1^{\mathrm{t}}(X)$ and $\pi_1(X)^\ell$ for ℓ different from the characteristic are torsion free and center free.*

Proof: We recall that smooth hyperbolic curves are algebraic and pro-ℓ $K(\pi, 1)$ spaces for all ℓ by Appendix A.4.1. Furthermore, the Riemann–Hurwitz formula for a cover Y/X yields

$$\chi(Y, \mathbb{Q}_\ell) = \deg(Y/X)\chi(X, \mathbb{Q}_\ell) - \text{wild contribution}.$$

Thus Y is again smooth hyperbolic as the wild contribution is non-negative. The asserted vanishing of the center follows from Proposition 4.2.2 and Corollary 4.2.4.

If $\pi_1(X)$ or $\pi_1^{\mathrm{t}}(X)$ contain torsion then there is an open subgroup H such that H^ℓ contains torsion for some prime number ℓ. We are thus reduced to the case of π_1^ℓ. We conclude by an argument of cohomological dimension as in Proposition 4.2.1 because $\mathrm{cd}_\ell(\pi_1^\ell) = \mathrm{cd}(X) < \infty$. □

For the sake of completeness, we give a list describing the fundamental group for curves which are not hyperbolic.

- The projective line $\mathbb{P}^1_K$ is simply connected.

- The fundamental group of an elliptic curve is abelian on two topological genrators.
- In characteristic 0, or concerning the tame quotient in all characteristics, the affine line $\mathbb{A}^1_K$ is simply connected and the fundamental group of $\mathbb{G}_{m,K}$ is pro-cyclic.
- In positive characteristic, the affine line $\mathbb{A}^1_K$ and $\mathbb{G}_{m,K}$ have center free fundamental groups. We may use wild ramification at infinity to find a cofinal system of étale covers by hyperbolic curves. Then Proposition 4.2.2 applies.

4.2.2 Uniqueness of extension of maps

Dominant maps between schemes are epimorphisms. Let $X \to Y$ and $X' \to Y'$ be dominant maps. If c is an isomorphims $X \cong X'$ then there is at most a unique isomorphism $Y \cong Y'$ compatible with c. In this section we use the vanishing of torsion and center (from Section 4.2.1) together with the locality property for fundamental groups to prove a group-theoretical counterpart of this phenomenon.

Definition 4.2.6. *Let Γ be a pro-finite group. We call Γ* **strongly center free** *if and only if for all open subgroups $U < V < \Gamma$ the centralizer $C_V(U)$ of U in V is trivial.*

Lemma 4.2.7. *Let Γ be a torsion free pro-finite group such that all open subgroups U of Γ are center free. Then Γ is strongly center free.*

Proof: It suffices to show that $C_\Gamma(U)$ is trivial for all open normal subgroups U. But we have an exact sequence

$$1 \to C_U(U) \to C_\Gamma(U) \to \Gamma/U$$

proving that the centralizer $C_\Gamma(U)$ is torsion and thus trivial. □

Corollary 4.2.8. *(1) Let X/K be a smooth hyperbolic curve over an algebraically closed field . Then $\pi_1(X)$, $\pi_1^{\mathrm{t}}(X)$ and $\pi_1(X)^\ell$, for ℓ different from the characteristic, are strongly center free.*

(2) Suppose the field K has positive characteristic and X is isomorphic to $\mathbb{A}^1_K$ or $\mathbb{G}_{m,K}$. Then $\pi_1(X)$ is strongly center free. □

Lemma 4.2.9. *Let Γ, Γ' be strongly center free pro-finite groups. Then any two isomorphisms of Γ with Γ' coincide if they do so after restriction to some open subgroup.*

Proof: Let α, β be isomorphisms which coincide on $U < \Gamma$. Write $\gamma = \alpha|_U = \beta|_U$. We may assume that U is a normal subgroup. By U being center free , we have

$$\Gamma = \mathrm{Aut}(U) \times_{\mathrm{Out}(U)} \Gamma/U \tag{4.2.7}$$

which is canonically induced by conjugation. The corresponding exterior representation $\rho : \Gamma/U \hookrightarrow \mathrm{Out}(U)$ is faithful as its kernel $C_\Gamma(U)$ is trivial by assumption. The same holds for Γ' with respect to $U' = \gamma(U)$.

Thus α (resp. β) can be decomposed as $\gamma(\cdot)\gamma^{-1} \times \overline{\alpha}$ (resp. $\gamma(\cdot)\gamma^{-1} \times \overline{\beta}$) for the induced maps $\overline{\alpha}, \overline{\beta} : \Gamma/U \to \Gamma'/U'$ and the fibre product decomposition (4.2.7) of Γ (resp. Γ').

Consider the following diagram.

$$\begin{array}{ccc} \mathrm{Out}(U) & \xrightarrow{\gamma(\cdot)\gamma^{-1}} & \mathrm{Out}(U') \\ \Big\uparrow{\scriptstyle \rho} & & \Big\uparrow{\scriptstyle \rho'} \\ \Gamma/U & \xrightarrow[\overline{\alpha},\, \overline{\beta}]{} & \Gamma'/U' \end{array}$$

The maps $\overline{\alpha}$ and $\overline{\beta}$ coincide because both are restrictions of $\gamma(\cdot)\gamma^{-1} : \mathrm{Out}(U) \to \mathrm{Out}(U')$ and the exterior representations are faithful. This proves the lemma. □

Lemma 4.2.10. *Let $f : X \to Y$ be a dominant map of finite type between normal connected schemes. Then $\pi_1(f)$ is an open map. If $\pi_1^{\mathrm{t}}(f)$ is defined, it is open as well.*

Proof: It clearly suffices to show that $\pi_1(f)$ has open image. Let L (resp. K) be the residue field at the generic point of X (resp. Y). Then the absolute Galois groups of the respective function fields surject onto the respective fundamental group. Hence it is sufficient to prove that the image under restriction res : $G_L \to G_K$ for any finitely generated field extension L/K has finite index.

We choose a transcendence basis $\mathfrak{T}$ for L/K, and let $\tilde{K}$ denote the relative algebraic closure of K in L. The index of the image of res is the degree of $\tilde{K}/K$ which equals the degree of $\tilde{K}(\mathfrak{T})/K(\mathfrak{T})$ which is finite being majorated by the degree of $L/K(\mathfrak{T})$. □

Corollary 4.2.11. *Let $f : X \to Y$ and $f' : X' \to Y'$ be dominant maps of finite type between normal, connected schemes X, X' and smooth curves Y, Y' over an algebraically closed field. Let γ, α, β be isomorphisms such that the following "two" diagrams*

$$\begin{array}{ccc} \pi_1(X) & \xrightarrow{\pi_1(f)} & \pi_1(Y) \\ {\scriptstyle \gamma}\Big\downarrow & & \Big\downarrow{\scriptstyle \alpha,\ \beta} \\ \pi_1(X') & \xrightarrow{\pi_1(f')} & \pi_1(Y') \end{array}$$

commute. Then $\alpha = \beta$. If the tame fundamental group is defined the analogous statement holds.

Proof: This follows immediately from Lemma 4.2.10 and Lemma 4.2.9 as either the (tame) fundamental groups of the curves are strongly center free by Corollary 4.2.8 or torsion free abelian (then the corollary holds for trivial reasons). □

4.3 Control

Anabelian Geometry tries to describe the geometry of certain varieties entirely in group-theoretical terms by means of their fundamental groups. Geometric description or further group-theoretical data is often in a natural way extracted from the fundamental group. The notion of "control" yields a way of formalising this phenomenon (compare also to §1 [Ta99]).

Definition 4.3.1. *Let $\mathscr{C}$ be a category and let $F : \mathscr{C} \to \mathscr{D}$, $G : \mathscr{C} \to \mathscr{E}$ be functors. A* **control** $c : F \rightsquigarrow G$ *consists for all pairs of objects $X, Y \in \mathscr{C}$ of a map*

$$c_{X,Y} : \mathrm{Isom}_{\mathscr{D}}(FX, FY) \to \mathrm{Isom}_{\mathscr{E}}(GX, GY)$$

subject to the following conditions:

(i) For composable morphisms φ, ψ functoriality $c_{X,Z}(\varphi\psi) = c_{Y,Z}(\varphi)c_{X,Y}(\psi)$ holds.

(ii) For a morphism $h : X \to Y$ in $\mathscr{C}$ the compatibility $c_{X,Y}(Fh) = Gh$ holds.

One says "F controls G (for objects from $\mathscr{C}$)" if there is a control $F \rightsquigarrow G$. Furthermore, there is an obvious modification by replacing Isom *with* Hom.

Examples. (1) Anabelian Geometry deals with the problem for which category of varieties the functor π_1 controls the identity functor, i.e., $\mathscr{C} = \mathscr{E} = \mathsf{Anab}_K \subset \mathsf{Var}_K$ and $\mathscr{D} = \mathscr{G}(G_K)$ or $\mathrm{EXT}[G_K]$. Strictly speaking, one further claims that the control-map is bijective, but this follows usually by general considerations from being a retraction.

(2) The fundamental group of topological spaces controls the first singular homology. This is due to the presence of a functor, the abelianisation of groups, that yields naturally a factorisation $(\ )^{\mathrm{ab}} \circ \pi_1 = \mathrm{H}_1(\ \ , \mathbb{Z})$ such that the control is based on the effect of this functor on morphisms. Of course this generalises.

(3) If there is a functor $H : \mathscr{D} \to \mathscr{E}$ such that naturally $HF \cong G$ then the effect of H on morphisms yields a control $F \rightsquigarrow G$. The main difficulty that inhibits a control to originate from a functor consists in choosing the correct categories. A control that is based on a functor obeys the Hom-version. But if all categories involved are groupoids, i.e., all morphisms are isomorphisms, and F is essentially surjective (replace $\mathscr{D}$ by the essential image of F) a control is not far from being a functor.

(4) Numerical invariants fit into this pattern by regarding $\mathbb{N}$ as the objects of a category with identities as the only morphisms. Let, for example, K be a fixed algebraically closed field and let ℓ be a fixed prime number different from the characteristic. Let $\mathscr{C}$ be the groupoid associated to the category of smooth, proper varieties over K that furthermore are pro-ℓ $K(\pi, 1)$ spaces. Then there is a functor $\dim : \mathscr{C} \to \mathbb{N} = \mathscr{E}$ that assigns to a variety its dimension. The functor dim is controlled by the fundamental group (as a functor $\mathscr{C} \to \mathscr{G} = \mathscr{D}$) due to the fact that $\mathrm{cd}_\ell\left(\pi_1^\ell(X)\right) = 2\dim(X)$.

Chapter 5

Projective Anabelian Curves

Our main result is Theorem 5.1.1. This chapter contains some general steps towards its proof — the parts more specific to curves will be covered by Chapters 6 and 7.

Definition 5.0.1. *Let S be a scheme. A* **proper, smooth curve** *X over S is a proper, smooth map $f : X \to S$ of finite presentation with geometrically connected fibres of dimension* 1*. A* **smooth curve** *X over S is a smooth map $f : X \to S$ such that locally on S there exists a smooth compactification $\overline{f} : \overline{X} \to S$ by a proper, smooth curve where the boundary $\overline{X} \setminus X$ is finite, relative étale over S.*

The **genus** *g of the fibre and the* **degree** *n of the boundary divisor are locally constant functions on S. The S-curve is called* **hyperbolic** *if the Euler-characteristic $\chi = 2-n-2g$ is negative.*

The curve X/S is called **isotrivial** *if there is a finite surjective $S' \to S$ such that the base extension $X' = X \times_S S'$ is isomorphic to the base extension of a curve over a finite field.*

5.1 Statement of the theorem

Theorem 5.1.1. *Let K be a finitely generated field of characteristic $p \geq 0$ with algebraic closure $\overline{K}$. Let X, X' be smooth, hyperbolic, geometrically connected curves over K. Assume that at least one of them is not isotrivial. Then the functor π_1 induces a natural bijection*

$$\pi_1 : \mathrm{Isom}_{K,F_K^{-1}}\big(X, X'\big) \xrightarrow{\sim} \mathrm{Isom}_{G_K}\big(\pi_1(X_{\overline{K}}), \pi_1(X'_{\overline{K}})\big) \tag{5.1.1}$$

of finite sets.

Corollary 5.1.2. *Let K be a finitely generated field of characteristic $p \geq 0$ with algebraic closure $\overline{K}$. Let X, X' be smooth, hyperbolic, geometrically connected curves over K. Assume that at least one of them is not isotrivial. Then the following are equivalent:*

(a) The fundamental groups of X and X' are isomorphic as exterior G_K-modules.

(b) The curves X and X' are isomorphic up to a twist by Frobenius, i.e., there are natural numbers m, m' such that $X(m) \cong X'(m')$ as curves over K.

□

Remark. **(1)** For isotrivial curves the statement of Theorem 5.1.1 has to be modified as follows. The map π_1 can at most be dense injective with respect to natural topologies (discrete versus pro-finite). There is an action by $\mathbb{Z} = \langle F_K^f \rangle$ on the geometric side. The group-theoretical side is a set with an action by $\hat{\mathbb{Z}}$, which is the Galois group of a finite field. Both actions are free and compatible in the obvious way. For affine hyperbolic curves Theorem 0.5 [Ta97] essentially says that π_1 induces a bijection on the respective sets of orbits, cf. [Sx02]. For affine hyperbolic curves also Corollary 5.1.2 remains unchanged.

(2) Theorem 5.1.1 is known in the following cases. Tamagawa's Theorem 0.3 [Ta97] deals with affine hyperbolic curves and K of characteristic 0. Mochizuki has proven even stronger results for hyperbolic curves over sub-p-adic fields, cf. Theorem A [Mz99]. His work covers the characteristic 0 case and includes projective curves.

(3) Finally, Theorem 1 [Sx02] contains a proof for the case of affine hyperbolic curves in arbitrary characteristic concerning their tame fundamental group. But the G_K-module $\pi_1(X_{\overline{K}})$ controls whether X is affine or complete. Moreover, it controls which covers are tamely ramified and thus the quotient $\pi_1^{\mathrm{t}}(X_{\overline{K}})$. By the methods of Section 5.3 that reduces the proof of Theorem 5.1.1 to the construction of a retraction one easily derives from Theorem 1 [Sx02] also the apparently weaker statement concerning the full étale fundamental group. This yields Theorem 5.1.1 for affine hyperbolic curves over finitely generated fields of positive characteristic.

The results stated in (2) and (3) above reduce the proof of Theorem 5.1.1 to the following theorem.

Theorem 5.1.3. *Let K be a finitely generated field of characteristic $p > 0$ with algebraic closure $\overline{K}$. Let X, X' be smooth, proper, geometrically connected curves over K of genus $g \geq 2$, i.e., they are complete hyperbolic. Assume that at least one of them is not isotrivial. Then the functor π_1 induces a natural bijection*

$$\pi_1 : \mathrm{Isom}_{K,F_K^{-1}}\big(X, X'\big) \xrightarrow{\sim} \mathrm{Isom}_{G_K}\big(\pi_1(X_{\overline{K}}), \pi_1(X'_{\overline{K}})\big)$$

of finite sets.

5.2 Injectivity and finiteness

In the sequel we discuss formal reduction steps for the proof of bijectivity of the map π_1 and the finiteness assertion in Theorem 5.1.3. We will only use F-cohomological rigidity (to be defined below) and the properties of algebraic $K(\pi, 1)$ spaces. In fact, these arguments are not specific to smooth curves. Hence we keep the exposition more general.

Definition 5.2.1. *Let $\overline{X}$ be a connected variety over an algebraically closed field $\overline{K}$. We call it F-**cohomologically rigid** if the natural map*

$$\mathrm{Aut}_{\overline{K},F_{\overline{K}}^{-1}}(\overline{X}) \to \mathrm{Aut}^{\mathrm{opp}}\big(\mathrm{H}^*(\overline{X}, \mathbb{Z}/n)\big)$$

*is injective for all $n \gg 0$ prime to the characteristic. If n is big enough, such that injectivity holds, we call $\overline{X}$ F-**cohomologically rigid with coefficients in** $\mathbb{Z}/n$.*

A geometrically connected variety X/K is called F-cohomologically rigid (with coefficients in $\mathbb{Z}/n$) if $X_{\overline{K}}$ is F-cohomologically rigid (with coefficients in $\mathbb{Z}/n$).

Clearly, if a variety is F-cohomologically rigid with coefficients in $\mathbb{Z}/n$ then it is also F-cohomologically rigid with coefficients in $\mathbb{Z}/N$ for all integral multiples N of n. We will see later that proper, non-isotrivial, hyperbolic curves are F-cohomologically rigid with coefficients in $\mathbb{Z}/n$ for all $n \geq 3$, cf. Corollary 7.3.4. By Corollary B.3.5 a smooth, hyperbolic curve is F-cohomologically rigid if and only if it is not isotrivial.

Lemma 5.2.2. *(1) Let X, X' be geometrically connected varieties over the field K. Assume that $X_{\overline{K}}$ and $X'_{\overline{K}}$ are F-cohomologically rigid and algebraic $K(\pi,1)$ spaces. Then the natural map*

$$\pi_1 : \mathrm{Isom}_{K,F_K^{-1}}\big(X, X'\big) \to \mathrm{Isom}_{G_K}\big(\pi_1(X_{\overline{K}}), \pi_1(X'_{\overline{K}})\big)$$

is injective.

(2) If, moreover, π_1 is bijective then both sets of isomorphisms are finite.

Proof: For (1) we choose $n \gg 0$ such that $X_{\overline{K}}$ and $X'_{\overline{K}}$ are F-cohomologically rigid with coefficients in $\mathbb{Z}/n$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Isom}_{K,F_K^{-1}}\big(X, X'\big) & \xrightarrow{\pi_1} & \mathrm{Isom}_{G_K}\big(\pi_1(X_{\overline{K}}), \pi_1(X'_{\overline{K}})\big) \\ \big\downarrow{\scriptstyle \mathrm{H}^*} & & \big\downarrow \\ \mathrm{Isom}_{G_K}\big(\mathrm{H}^*(X'_{\overline{K}}), \mathrm{H}^*(X_{\overline{K}})\big) & = & \mathrm{Isom}_{G_K}\big(\mathrm{H}^*(\pi_1(X'_{\overline{K}})), \mathrm{H}^*(\pi_1(X_{\overline{K}}))\big)\ . \end{array} \tag{5.2.2}$$

Here all coefficients for cohomology are constant $\mathbb{Z}/n$. The map H^* is defined by the topological invariance of the étale site and is injective due to F-cohomological rigidity. As $X_{\overline{K}}$ and $X'_{\overline{K}}$ are algebraic $K(\pi,1)$ spaces, we have equality in the bottom line. Thus π_1 is injective.

For (2), we argue by Th. finitude 1.1 [SGA $4\frac{1}{2}$] that $\mathrm{H}^*(X_{\overline{K}}, \mathbb{Z}/n)$ is finite and use the diagram (5.2.2) again. □

In the next section we need center free G-torsors of fundamental groups, cf. Section 4.1. Although we have already dealt with center freeness results in Section 4.2.1, we prove the following lemma, because we have placed ourselves in a different axiomatic framework. Here the proof will rely on F-cohomological rigidity and the property of being an algebraic $K(\pi,1)$ space. In particular, we reprove that proper hyperbolic curves have center free fundamental groups, cf. Corollary 4.2.5.

Lemma 5.2.3. *Let X/K be a geometrically connected variety such that $X_{\overline{K}}$ is an algebraic $K(\pi,1)$ space and all finite étale covers are F-cohomologically rigid. Then $\pi_1(X_{\overline{K}})$ is a center free pro-finite group.*

Proof: The proof is analogous to the proof of Lemma 4.2.2. Let σ be an element of the center of $\pi_1(X_{\overline{K}})$. We identify $\pi_1(X_{\overline{K}})$ with the automorphism group of a universal pro-étale cover $\hat{X}/X_{\overline{K}}$. The restriction $\sigma|_Y$ to a G-Galois subcover $Y \to X_{\overline{K}}$ acts by inner automorphisms on $\pi_1(Y)$ as this action comes from the exact sequence

$$1 \to \pi_1(Y) \to \pi_1(X_{\overline{K}}) \to G \to 1\ .$$

Hence the action is trivial on $\mathrm{H}^*(\pi_1(Y_{\overline{K}}), \mathbb{Z}/n)$, which equals $\mathrm{H}^*(Y_{\overline{K}}, \mathbb{Z}/n)$ as $Y_{\overline{K}}$ is also an algebraic $K(\pi,1)$ space by Proposition A.2.3. The F-cohomological rigidity of Y thus forces $\sigma|_Y$ to be the identity. □

5.3 Surjectivity

We proceed with our discussion of formal reduction steps for the proof of Theorem 5.1.3.

Let $\mathscr{V}$ be a class of geometrically connected varieties over finite separable extensions of a field K. We assume that: (i) $\mathscr{V}$ is stable under finite étale covers, (ii) all $X \in \mathscr{V}$ are F-cohomologically rigid, and (iii) for all $X \in \mathscr{V}$ the geometric fibres $X_{\overline{K}}$ are algebraic $K(\pi, 1)$ spaces.

In the proof of Theorem 5.1.3 $\mathscr{V}$ coincides with the class of proper hyperbolic curves that are not isotrivial, cf. Lemma 7.3.1, Corollary 7.3.4 and Proposition A.4.1.

By Lemma 5.2.3 any variety in $\mathscr{V}$ has center free geometric fundamental group. Hence we may apply the theory of center free torsors of Section 4.1. Let $X, X' \in \mathscr{V}$ be geometrically connected varieties over K. Let L/K be a finite seperable extension. Suppose given étale G-torsors $Y \to X_L$ (resp. $Y' \to X'_L$) which are also geometrically connected over L. Recall from Lemma 4.1.6 that we get a natural map

$$\mathrm{Aut}(\cdot) \times_{\mathrm{Out}(\cdot)} G \ : \ \mathrm{Isom}^{G\text{-equiv}}_{G_L}\big(\pi_1(Y_{\overline{K}}), \pi_1(Y'_{\overline{K}})\big) \to \mathrm{Isom}_{G_L}\big(\pi_1(X_{\overline{K}}), \pi_1(X'_{\overline{K}})\big)$$

from G-equivariant maps of center free torsors to maps of the base.

Lemma 5.3.1. *Let $\mathscr{V}$ be as above. Suppose that there exists a map*

$$\begin{array}{rcl} \lambda = \lambda_{L,X,X'} : \mathrm{Isom}_{G_L}\big(\pi_1(X_{\overline{K}}), \pi_1(X'_{\overline{K}})\big) & \to & \mathrm{Isom}_{L,F_L^{-1}}(X, X') \\ \alpha & \mapsto & \lambda(\alpha) \end{array}$$

for all finite seperable extensions L/K and all $X, X' \in \mathscr{V}$ which are geometrically connected over L. Suppose moreover, that $\pi_1(\lambda(\alpha))$ and α coincide on cohomology with coefficients in $\mathbb{Z}/n$ for all $n \gg 0$ prime to the characteristic (the bound on n may depend on X, X'). Then

(i) λ is a retraction: $\lambda \circ \pi_1 = \mathrm{id}$,

(ii) λ is functorial: $\lambda(\alpha_1\alpha_2) = \lambda(\alpha_1)\lambda(\alpha_2)$,

(iii) for étale G-Galois covers $Y \to X$ and $Y' \to X'$ of varieties in $\mathscr{V}$ such that Y, Y' are geometrically connected over L the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Isom}^{G\text{-equiv}}_{G_L}\big(\pi_1(Y_{\overline{K}}), \pi_1(Y'_{\overline{K}})\big) & \xrightarrow{\lambda} & \mathrm{Isom}^{G\text{-equiv}}_{L,F_L^{-1}}(Y, Y') \\ {\scriptstyle \mathrm{Aut}(\cdot)\times_{\mathrm{Out}(\cdot)}G}\big\downarrow & & \big\downarrow{\scriptstyle \cdot/G} \\ \mathrm{Isom}_{G_L}\big(\pi_1(X_{\overline{K}}), \pi_1(X'_{\overline{K}})\big) & \xrightarrow{\lambda} & \mathrm{Isom}_{L,F_L^{-1}}(X, X')\ . \end{array} \tag{5.3.3}$$

Proof: Replace L by K. Parts (i) and (ii) are obvious by comparing f with $\lambda(\pi_1(f))$ (resp. α_1, α_2 and $\alpha_1\alpha_2$) on cohomology with appropriate coefficients.

By (i) and (ii) the upper row of diagram (5.3.3) is well defined. Part (iii) then claims that for $G \times G_L$ equivariant maps β if we set $\alpha = \mathrm{Aut}(\beta) \times_{\mathrm{Out}} G$ the induced map on the quotients $X = Y/G$ and $X' = Y'/G$ is $\lambda(\beta)/G = \lambda(\alpha)$. To verify this, we use the same strategy and compare $\lambda(\beta)/G$ with $\lambda(\alpha)$ on cohomology with appropriate coefficients $\mathbb{Z}/n$. If n is prime to the order of G the restriction

$$\mathrm{res} : \mathrm{H}^*\big(\pi_1(X_{\overline{K}}), \mathbb{Z}/n\big) \hookrightarrow \mathrm{H}^*\big(\pi_1(Y_{\overline{K}}), \mathbb{Z}/n\big)$$

is injective by a corestriction argument. As the effect of $\lambda(\beta)/G$ and $\lambda(\alpha)$ on cohomology are both compatible under restriction with the effect of β, the commutativity of (iii) follows. □

Lemma 5.3.2. *Let $\mathscr{V}$ and λ be as in Lemma 5.3.1 and assume that λ has the properties (i), (ii) and (iii) of the conclusion of Lemma 5.3.1. Then λ is an inverse of π_1.*

Proof: We need to show that $\pi_1 \circ \lambda = \mathrm{id}$. Let $\alpha : \pi_1(X_{\overline{K}}) \to \pi_1(X'_{\overline{K}})$ be a map of exterior G_K-modules. Fix a representative as isomorphism of pro-finite groups and fix fibre functors such that α can be regarded as a functor $\alpha^* : \mathsf{Rev}(X'_{\overline{K}}) \to \mathsf{Rev}(X_{\overline{K}})$. For an open subgroup $H < \pi_1(X_{\overline{K}})$ let H' denote its image under α. Let Y_H (resp. $Y'_{H'}$) denote the cover corresponding to $\pi_1(X_{\overline{K}})/H$ (resp. $\pi_1(X'_{\overline{K}})/H'$). Then $\alpha^*(Y'_{H'}) = Y_H$.

The functor $\lambda(\alpha)^* : \mathsf{Rev}(X'_{\overline{K}}) \to \mathsf{Rev}(X_{\overline{K}})$ is naturally isomorphic to α^* via $\lambda(\alpha|_H) : \alpha^*(Y'_{H'}) = Y_H \xrightarrow{\sim} Y'_{H'}$ where H runs through the system of all open subgroups of $\pi_1(X_{\overline{K}})$. The conditions (i) and (ii) guarantee that G-equivariant maps of groups become G-equivariant maps of curves and condition (iii) yields the compatibility of the system of the $\lambda(\alpha|_H)$ for Galois covers. But covers are epimorphisms and Galois covers are cofinal in the system of all covers. Thus Galois covers suffice to establish that all the $\lambda(\alpha|_H)$ are compatible.

As isomorphic functors between Galois categories differ only by an inner automorphism of the respective fundamental groups it follows that $\pi_1(\lambda(\alpha)) = \alpha$ and the proof of the lemma is complete. □

To conclude with a proof of Theorem 5.1.3 we are left with the task of constructing a retraction λ as in Lemma 5.3.1. This will be done in Section 7.3.3.

Chapter 6

Components of the Special Fibre

Let X/S be a generically smooth, proper, stable, but not smooth curve over a "good" discrete valuation ring. Following Mochizuki, we study in this chapter how the fundamental group of the generic fibre controls the set of irreducible components of the special fibre. In principle, we therefore control the logarithmic specialisation map. The Van Kampen Theorem provides us with an amalgamation formula for the logarithmic fundamental group of the special fibre. A summand of the amalgamation may be controlled as a stabilizer and essentially consists of the tame fundamental group of the respective component minus the double points. The complexity of the reasoning results from recovering the π_1^{t} of the smooth locus of an irreducible component not only as a group but as an exterior Galois module.

Definition 6.0.1. *Let S be a scheme. A* **semistable curve** *X over S is a flat map $f : X \to S$ of finite presentation such that all geometric fibres are connected and locally in the étale topology isomorphic to either (a) $\mathrm{Spec}(k[t])$ at $t = 0$ or (b) $\mathrm{Spec}(k[x, y]/xy)$ at $x, y = 0$.*

Points of type (b) are called **double points** *and form the* **singular set** *$\mathrm{Sing}(X/S)$ of the map f. $\mathrm{Sing}(X/S)$ is the closed subscheme defined by the first Fitting ideal of the sheaf $\Omega^1_{X/S}$. The map $\mathrm{Sing}(X/S) \to S$ is quasi-finite, unramified and of finite presentation.*

A semistable curve X over S is called **stable** *if it is proper and if the group of automorphisms of any geometric fibre is finite. Equivalently, any rational (resp. elliptic) component contains at least 3 (resp. 1) double points.*

6.1 Detecting irreducible components

The main result of this section comes from §1 [Mz96]. We recall it here for the sake of completeness. Let k be a field with a fixed algebraic closure $\overline{k}$. Let X/k be a semistable, proper and connected curve. Let $X_{\overline{k}} = \bigcup X_i$ denote its geometric decomposition into irreducible components. Let g_{X_i} be the genus of the normalisation $\tilde{X}_i$. The dual graph Γ of X/k has the set of irreducible components $\Gamma_0 = \{X_i\}$ as vertex set and the set of double points $\Gamma_1 = \mathrm{Sing}\big(X/k\big)(\overline{k})$ as set of edges. An edge connects the two vertices that pass through it. The topological realisation of the dual graph is a one dimensional CW-complex which is also denoted by Γ. Its Euler-characteristic is $\chi = \#\Gamma_0 - \#\Gamma_1$. Cohomology of Γ

is computed using its CW-chain complex

$$C_\bullet^{\mathrm{CW}}(\Gamma,\mathbb{Z}) = \Big[\mathbb{Z}[\Gamma_0] \overset{\partial}{\leftarrow} \mathbb{Z}[\Gamma_1]\Big] \ .$$

The category $\mathsf{Rev}(X)$ of étale covers of X has a full Galois subcategory $\mathsf{Rev}^{\mathrm{mock}}(X)$ of covers described by the property that over $X_{\overline{k}}$ all generic points of irreducible components are covered trivially. The objects of $\mathsf{Rev}^{\mathrm{mock}}$ are of purely combinatorial nature and correspond uniquely to finite covers of the dual graph. The associated fundamental group $\pi_1^{\mathrm{mock}}(X)$ therefore naturally satisfies

$$\mathrm{H}^1_{\mathrm{CW}}(\Gamma,\mathbb{Z}_\ell) \cong \mathrm{Hom}(\pi_1^{\mathrm{mock}}(X_{\overline{k}}),\mathbb{Z}_\ell) \ .$$

Let f be the finite surjective map $\coprod \tilde{X}_i \to X_{\overline{k}}$, and let ℓ be a prime number different from the residue characteristic. Then we obtain a short exact sequence of ℓ-adic sheaves on $X_{\overline{k}}$ in the étale topology $X_{\overline{k},\acute{e}t}$

$$0 \to \underline{\mathbb{Z}_\ell}_{X_{\overline{k}}} \to f_*\big(\bigoplus_{i\in\Gamma_0} \underline{\mathbb{Z}_\ell}_{\tilde{X}_i}\big) \to \bigoplus_{x\in\Gamma_1} i_{x,*}\underline{\mathbb{Z}_\ell}_{\overline{k}} \to 0 \ .$$

The corresponding long exact sequence yields

$$0 \to \mathrm{H}^1_{\mathrm{CW}}(\Gamma,\mathbb{Z}_\ell) \to \mathrm{H}^1_{et}(X_{\overline{k}},\mathbb{Z}_\ell) \to \bigoplus_{i\in\Gamma_0} \mathrm{H}^1_{et}(\tilde{X}_i,\mathbb{Z}_\ell) \to 0$$

which is $\mathrm{Hom}(-,\mathbb{Z}_\ell)$-dual to

$$0 \to \bigoplus_{i\in\Gamma_0} \pi_1(\tilde{X}_i)^{\mathrm{ab},\ell} \to \pi_1(X_{\overline{k}})^{\mathrm{ab},\ell} \to \pi_1^{\mathrm{mock}}(X_{\overline{k}})^{\mathrm{ab},\ell} \to 0 \ . \tag{6.1.1}$$

Let l/k be a finite extension such that X_l splits into geometrically irreducible components. Then the short exact sequence (6.1.1) is compatible with the natural Galois action by $G_l < G_k$.

For the rest of this section, let $k = \mathbb{F}_q$ be a finite field of characteristic p.

Theorem 6.1.1 (Mochiziki). *Let X/k be a semistable, proper, connected curve, $k = \mathbb{F}_q$, $\ell \neq p$, and let $X_{\overline{k}} = \bigcup X_i$ be the decomposition into irreducible components. Let $\tilde{X}_i$ be the normalisation of X_i. Then the exterior G_k-module $\pi_1(X_{\overline{k}})^\ell$ controls group-theoretically the following exact sequence*

$$0 \to \bigoplus_{i\in\Gamma_0} \pi_1(\tilde{X}_i)^{\mathrm{ab},\ell} \to \pi_1(X_{\overline{k}})^{\mathrm{ab},\ell} \to \pi_1^{\mathrm{mock}}(X_{\overline{k}})^{\mathrm{ab},\ell} \to 0 \tag{6.1.2}$$

including the decomposition of the kernel as a direct sum.

Corollary 6.1.2. *The exterior G_k-module $\pi_1(X_{\overline{k}})^\ell$ controls the set $\{X_i \,|\, g_{X_i} > 0\}$ and the numbers $1-\chi$ and $\sum_i g_{X_i}$.*

Proof: By abelianisation we recover the genuine G_k-module $\pi_1(X_{\overline{k}})^{\mathrm{ab},\ell}$. It carries a natural descending filtration $F^\bullet$, the filtration by the Frobenius weight, such that F^n

equals the largest submodule such that all pure subquotients have weight wt $\leq n$. The weight of the constituents in (6.1.2) are

$$\mathrm{wt}\left(\pi_1(\tilde{X}_i)^{\mathrm{ab},\ell}\right) = -1 \quad \text{and} \quad \mathrm{wt}\left(\pi_1^{\mathrm{mock}}(X_{\bar{k}})^{\mathrm{ab},\ell}\right) = 0 \ .$$

Hence the short exact sequence (6.1.2) of the theorem is group-theoretically controlled already on the abelian level by the weight filtration as

$$0 \to F^{-1} \to F^0 \to F^0/F^{-1} \to 0 \ .$$

The ranks of F^{-1} and F^0/F^{-1} as $\mathbb{Z}_\ell$-modules give the numerical data $1-\chi$ and $2\sum g_{X_i}$.

From now on we need also nonabelian parts of $\pi_1(X_{\bar{k}})^\ell$, because we also need the abelianisation of open subgroups. However, the maximal pro-ℓ metabelian quotient endowed with its inherited exterior Galois action suffices.

For any finite abelian group A the set of étale connected A-torsors over $X_{\bar{k}}$, up to isomorphism, is in natural bijection with the set of surjective continuous homomorphisms $\varphi : \pi_1(X_{\bar{k}}) \twoheadrightarrow A$. Let $X(\varphi) \to X_{\bar{k}}$ be the corresponding torsor. All data associated to $X(\varphi)$ will be denoted by the same symbol as for X but $\cdot(\varphi)$ added. The fundamental group of $X(\varphi)$ is naturally isomorphic to the kernel of φ. If A is an ℓ-primary group then φ factors through $\pi_1(X_{\bar{k}})^{\mathrm{ab},\ell}$ and we will consider φ the like.

Let us first consider étale connected $\mathbb{Z}/\ell$-torsors $X(\varphi)$ which are not mock, i.e., φ does not factorise through $\pi_1^{\mathrm{mock}}(X_{\bar{k}})^{\mathrm{ab},\ell}$. Then $\#\Gamma_1(\varphi) = \ell\#\Gamma_1$ and as at least one component is indecomposed $\#\Gamma_0(\varphi) \leq \ell(\#\Gamma_0 - 1) + 1$. Hence we get the inequality

$$1-\chi(\varphi) = 1 - \#\Gamma_0(\varphi) + \#\Gamma_1(\varphi) \geq 1 - \big(\ell(\#\Gamma_0 - 1) + 1\big) + \ell\#\Gamma_1 = \ell(1-\chi) \ . \quad (6.1.3)$$

Equality in (6.1.3) holds precisely when just one component X_{i_0} is indecomposed. This is equivalent to

$$\ker(\varphi) \supseteq \bigoplus_{i \neq i_0} \pi_1(\tilde{X}_i)^{\mathrm{ab},\ell} \ .$$

Equality in (6.1.3) can be checked group-theoretically. Let $\mathrm{T}^{\min}_{\mathbb{Z}/\ell}$ denote the set of such minimally indecomposed $\mathbb{Z}/\ell$-torsors. For $\varphi, \varphi' \in \mathrm{T}^{\min}_{\mathbb{Z}/\ell}$ we would like to decide the equivalence relation defined by $\varphi \sim \varphi' \iff$ the torsors $X(\varphi)$ and $X(\varphi')$ are indecomposed over the same irreducible component of X. That is achieved by the following lemma.

Lemma 6.1.3 (Mochizuki). *Minimal torsors $\varphi, \varphi' \in \mathrm{T}^{\min}_{\mathbb{Z}/\ell}$ are equivalent if and only if for all $\alpha, \beta \in \mathbb{Z}/\ell$ such that $\alpha\varphi + \beta\varphi' : \pi_1(X_{\bar{k}})^{\mathrm{ab},\ell} \to \mathbb{Z}/\ell$ is surjective we have $1 - \chi(\alpha\varphi + \beta\varphi') = \ell(1-\chi)$.*

Proof: The $A \times A'$-torsor $X(\varphi) \times X(\varphi')$ is described by $(\varphi, \varphi') : \pi_1(X_k) \to A \times A'$. Now we just consider decomposition groups for the $\mathbb{Z}/\ell \times \mathbb{Z}/\ell$-torsor $X(\varphi) \times X(\varphi')$. The $\alpha\varphi + \beta\varphi'$ belong to all connected $\mathbb{Z}/\ell$-quotient torsors and are required to be indecomposed over exactly a single component by the condition of the lemma. □

Every component of X with positive genus appears in this way as an equivalence class of minimal $\mathbb{Z}/\ell$-torsors, those with $g = 0$ do not. Hence we can naturally identify

$$\mathrm{T}^{\min}_{\mathbb{Z}/\ell}/_{\sim} = \{X_i \mid g_{X_i} > 0\} \ .$$

Now we turn to étale connected $\mathbb{Z}/\ell^n$-torsors φ. We declare φ minimal and write $\varphi \in \mathrm{T}^{\min}_{\mathbb{Z}/\ell^n}$ if each $\mathbb{Z}/\ell$-subquotient torsor is minimal in the above sense. The structure of the cyclic group and the behaviour of decomposition subgroups shows that by composing with the natural projection $\mathrm{pr} : \mathbb{Z}/\ell^n \twoheadrightarrow \mathbb{Z}/\ell$ we get a surjection

$$\mathrm{pr} : \mathrm{T}^{\min}_{\mathbb{Z}/\ell^n} \twoheadrightarrow \mathrm{T}^{\min}_{\mathbb{Z}/\ell}/_{\sim} = \{X_i \,|\, g_{X_i} > 0\} \ .$$

and that in $X(\varphi)$ precisely the component $\mathrm{pr}(\varphi) = (\mathrm{pr} \circ \varphi \bmod \sim)$ is indecomposed but this one completely.

Sufficiently many such torsors exist. It follows that for a component X_i we have

$$\pi_1(\tilde{X}_i)^{\mathrm{ab},\ell} = \bigcap_n \bigcap_{\substack{\varphi \in \mathrm{T}^{\min}_{\mathbb{Z}/\ell^n} \\ \mathrm{pr}(\varphi) \neq X_i}} \left(\ker(\varphi) \cap F^{-1}\right) \subseteq \pi_1(X_{\overline{k}})^{\mathrm{ab},\ell} = F^{-1} \ ,$$

whence the direct summands of the direct sum decomposition in (6.1.2) can be group-theoretically described and the proof of Theorem 6.1.1 is complete. □

6.2 Application of the Van Kampen Theorem

This section uses logarithmic geometry. For the basic notions we refer to Chapter 3. We prove an amalagamation formula for the logarithmic fundamental group of the special fibre of a stable curve. This allows for a recovering of the π_1^{t} of the smooth locus of an irreducible component as an exterior Galois module.

6.2.1 Semistable curves over the log point

Let s be the spectrum of a field k endowed with the fs-log structure induced from the chart $\mathbb{N} \to k;\ (n > 0) \mapsto 0$. We call s the standard log point and let π denote the global section $1 \in \mathbb{N} \subset M_s(s)$. Let $\overline{s} = \mathrm{Spec}(\overline{k})$ be a geometric point and $\tilde{s}$ a log geometric point above $\overline{s}$. The log fundamental group $\pi_1^{\log}(s, \tilde{s})$ will be denoted by $\pi^{\log}$. From the map $\varepsilon : s \to \mathring{s}$ we obtain the canonical short exact sequence

$$1 \to I^{\log} \to \pi^{\log} \to G_k \to 1$$

such that $I^{\log} = \hat{\mathbb{Z}}(1)\big(\overline{k}\big)$ as a G_k-module.

Definition 6.2.1. *Let $f : X \to S$ be a morphism of fs-log schemes, $\overline{x} \in X$ a geometric point with image $\overline{s} \in S$. The* **relative dimension** $\dim_{\overline{x}}(f)$ *of f in $\overline{x}$ is defined as*

$$\mathrm{rk}_{\mathbb{Z}} \left(\overline{M}^{\mathrm{gp}}_X / f^*\overline{M}^{\mathrm{gp}}_S\right)_{\overline{x}} + \dim_{\overline{x}} \left(\mathcal{O}^{\mathrm{sh}}_{X,\overline{x}}/\alpha(M_{X,\overline{x}} \setminus M^*_{X,\overline{x}}) \to \mathcal{O}^{\mathrm{sh}}_{S,\overline{s}}/\alpha(M_{S,\overline{s}} \setminus M^*_{S,\overline{s}})\right)$$

where the relative dimension in the classical case is as in 0 14.1.2 and IV 17.10.1 [EGA$_{IV}$].

Lemma 6.2.2. *Let X/s be log-smooth of relative dimension* 1. *Then étale locally X/s is isomorphic to the fibre in $\mathbb{N}_{>0} = 0$ of one of the following local models:*

(i) $\mathrm{Spec}(k[t]\big[\mathbb{N}\big]) \to \mathrm{Spec}(k[\mathbb{N}]),\ 1 \mapsto e$ *for some $e \in \mathbb{N}$ at $t = 0$, or*

(ii) $\mathrm{Spec}(k[Q]) \to \mathrm{Spec}(k[\mathbb{N}]), 1 \mapsto q$ *for some non-vanishing* $q \in Q$ *at* $Q \setminus \{0\} = 0$ *for some unit free fs monoid* $Q \subset Q^{\mathrm{gp}} = \mathbb{Z}^2$*. Moreover,* q *is not contained in* $p \cdot Q$ *where* p *is the residue characteristic.*

Proof: This follows easily from 3.5 [Ka89]. □

Lemma 6.2.3. *Let* X/s *be connected, log-smooth and of relative dimension* 1*. Then* $\mathring{X}_{\mathrm{red}}/\mathring{s}$ *is a semistable curve over* k*.* □

Definition 6.2.4. *Let* X/s *be log-smooth of relative dimension* 1*. Any point of type (ii) as in Lemma 6.2.2 which is not a double point of* $\mathring{X}_{\mathrm{red}}/\mathring{s}$ *is called a* **tail**.

Let $\mathrm{Sing}(X/s)$ *denote the reduced induced fs-log scheme structure on the disjoint union of the locus of all tails together with the locus of all double points* $\mathrm{Sing}\left(\mathring{X}_{\mathrm{red}}/\mathring{s}\right)(\overline{k})$ *in* X*.*

Lemma 6.2.5. *Let* X/s *be log-smooth of relative dimension* 1*. The image of the map* $M_X \to \mathcal{O}_{X_{\mathrm{red}}}$ *contains locally at any tail or double point* $\overline{x}$ *for any branch passing through* $\overline{x}$ *a uniformizers at* $\overline{x}$*.* □

Lemma 6.2.6. *Let* X/s *be log-smooth and of relative dimension* 1*. A point* $\overline{x} \in X$ *of type (ii) as in Lemma 6.2.2 is a tail if and only if the chart* $\mathbb{N} \to Q$ *of the local model maps* 1 *to a multiple of an essential generator of* Q *as an fs-monoid.*

Proof: By the choice of an isomorphism $Q^{\mathrm{gp}} = \mathbb{Z}^2$ the monoid Q gets identified with the points of the standard lattice within a rationally defined angle with vertex at the origin. The essential generators as a fs-monoid are the points on the boundary with relatively prime integral coordinates. □

Lemma 6.2.7. *Let* X/s *be log-smooth and of relative dimension* 1*. If* $Y \to X$ *is a Kummer log-étale map, then* Y/s *is log-smooth and of relative dimension* 1 *as well. If, moreover,* $Y \to X$ *is surjective then* Y *has tails if and only if* X *has tails.*

Proof: This follows easily from Proposition 3.1.10. □

As in the classical situation, for a quasi-compact X/s that is log-geometrically connected there is a natural homotopy short exact sequence:

$$1 \to \pi_1^{\log}(X_{\tilde{s}}) \to \pi_1^{\log}(X) \to \pi^{\log} \to 1 \ . \tag{6.2.4}$$

We will study this sequence for a log-geometrically connected log-smooth X/s of relative dimension 1. Let $\mathrm{Irr}(X) = \{X_i\}$ be the set of irreducible components of X, endowed with the reduced induced subscheme structure and the induced fs-log structure. Let Γ_X denote the dual graph of $\mathring{X}_{\mathrm{red}}$. For simplicity we assume that all components are smooth over k and geometrically connected, hence log-geometrically irreducible, and that all double points are already k-rational. The natural map $f : \coprod X_i \to X$ and its log-geometric fibre $f_{\tilde{s}} : \coprod X_{i,\tilde{s}} \to X_{\tilde{s}}$ are effective descent morphisms for $\mathsf{Rev}^{\log}$ by Theorem 3.2.20. The maps f and $f_{\tilde{s}}$ are furthermore composed in a natural way as a disjoint union of monomorphisms. We may therefore use the Van Kampen Theorem 2.2.9 through an obvious logarithmic version of Corollary 2.3.3. We obtain a natural isomorphism of the homotopy exact sequence (6.2.4) with the following.

$$1 \to \frac{\left(\mathop{\ast}\limits_{\mathrm{Irr}(X)} \pi_1^{\log}(X_{i,\tilde{s}}) * \widehat{\pi}_1(\Gamma_X)\right)}{\text{edge relation}} \to \frac{\left(\mathop{\ast}\limits_{\mathrm{Irr}(X)} \pi_1^{\log}(X_i) * \widehat{\pi}_1(\Gamma_X)\right)}{\text{edge relation}} \to \pi^{\log} \to 1 \tag{6.2.5}$$

Here it is not necessary to consider cocycle relations as there are no triple intersections of components by Lemma 6.2.3 and thus no faces in the 2-complex of the discretised descent datum. The edge relations are more carefully described as follows.

Let $x \in \mathrm{Sing}(X/s)$ be a double point and X_0, X_1 be the components passing through x. As a connected component of degree 1 in the decent datum, x has to be considered as an fs-log scheme with strict inclusion to X, i.e., there is a unit free fs monoid $Q \subset \mathbb{Z}^2$, a chart $Q \to k$, $(Q \setminus \{0\}) \mapsto 0$, and a non-vanishing element $q \in Q$ which is the image of π under $x \to s$. Thus $\pi_1^{\log}(x_{\tilde{s}}) = \mathrm{Hom}(Q^{\mathrm{gp}}/\langle q \rangle, \hat{\mathbb{Z}}(1)(\overline{k}))$ is free of rank 1 over $\hat{\mathbb{Z}}(1)(\overline{k})$.

The inclusions $i_\nu : x \hookrightarrow X_\nu$ for $\nu = 0, 1$ induce maps of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1^{\log}(x_{\tilde{s}}) & \longrightarrow & \pi_1^{\log}(x) & \longrightarrow & \pi^{\log} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \pi_1^{\log}(X_{\nu,\tilde{s}}) & \longrightarrow & \pi_1^{\log}(X_\nu) & \longrightarrow & \pi^{\log} & \longrightarrow & 1 . \end{array}$$

We will show in Corollary 6.2.9 that the vertical maps are in fact injective. Of course, these maps have to be constructed using the same base points and paths between those that are implicit in the Van Kampen Theorem.

To the double point x corresponds an edge $\vec{e}_x$ of Γ_X that yields an element of $\widehat{\pi}_1(\Gamma_X)$. The edge relations now consist of the following: for all $\gamma \in \pi_1^{\log}(x)$ (resp. $\gamma \in \pi_1^{\log}(x_{\tilde{s}})$) we impose

$$i_{0,\#}(\gamma)\vec{e}_x = \vec{e}_x i_{1,\#}(\gamma)$$

where $i_{\nu,\#}$ for $\nu = 0, 1$ is the respective map induced by i_ν on fundamental groups.

The left most group of the sequence (6.2.5) is essentially the same as the group π_1^{kum} of Theorem 2.8 [Saï97]. With this point of view we might have entitled this section "Galois equivariant moderate covers" as (6.2.5) expresses an arithmetic version of loc. cit.

6.2.2 The stabilizer

As above, we discuss a log-smooth X/s of relative dimension 1 over the log point. For simplicity, we assume that all irreducible components $X_i \subset X$ with the reduced induced subscheme structure are smooth over k and geometrically connected, hence log-geometrically irreducible (with the induced fs-log structure), and that all double points are k-rational.

Lemma 6.2.8. *Let $D_i = X_i \cap \mathrm{Sing}(X/s)$ be the divisor of double points and tails on X_i and $M_i = M(\log D_i)$ the associated fs-log structure on $\mathring{X}_i$. Then the following holds:*

(1) There is a unique map of fs-log schemes $f : X_i \to (\mathring{X}_i, M_i)$ such that the underlying map of schemes is the identity.

(2) The map f of (a) induces a map of homotopy short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1^{\log}(X_{i,\tilde{s}}) & \longrightarrow & \pi_1^{\log}(X_i) & \longrightarrow & \pi^{\log} & \longrightarrow & 1 \\ & & \| & & \twoheaddownarrow & & \twoheaddownarrow & & \\ 1 & \longrightarrow & \pi_1^{\mathrm{t}}(X_{i,\overline{s}}, D_{i,\overline{s}}) & \longrightarrow & \pi_1^{\mathrm{t}}(X_i, D_i) & \longrightarrow & G_k & \longrightarrow & 1 . \end{array}$$

Proof: (1) By uniqueness, it suffices to construct f étale locally. Away from the double points and tails f coincides with $\varepsilon : X_i \to \mathring{X}_i$. At $\overline{x} \in \mathrm{Sing}(X/s) \cap X_i$, the log structure M_i has a chart $\mathbb{N} \to \mathcal{O}^{\mathrm{sh}}_{X_i,\overline{x}}$ that maps 1 to a uniformizer of X_i at $\overline{x}$. By Lemma 6.2.5, uniformizers are in the image of α for the fs-log structure on X_i and they are easily verified to have a unique preimage.

(2) The map f of (1) is compatible with $\varepsilon : s \to \mathring{s}$ and $\pi_1^{\log}(\mathring{X}_i, M_i) = \pi_1^{\mathrm{t}}(X_i, D_i)$ by Example (4) at the end of Section 3.3.1. Thus it remains to prove the asserted isomorphy between the geometric parts of the homotopy exact sequences.

Let $U = X_i \setminus D_i$ with the induced fs-log structure from X_i. Then surjectivity of $\pi_1^{\log}(f)$ follows from surjectivity of $\pi_1^{\log}(U) \twoheadrightarrow \pi_1(\mathring{U})$ and $\pi_1(\mathring{U}) \twoheadrightarrow \pi_1^{\mathrm{t}}(X_i, D_i)$.

Concerning injectivity we now may work strict étale local. The lemma now follows from Proposition 3.1.10. □

Corollary 6.2.9. *The maps $\pi_1^{\log}(x_{\tilde{s}}) \to \pi_1^{\log}(X_{i,\tilde{s}})$ (resp. $\pi_1^{\log}(x) \to \pi_1^{\log}(X_i)$), of edge groups into vertex groups of the pro-finite amalgamation data in the above application of the Van Kampen Theorem are injective.* □

Corollary 6.2.10. *The exterior $\pi^{\log}$-representation $\pi_1^{\log}(X_{i,\tilde{s}})$ controls the exterior G_k-representation $\pi_1^{\mathrm{t}}(X_{i,\overline{s}}, D_{i,\overline{s}})$.*

Proof: $I^{\log}$ just acts trivial. □

Proposition 6.2.11. *The natural map $i_\# : \pi_1^{\log}(X_i) \to \pi_1^{\log}(X)$ is injective. Its image consists of the stabilizer of the monodromy action of $\pi_1^{\log}(X)$ on the set of components $\varprojlim_{Y/X} \mathrm{Irr}(Y)$ of a universal két cover at a component $\hat{X}_i$ above X_i which is distinguished by a choice of base points.*

Proof: The stabilizer of $\hat{X}_i$ is by construction the image of $i_\#$ when X_i is equipped with a universal két cover dominating $\hat{X}_i$ as a base point. It remains to prove injectivity.

According to the Van Kampen Theorem we have a decomposition of $\pi_1^{\log}(X)$ as a pro-finite amalgam of pro-finite goups along a finite graph. The map $i_\#$ therefore is a projective limit of the following kind. There is a projective system of data for amalgamation of finite groups along a fixed finite graph. The map $i_\#$ is the projective limit of the natural maps of the vertex group at a fixed vertex to the pro-finite completion of the respective amalgam.

We use the fact that the maps of edge groups to vertex groups are in fact injective, cf. Corollary 6.2.9. By an easy modification of II 2.7 [Di80] and II 2.6 [Di80], the natural map of the vertex group into the pro-finite completion of the amalgam along a finite graph of finite groups with injective edge group inclusions is injective. This proves the proposition as a projective limit of injective maps is again injective. □

6.3 Control of logarithmic specialisation

We discuss the logarithmic specialisation map $\mathrm{sp}_{\log}$ of Theorem 3.4.6 from a group-theoretical point of view. We prove a logarithmic analogue of Tamagawa's characterisation of the classical specialisation map sp. Theorem 6.3.5 will be used for the degeneration argument in the prove of Theorem 5.1.3.

In this section, S denotes the spectrum of an excellent henselian discrete valuation ring R with residue field $R/(\pi) = k$ and field of fractions K. Let η (resp. s) be its generic (resp. its closed) point and $\overline{\eta}$ (resp. $\overline{s}$) a geometric point above η (resp. s) compatible with a fixed strict henselisation $\tilde{S}$ of S in $\overline{s}$.

Specialisation. For the convenience of the reader, we restate Tamagawa–Oda's theorem on control of the specialisation map (proper case).

Theorem 6.3.1 (Tamagawa–Oda, control of sp**).** *Let $X/S, X'/S$ be proper, smooth, hyperbolic curves with geometrically connected fibres. Let ℓ be a prime number different from the residue characteristic of S. The kernel of the specialisation map $\mathrm{sp} : \pi_1(X_\eta) \twoheadrightarrow \pi_1(X_s)$ is the intersection of those open $H \subset \pi_1(X_\eta)$ which satisfy*

(i) the image of H in G_K contains the inertia group I.

(ii) the image of I in $\mathrm{Out}(\overline{H^\ell})$ is trivial. Here $\overline{H} = H \cap \pi_1(X_{\overline{\eta}})$.

Moreover, there is a natural map

$$\mathrm{Isom}_{G_K}\big(\pi_1(X_{\overline{\eta}}), \pi_1(X'_{\overline{\eta}})\big) \to \mathrm{Isom}_{G_k}\big(\pi_1(X_{\overline{s}}), \pi_1(X'_{\overline{s}})\big)$$

that yields the control of the specialisation map.

Proof: This is Theorem 5.7 [Ta97]. □

The proof of Theorem 6.3.1 uses known criteria for good reduction of a proper curve X_η via its jacobian, minimal semistable models and the combinatorics of the dual graphs for a $\mathbb{Z}/\ell\mathbb{Z}$-cover.

In the rest of this section we prove a logarithmic analogue of Theorem 6.3.1. For that concern we endow S with its standard fs-log structure $M(\log s)$ and let $\tilde{s}$ be a log geometric point over $\overline{s}$. We assume furthermore that the residue field is finite.

Lemma 6.3.2. *Let X/S be a semistable curve which is endowed with the standard fs-log structure $M(\log X_s)$. Let Y/X be a két cover. Then the reduced special fibre $Y_{s,\mathrm{red}}$ is a semistable curve over k.*

Proof: This follows imediately from Lemma 6.2.3 and Lemma 6.2.7. □

Proposition 6.3.3. *Let X/S be a proper, stable curve of genus $g \geq 2$. Then the G_K-module $\pi_1(X_{\overline{\eta}})$ controls the set of irreducible components $\mathrm{Irr}(X_{\overline{s}})$ of the special fibre.*

Proof: It suffices to work with $\pi_1(X_{\overline{\eta}})^\ell = \pi_1^{\log}(X_{\tilde{s}})^\ell$. Let Y_η/X_η be an étale Galois cover with an ℓ-group as geometric monodromy group. By Theorem 3.4.6 it extends to a két cover Y/X. The reduced special fibre $Y_{s,\mathrm{red}}$ of Y is a semistable curve over s by Lemma 6.3.2. Let Γ_Y denote the dual graph of $Y_{s,\mathrm{red}}$. From (6.2.5) and Lemma 6.2.8 we obtain an exact sequence of $G_K^{\mathrm{t}} = G_K/P \cong \pi^{\log}$ modules

$$0 \to \mathrm{H}_1^{\mathrm{CW}}\big(\Gamma_Y, \mathbb{Z}_\ell(1)\big) \to \pi_1^{\log}(Y_{\tilde{s},\mathrm{red}})^{\mathrm{ab},\ell} \to \pi_1(Y_{\overline{s},\mathrm{red}})^{\mathrm{ab},\ell} \to 0 \ . \qquad (6.3.6)$$

This sequence is characteristic by the theory of weights: $\pi_1(Y_{\overline{s},\mathrm{red}})^{\mathrm{ab},\ell}$ is the largest quotient of $\pi_1^{\log}(Y_{\tilde{s},\mathrm{red}})^{\mathrm{ab},\ell}$ for which G_K^{t} acts through the quotient G_k with mixed weights $-1, 0$. Thus the G_K-module $\pi_1(Y_{\overline{\eta}})$ controls the G_k-module $\pi_1(Y_{\overline{s},\mathrm{red}})^{\mathrm{ab},\ell}$ and by Theorem 6.1.1 also $\{Y_i \in \mathrm{Irr}(Y_{\overline{s}}) \,|\, g_{Y_i} > 0\}$.

For varying két covers Y/X, there are natural maps between the respective pro-ℓ abelianisation sequences (6.3.6). These maps are compatible with the direct sum decompositions used in Theorem 6.1.1. Hence, for Y'/X of the above type dominating Y/X, we control the natural inclusion

$$\{Y_i \in \mathrm{Irr}(Y_{\overline{s}}) \,|\, g_{Y_i} > 0\}/\overline{\mathrm{Gal}}(Y/X) \hookrightarrow \{Y_i' \in \mathrm{Irr}(Y_{\overline{s}}') \,|\, g_{Y_i'} > 0\}/\overline{\mathrm{Gal}}(Y'/X)$$

where $\overline{\mathrm{Gal}}$ denotes the respective geometric monodromy group. In the limit we recover

$$\mathrm{Irr}(X_{\overline{s}}) = \varinjlim_{\substack{Y/X \\ \text{geometric } \ell \text{ Galois}}} \{Y_i \in \mathrm{Irr}(Y_{\overline{s}}) \,|\, g_{Y_i} > 0\}/\overline{\mathrm{Gal}}(Y/X)$$

as any rational component C gains genus in a suitable covering and thus becomes visible. Indeed, using $\mathbb{Z}/\ell$ mock-covers we may assume that the component is smooth. Then Lemma 6.2.8 and Proposition 6.2.11 apply to show that all covers of C with tame ramification in the double points appear. The component minus the double points being smooth hyperbolic (stability!) these covers have genus tending to ∞. □

Corollary 6.3.4. *Let X/S be a proper, stable curve of genus $g \geq 2$. Then the G_K-module $\pi_1(X_{\overline{\eta}})$ controls (1) the Euler-characteristic $\chi(\Gamma_X)$ of the dual graph of the special fibre, (2) the cardinality $\#\Gamma_0(X_{\overline{s}})$ of the set of vertices, and (3) the cardinality $\#\,\mathrm{Sing}\big(X/S\big)(\overline{k})$ of the set of geometric double points.*

Proof:

(1) $\chi(\Gamma_X) = \mathrm{rk}_{\mathbb{Z}_\ell} \mathrm{H}_1^{\mathrm{CW}}\big(\Gamma_X, \mathbb{Z}_\ell(1)\big)$,

(2) $\#\Gamma_0(X_{\overline{s}}) = \#\,\mathrm{Irr}(X_{\overline{s}})$,

(3) $\#\,\mathrm{Sing}\big(X/S\big)(\overline{k}) = \#\Gamma_1(X_{\overline{s}}) = \#\Gamma_0(X_{\overline{s}}) - \chi(\Gamma_X)$. □

Theorem 6.3.5 (control of $\mathrm{sp}_{\log}$). *Let X/S be a proper, stable curve of genus $g \geq 2$ which is not smooth.*

Then the G_K-module $\pi_1(X_{\overline{\eta}})$ controls the G_K^t-module $\pi_1^{\log}(X_{\tilde{s}})$.

Proof: The kernel of the logarithmic specialisation map consists of the intersection of all open normal subgroups such that the Conditions (i)-(iii) of Theorem 3.4.14 are satisfied. We need to control these conditions. As (i) and (ii) are obviously group-theoretically controlled, we treat (iii). Let $H \lhd \pi_1(X_\eta)$ be the normal subgroup corresponding to $Y_\eta \to X_\eta$ such that (i) and (ii) hold. By a két base extension we may assume that Y_η has a stable model Y/S. Uniqueness of the stable model implies that the action of the Galois group G extends to Y. We need to show that the stabilizers of the double points under the action of G do not contain p-groups. Equivalently, we may verify that any $\mathbb{Z}/p \subset G$ acts freely on $\mathrm{Sing}(Y/S)(\overline{k})$. By Raynaud's theorem in the appendix of [Ra90], see Theorem 7.1.4, the quotient $X' = Y'/(\mathbb{Z}/p)$ is also semistable over S. By weight considerations we control the Euler-characteristic $\chi(\Gamma_{X'})$ of the dual graph of its special fibre, but we also control the set of vertices as these are just the quotient of the $\mathbb{Z}/p$ action on $\mathrm{Irr}(Y_{\overline{s}})$. Hence we control group-theoretically whether

$$\mathrm{Sing}(Y/S)(\overline{k}) = p \cdot \mathrm{Sing}(X'/S)(\overline{k})$$

is valid which is equivalent to $\mathbb{Z}/p$ acting freely on the set of double points. Thus (iii) is a condition controlled by the G_K-module $\pi_1(X_{\overline{\eta}})$. □

Proposition 6.3.6. *Let X/S be a proper, stable curve of genus $g \geq 2$ which is not smooth, such that the reduced irreducible components $X_{i,s}$ of X_s are smooth over k, geometrically connected, and all double points are k-rational. Let D_i be the divisor* $\mathrm{Sing}(X/S) \cap X_{i,s}$ *on $X_{i,s}$. Then the exterior G_K-module $\pi_1(X_{\overline{\eta}})$ controls the exterior G_k-module $\pi_1^{\mathrm{t}}\big(X_{i,\overline{s}}, D_{i,\overline{s}}\big)$.*

Proof: First of all by Theorem 6.3.5 we control $\pi_1^{\log}(X_s)$. For a cofinal system of két covers Y/X we control $\mathrm{Irr}(Y_s)$ by Proposition 6.3.3 as a set. Therefore we control the action of $\pi_1^{\log}(X_s)$ on $\varprojlim_{Y/X,\text{két}} \mathrm{Irr}(Y_s)$ which is the pro-finite set of components of a universal két cover of X_s. We choose a component $\hat{X}_{i,s}$ above $X_{i,s} \in \mathrm{Irr}(X_s)$ and consider the stabilizer of the $\pi_1^{\log}(X_s)$ action. Due to Proposition 6.2.11 this stabilizer is nothing but $\pi_1^{\log}(X_{i,s})$ which in turn by Corollary 6.2.10 controls the exterior G_k module $\pi_1^{\mathrm{t}}\big(X_{i,\overline{s}}, D_{i,\overline{s}}\big)$. □

Chapter 7

The Use of Moduli Spaces

Here we prove Theorem 5.1.1. Our strategy will be to extend our curve to a family X/S and control its representing map $S \to \mathscr{M}$ into a fine moduli scheme in terms of the fundamental group of the generic fibre.

We face three problems. (1) Anabelian Geometry over finite fields so far deals with π_1^{t} of *affine* hyperbolic curves. (2) Anabelian Geometry over finite fields only controls curves up to a Frobenius twist that a priori depends on the point $s \in S$. (3) We need extra structure to obtain a fine moduli scheme of curves.

To overcome these problems we involve (3) level structures, (2) topological coincidence of maps, Theorem 1.2.1, and (1) degenerating Galois covers with stable non-smooth reduction at s together with the results of Chapter 6.

7.1 Degeneration

7.1.1 The fundamental group in families

In characteristic 0 the fundamental group of a smooth curve does only depend on its type (g, r): the genus g and the number r of cusps or points at infinity. As these discrete invariants are locally constant in families the fundamental group does not vary locally although the isomorphism class of the curve possibly does. The situation behaves the opposite in positive characteristic.

It is known that there is a course moduli variety $M_g/\mathbb{F}_p$ for curves of genus g in characteristic p, cf. Theorem 5.11 [GIT94]. It has the property that for an algebraically closed field Ω of characteristic p there is a natural bijection

$$M_g(\Omega) = \{X/\Omega \mid \text{ smooth, proper curves of genus } g \ \}/_{\text{isom}} .$$

Additionally, for every family X/S of smooth proper curves of genus g over a base $S/\mathbb{F}_p$ there is a characteristic map $\zeta_X : S \to M_g$ such that a geometric point $\omega \in S(\Omega)$ is mapped to $X_\omega \in M_g(\Omega)$.

In the base S of a family X/S we look closer at a pair of points $t_0, t_1 \in S$ such that $t_1 \in \overline{\{t_0\}} =: T$. We choose appropriate geometric points $\bar{t}_0, \bar{t}_1$ which are tied together by fixing a strict henselisation $\mathcal{O}^{\mathrm{sh}}_{T,t_1}$. Along this strict henselisation one constructs the specialisation map $sp : \pi_1(X_{\bar{t}_0}) \twoheadrightarrow \pi_1(X_{\bar{t}_1})$ of exterior $G_{\kappa(t_0)}$-modules where this structure on the second group comes from the restriction $G_{\kappa(t_0)} \twoheadrightarrow G_{\kappa(t_1)}$. The question arises when sp is an isomorphism.

Theorem 7.1.1 (Tamagawa). *Let $S/\mathbb{F}_p$ be irreducible with generic point η and X/S be a smooth, proper curve of genus $g \geq 2$ such that the corresponding $\zeta_X : S \to M_g$ is finite. Then for all but finitely many closed point $s \in S$ the specialisation map*

$$\mathrm{sp} : \pi_1(X_{\overline{\eta}}) \twoheadrightarrow \pi_1(X_{\overline{s}})$$

is not an isomorphism.

Proof: This is Tamagawa's Theorem 0.3 [Ta02]. It even suffices to consider the following. For a suitable prime-to-p cyclic cover the p-rank of the generic fibre exceeds the p-rank of the special fibre. Hence the theorem also holds on the metabelian level. Note that the above statement is obvious in the mixed characteristic case exploited by Mochizuki in [Mz96]. □

7.1.2 Stable reduction

The preceeding theorem essentially provides us with covers without good reduction. If we allow ourselves finite base extensions the following well known theorem says that the situation is not too bad.

Fix a prime ℓ different from the residue characteristic.

Theorem 7.1.2 (stable reduction, Neron–Ogg–Shafarevic, Serre–Tate, Grothendieck, Raynaud, Artin–Winters, T. Saito). *Let S be the spectrum of an excellent, strict henselian discrete valuation ring with generic point η and closed point s. Let $K = \kappa(\eta)$ be the field of fractions, $\overline{K}/K$ an algebraic closure with Galois group G_K. Let X_K/K be a smooth proper curve of genus $g \geq 2$. Then the following are equivalent:*

(a) the action of G_K on $\mathrm{H}^1_{et}(X_{\overline{K}}\mathbb{Q}_\ell) = \mathrm{Hom}\big(\pi_1(X_{\overline{K}}), \mathbb{Q}_\ell\big)$ is unipotent,

(b) the minimal regular model of X_K/K over S is semistable.

Proof: Theorem 1 [Sai87] using vanishing cycles; 1.3 and 1.4 [Ab00]. □

We justify the name Theorem of Stable Reduction by the following. First, the theory of minimal models states that in case (*b*) we have also a stable model X/S for X_K/K. Second, the action of G_K is always quasi-unipotent, i.e., a subgroup of finite index acts unipotently by the Monodromy Theorem. Therefore, after eventually a finite base extension, the reduction becomes stable.

Theorem 7.1.3 (monodromy theorem, Grothendieck). *In the above situation the action of G_K on $\mathrm{H}^1_{et}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is quasi-unipotent of rank 2.*

Proof: appendix B [Ab00]; appendix [ST68]. □

Let $S = \{\eta, s\}$ be the spectrum of a henselian discrete valuation ring. Let X/S be a smooth proper curve of genus $g \geq 2$, and Y/S a proper stable curve such that there is a G-Galois cover $Y_\eta \to X_\eta$ of the generic fibres. The rest of this section describes the connection between the special fibres.

The uniqueness of the stable model prolongates the G-action to the stable model Y/S. The quotient Y/G is again semistable over S by the following theorem:

Theorem 7.1.4 (Raynaud). *Let S be the spectrum of an excellent, henselian discrete valuation ring, Y/S a generically smooth, semistable curve, and G a finite group acting on Y/S. Then the quotient Y/G exists and is semistable over S.*

Proof: Appendix [Ra90], the assumption of completeness can be weakend to excellent and henselian. □

Again by the theory of minimal models of surfaces there is a birational map $Y/G \to X$ over S whose fibres are at most trees of $\mathbb{P}^1$ in the special fibre. Hence the special fibres are not "too far apart" from the viewpoint of étale covers. If we apply the following lemma to the special fibre of Y/G the topological invariance of the fundamental group yields a similar assertion for Y_s/G and $(Y/G)_s$.

Lemma 7.1.5. *Let Y be a scheme and G a finite group of automorphisms such that the quotient $X = Y/G$ exists and $p : Y \to Y/G$ is affine. Let $X' \hookrightarrow X$ be an immersion and $Y' = X' \times_X Y$.*

Then the G-action prolongates to Y', the quotient $p' : Y' \to Y'/G$ exists and the induced map $f : Y'/G \to X'$ is a universal homeomorphism.

Proof: By V §1 [SGA 1] p and p' are topologically quotient maps. Thus f is bijective. Integral dependence shows that p, p' are universally closed and by the factorization $Y' \twoheadrightarrow Y'/G \xrightarrow{f} X'$ of the base change of p the same holds for f.

Let Ω be an algebraically closed field. Then by the next lemma

$$\big(Y'/G\big)(\Omega) = Y'(\Omega)/G = X'(\Omega)$$

which implies that f is radicial.

Being radicial, integral and surjective f is a universal homeomorphism by 18.12.11 [EGA_{IV}]. □

Lemma 7.1.6. *Let Y be a scheme and G a finite group of automorphisms such that the quotient $X = Y/G$ exists and $p : Y \to Y/G$ is affine. Let Ω be an algebraically closed field. Then $Y(\Omega)/G = X(\Omega)$.*

Proof: This follows from V §1.1 (ii), (iii) [SGA 1] as pointed out in §2 [SGA 1]. Flat base change and taking finite quotients commute, so we may replace X by the strict henselisation in a point $\overline{x} \in X(\Omega)$ and discuss its preimages. Y becomes a product of local strict henselian schemes which are permuted transitively by G, because it is the spectrum of the limit of finite $\mathcal{O}^{\text{sh}}_{X,\overline{x}}$-algebras whose cardinality of the topological special fibre is bounded by $\#G$, cf. §18 [EGA_{IV}]. Replacing Y by a connected component and G by its stabilizer we reduce to a local situation with pure inseparable residue field extension. Then the statement has become trivial. □

To summarize the above: the special fibre of $Y \to X$ factorises in first a quotient map by G, then a universal homeomorphism, and finally a contraction of trees of $\mathbb{P}^1$'s.

7.1.3 The use of degeneration

The aim of this section is to prove the following theorem.

Theorem 7.1.7. *Let $S/\mathbb{F}_p$ be irreducible of finite type with generic point η. Let X/S be a smooth, proper curve of genus $g \geq 2$ such that the characteristic map $\zeta_X : S \to M_g$ has image of dimension at least 1. Then for closed points s in an open dense subset of S the exterior $G_{\kappa(\eta)}$-module $\pi_1(X_{\overline{\eta}})$ controls the isomorphism type of the fibre X_s up to twists by Frobenius.*

Proof: By Theorem 7.1.1 we restrict to an open part of S where sp from the generic point to the closed points is not an isomorphism. We may further shrink the base to assume S regular, cf. 7.8.3 [EGA$_{\mathrm{IV}}$]. The idea is now to specialise π_1 by Theorem 6.3.1 along a chain of henselian discrete valuation rings from η to s. A system of regular parameters produces points $t_o = \eta, \ldots, t_i, \ldots, t_{\dim S} = s$, such that $\mathcal{O}^{\mathrm{h}}_{\overline{\{t_i\}}, t_{i+1}}$ is a henselian discrete valuation ring.

The choice of the regular parameters has to be performed with some care so that in the last step sp is still not an isomorphism: choose a cover with bad reduction at s and do not leave the open part of S where it has good reduction until the last step. Here the shunting into general position is done by Lemma 7.1.9 which we postpone.

This reduces the proof to the following local version of the theorem. □

Theorem 7.1.8. *Let $s \in C$ be a closed point on a smooth curve over $\mathbb{F}_p$. Let S be the spectrum of $\mathcal{O}^{\mathrm{h}}_{C,s}$ with generic point η. Let X/S be a smooth, proper curve of genus $g \geq 2$ such that* $\mathrm{sp} : \pi_1(X_{\overline{\eta}}) \twoheadrightarrow \pi_1(X_{\overline{s}})$ *is not an isomorphism.*

Then the exterior $G_{k(\eta)}$-module $\pi_1(X_{\overline{\eta}})$ controls the isomorphism type of the special fibre X_s up to twists by Frobenius.

Proof: We choose a degenerating G-Galois cover $Y_\eta \to X_\eta$ coresponding to a suitable exterior Galois submodule. Let us assume for the moment that Y_η has stable, but not smooth reduction Y/S such that the irreducible components $Y_{s,i}$ of the special fibre Y_s are smooth and geometrically connected as curves over $\kappa(s)$. Let $U_i = Y_{s,i} \setminus \mathrm{Sing}(Y/S)$ be the regular locus of $Y_{s,i}$. By Proposition 6.3.6 we control the set $\Gamma_0(Y_s) = \{Y_{s,i}\}$ of irreducible components together with the respective tame fundamental groups $\pi_1^{\mathrm{t}}(U_{i,\overline{s}})$ as exterior $G_{\kappa(s)}$-modules. Anabelian Geometry for affine hyperbolic curves over finite fields [Ta97, Sx02] gives us control over U_i up to a Frobenius twist.

The action of G on this data is preserved due to the functorial nature of the notion of control. Let G_i denote the stabilizer of the component $Y_{s,i}$ under the G-action. It follows that we control the quotient U_i/G_i. Applying π_1^{t} gives us control over $\pi_1^{\mathrm{t}}\big((U_i/G_i)_{\overline{s}}\big)$ as an exterior $G_{\kappa(s)}$-module.

According to the last section the map $Y_s \to X_s$ factorises as a quotient map by G, a universal homeomorphism and a contraction of trees of $\mathbb{P}^1$'s. Due to this using the topological invariance of the tame fundamental group and control of the genus of the smooth compactification, just one of the $\pi_1^{\mathrm{t}}\big((U_i/G_i)_{\overline{s}}\big)$ describes a curve of genus ≥ 2, which is an affine open part U of the special fibre X_s. But controlling $\pi_1^{\mathrm{t}}(U_{\overline{s}})$ as an exterior $G_{\kappa(s)}$-module controls U up to twist by Frobenius by Anabelian Geometry over finite fields. Compactifying, we control X_s up to a twist by Frobenius.

Finally, the assumptions made for Y are valid after possibly a finite extension S'/S of schemes of valuation rings, cf. Theorem 7.1.2, and the replacement by a $\mathbb{Z}/\ell$ mock-cover for the purpose of disentangling selfintersecting components. We may assume S'/S Galois. But then this Galois group acts on the situation in a compatible way with the control of the special fibre. Galois descent for quasi-projective varieties enables us also to control the original special fibre, cf. the sheaf Lemma 4.1.3. □

Lemma 7.1.9 (shunting). *Let T be a regular, local, noetherian scheme with closed point $s \in T$. Let $U \subset T$ be an open subset not containing s. Then there is a regular sequence of parameters $x_1, \ldots, x_d \in \mathcal{O}_{T,s}$ such that $(x_1, \ldots, x_d) = \mathfrak{m}$ is the prime ideal corresponding to s and for $i < d$ the points corresponding to the prime ideals $(x_1, \ldots x_i)$ lie in U.*

Proof: By induction it suffices to construct $x_1 \in \mathfrak{m} \setminus \mathfrak{m}^2$. Let Z^0 be the set of generic points for $Z = T \setminus U$. If $Z = \{s\}$ then any regular sequence of parameters satisfies our needs. Otherwise $\mathfrak{m} \setminus \left(\mathfrak{m}^2 \cup \bigcup_{\mathfrak{p} \in Z^0} \mathfrak{p}\right)$ is not empty by strong prime avoidance, cf. Lemma 3.3 [Ei94]. Any x_1 therin works as (x_1) has height 1 but is not a generic point of Z. □

7.2 Digestion with level structures

In the proof of Theorem 5.1.1 we need a fine moduli space of smooth curves that is a scheme. The standard construction introduces level structures. We discuss how the fundamental group controls level structures. The previous results of Section 7.1.3 are shown to be compatible with level structures in Section 7.2.4.

For the following compare with Definition 5.6 in [DM69] where Teichmüller structures for more general groups than $(\mathbb{Z}/n)^{2g}$ are defined.

7.2.1 Level structure for Galois modules

Let P be a pro-finite group with an exterior Γ-module structure for some pro-finite group Γ. Fix nonnegative integers n, g and consider $(\mathbb{Z}/n)^{2g}$ as a Γ-module with trivial action.

Definition 7.2.1. *A* **level** n **structure** *of genus g on P is a surjection $\varphi : P \twoheadrightarrow (\mathbb{Z}/n)^{2g}$ of Γ-modules. The number n is called the level.*

Of course, not all P carry such level n structures. But fundamental groups of proper smooth curves posses at least étale locally level n structures of their respective genus. Usually we suppress mentioning the genus and even the level of a level structure.

The essence of the definition does not depend on the action by Γ. On any finite quotient module a subgroup of finite index will act trivially, and often we are free to replace Γ by such an open subgroup.

Change of base. A morphism $u : \Gamma' \to \Gamma$ transforms Γ-modules P to Γ'-modules u^*P; the action of $\gamma' \in \Gamma'$ is by $u(\gamma)$. When P carries a level structure so does u^*P naturally.

7.2.2 Level structure for curves

For an abelian group A the constant étale sheaf on a scheme is denoted by the same letter. Let $p : X \to S$ be a smooth proper curve of genus g. Assume that n is invertible on S. Then $\mathrm{R}^1 p_*\mathbb{Z}/n$ is a locally constant sheaf on S_{et} with stalks isomorphic to $(\mathbb{Z}/n)^{2g}$, cf. Arcata V 3.1, III 3.5 [SGA $4\frac{1}{2}$].

Definition 7.2.2. *A* **level** n **structure** *on X/S is an isomorphism $\varphi : (\mathbb{Z}/n)^{2g} \cong \mathrm{R}^1 p_*\mathbb{Z}/n$ of étale sheaves on S_{et}. We call n its level.*

At least after étale localisation a level structure exists for every smooth proper curve. When they exist they form a torsor under $Gl_{2g}(\mathbb{Z}/n)(S)$.

Change of base. A base extension $u : S' \to S$ allows to pull back the curve $p : X \to S$ to a curve $p' : X' \to S'$. By proper base change, cf. Arcata IV 1.1 [SGA $4\frac{1}{2}$], we have $u^* \mathrm{R}^1 p_*\mathbb{Z}/n \cong \mathrm{R}^1 p'_*\mathbb{Z}/n$. Hence a level n structure on X/S naturally equipes X'/S' with a level structure.

7.2.3 Get together

Now we compare the two points of view in applying the first to the fundamental group of a fibre. Let $p : X \to S$ be a smooth proper curve of genus g. Assume S is connected and n still invertible on S. Then $\mathrm{R}^1\, p_*\mathbb{Z}/n$ being locally constant is determined by the action of $\pi_1(S, \overline{s})$ on the stalk $\big(\mathrm{R}^1\, p_*\mathbb{Z}/n\big)_{\overline{s}} = \mathrm{H}^1(X_{\overline{s}}, \mathbb{Z}/n)$ for any geometric point $\overline{s}$ of S. Furthermore

$$\mathrm{H}^1(X_{\overline{s}}, \mathbb{Z}/n) \cong \mathrm{Hom}\big(\pi_1(X_{\overline{s}}), \mathbb{Z}/n\big)$$

as both sides describe étale $\mathbb{Z}/n$ torsors. The isomorphism respects the $G_{\kappa(s)}$-action if we endow the Hom with the contragredient action. Hence we can describe a level n structure on X/S by a level n structure of genus g on $\pi_1(X_{\overline{s}})$ and vice versa. Of course, the latter factors through an isomorphism $\pi_1(X_{\overline{s}})^{\mathrm{ab}}/n \cong (\mathbb{Z}/n)^{2g}$.

Change of base. The concepts of change of base coincide. Let X'/S' be the base extension of X/S by a map $u : S' \to S$ of connected bases compatible with geometric points $\overline{s}'$ and $\overline{s}$. The base change isomorphism

$$\mathrm{R}^1\, p'_*\mathbb{Z}/n \cong u^*R^1p_*\mathbb{Z}/n$$

of étale cohomology is reflected by the isomorphism of $\pi_1(S', \overline{s}')$-modules

$$\mathrm{H}^1(X'_{\overline{s}'}, \mathbb{Z}/n) \cong (u_\#)^*\mathrm{H}^1(X_{\overline{s}}, \mathbb{Z}/n)$$

induced by projection. Here $u_\#$ is the canonical map $\pi_1(S', \overline{s}') \to \pi_1(S, \overline{s})$. Taking the $\mathrm{Hom}(\cdot, \mathbb{Z}/n)$ dual yields the natural map

$$\mathrm{pr}_\# : \pi_1(X'_{\overline{s}'})^{\mathrm{ab}}/n \cong (u_\#)^*\pi_1(X_{\overline{s}})^{\mathrm{ab}}/n$$

of $\pi_1(S', \overline{s}')$-modules. This map mediates level n structures group-theoretically.

Example. Base extension by the Frobenius map plays a prominent role and shall be studied in more detail. Let $u = F$ be the Frobenius in the above notation such that $X' = X(1)$ is the twist by Frobenius. We endow $S = S'$ with base points such that $\overline{s}' = \overline{s} \circ F \in S(\Omega)$ for some algebraically closed field Ω. Canonically $\pi_1(S, \overline{s}) = \pi_1(S, \overline{s}')$ holds. The Frobenius F is compatible with these base points and $F_\#$ becomes identity of $\pi_1(S, \overline{s})$ by Lemma 4.1.1. The composition of the following two maps of $\pi(S, \overline{s})$-modules

$$\pi_1(X_{\overline{s}})^{\mathrm{ab}}/n \xrightarrow{(F_S)_{\overline{s},\#}} \pi_1(X(1)_{\overline{s}})^{\mathrm{ab}}/n \xrightarrow{\mathrm{pr}_\#} (F_\#)^*\pi_1(X_{\overline{s}})^{\mathrm{ab}}/n$$

is by functoriality π_1 applied to F which yields identity by Lemma 4.1.1. Hence the base extension of level structures is as well mediated by the effect of the relative Frobenius map on fundamantal groups. Note: the map $\mathrm{pr}_\#$ above is not the base change of the projection $X(1) \to X$ but up to identification the projection $X(1)_{\overline{s}} = X_{\overline{s}}(1) \to X_{\overline{s}}$.

Specialisation. Let S be the spectrum of a henselian discrete vatuation ring with generic point η and closed point s, and let S^{sh} be a strict henselisation. Let n be prime to the residue characteristics. Let X/S be a smooth proper curve of genus g. As smooth morphisms are locally acyclic, cf. Arcata V 2.1, V 3.1 [SGA $4\frac{1}{2}$], the cospecialisation map

$$\mathrm{cosp}^* : \mathrm{H}^1(X_{\overline{\eta}}, \mathbb{Z}/n\mathbb{Z}) \xleftarrow{\sim} \mathrm{H}^1(X_{S^{\mathrm{sh}}}, \mathbb{Z}/n\mathbb{Z}) = \big(R^1p_*\mathbb{Z}/n\mathbb{Z}\big)_{\overline{s}} \xrightarrow{\sim} \mathrm{H}^1(X_{\overline{s}}, \mathbb{Z}/n\mathbb{Z})$$

is an isomorphism of abelian groups respecting the Galois action via restriction

$$G_{\kappa(\eta)} \twoheadrightarrow \pi_1(S, S^{\mathrm{sh}}) \xleftarrow{\sim} G_{\kappa(s)} \ .$$

Thus a level structure on the generic fibre specialises canonically to one of the special fibre.

Via the interpretation as groups classifying étale $\mathbb{Z}/n\mathbb{Z}$ torsors we see that cosp is the $\mathrm{Hom}(\cdot, \mathbb{Z}/n)$ dual of the specialisation map sp of fundamental groups showing that sp transports level structures like geometry does.

7.2.4 Compatibility

We convince ourselves that the result of Theorem 7.1.7 is compatible with level structures.

Theorem 7.2.3. *Let $S/\mathbb{F}_p$ be irreducible of finite type with generic point η. Let $n \in \mathbb{N}$ be invertible on S. Let X/S be a smooth, proper curve of genus $g \geq 2$ equipped with a level n structure. Assume that the characteristic map $\zeta_X : S \to M_g$ has image of dimension at least 1. Then for closed points s in an open dense subset of S the exterior $G_{\kappa(\eta)}$-module $\pi_1(X_{\overline{\eta}})$ controls the isomorphism type of the fibre X_s as smooth, proper curve with level n structure up to twists by Frobenius.*

Proof: This amelioration of Theorem 7.1.7 amounts to commutativity of

$$\begin{array}{ccc}
\mathrm{Isom}_{G_{\kappa(\eta)}}\big(\pi_1(X_{\overline{\eta}}), \pi_1(X'_{\overline{\eta}})\big) & \xrightarrow{\quad c \quad} & \mathrm{Isom}_{\kappa(s), F^{-1}_{\kappa(s)}}\big(X_s, X'_s\big) \\
\downarrow & & \downarrow \\
\mathrm{Isom}_{G_{\kappa(\eta)}}\big(\pi_1(X_{\overline{\eta}})^{\mathrm{ab}}/n, \pi_1(X'_{\overline{\eta}})^{\mathrm{ab}}/n\big) & \xrightarrow[\mathrm{sp}]{\sim} & \mathrm{Isom}_{G_{\kappa(\eta)}}\big(\pi_1(X_{\overline{s}})^{\mathrm{ab}}/n, \pi_1(X'_{\overline{s}})^{\mathrm{ab}}/n\big)
\end{array}$$

where c is the map claimed by control. Let α be an isomorphism of the π_1 of the generic fibre. We need to show that $\mathrm{sp} \circ \alpha = \pi_1(c(\alpha)) \circ \mathrm{sp}$.

By construction of c, see Theorem 7.1.8 for notations, various subquotients of Galois modules appear forming a diagram

$$\begin{array}{ccccccc}
 & & & & \pi_1^{\mathrm{t}}(Y_{\overline{s},i}) & \longrightarrow & \pi_1^{\mathrm{t}}(U_{\overline{s}}) \\
 & & & & \cap & & \Big\downarrow \\
\pi_1(Y_{\overline{\eta}}) & \xrightarrow{\mathrm{sp}_{\log}} & \pi_1^{\log}(Y_{S^{\mathrm{sh}}}) & \xleftarrow{\sim} & \pi_1^{\log}(Y_{\tilde{s}}) & & \Big\downarrow \\
\downarrow & & \downarrow & & & & \twoheaddownarrow \\
\pi_1(X_{\overline{\eta}}) & \xrightarrow{\mathrm{sp}} & \pi_1(X_{S^{\mathrm{sh}}}) & & \xleftarrow{\quad\sim\quad} & & \pi_1(X_{\overline{s}})
\end{array} \tag{7.2.1}$$

and a twin diagram with $'$ at the respective places.

Apart from the upper right corner in diagram (7.2.1), α induces maps of Galois modules connecting both diagrams by various results on control, cf. Theorem 6.3.5, Theorem 6.3.1 and Proposition 6.3.6. These are just restrictions of α to subquotients and therefore compatible. We call them α again.

The map $c(\alpha)$ comes from anabelian geometry for affine hyperbolic curves over finite fields and some geometric manipulations. The α between $\pi_1^{\mathrm{t}}(Y_{\overline{s},i})$ is $\pi_1^{\mathrm{t}}(\tilde{c})$ for some map $\tilde{c}$

a quotient of which composed with radicial maps and then compactified is $c(\alpha)$. However the following diagram commutes

$$\begin{array}{ccc} \pi_1^{\mathrm{t}}(Y_{\bar{s},i}) & \longrightarrow & \pi_1(X_{\bar{s}}) \\ {\scriptstyle \alpha=\pi_1^{\mathrm{t}}(\tilde{c})}\downarrow & & \downarrow \\ \pi_1^{\mathrm{t}}(Y'_{\bar{s},i}) & \longrightarrow & \pi_1(X'_{\bar{s}}) \end{array}$$

where the second vertical arrow is either α or $\pi_1(c(\alpha))$. By the property of *uniqueness of extension along geometric maps* for isomorphisms, Corollary 4.2.11, these two maps coincide. That implies the theorem. □

7.3 The proof of the main result

In Chapter 5 we reduced the proof of Theorem 5.1.1 to that of Theorem 5.1.3 and, moreover, to the following: F-cohomological rigidity, algebraic $K(\pi, 1)$ spaces (treated in appendix A), and the construction of a retraction λ. We now continue the proof.

7.3.1 Moduli of curves with level structure

The strategy which we are going to embark on for the proof of our main result will depend descisively on the availability of a fine moduli space for curves. We fix $g \geq 2$ and $n \geq 3$ prime to p. The contravariant functor on schemes $S/\mathbb{F}_p$

$$S \mapsto \left\{ (X/S, \varphi) \;\middle|\; \begin{array}{c} \text{smooth proper curves of genus } g \\ \text{with level } n \text{ structure } \varphi \end{array} \right\} /_{\text{isom.}}$$

is represented by a smooth scheme $\mathscr{M}_g[n]/\mathbb{F}_p$. Let $\mathscr{M}_g$ be the Deligne-Mumford stack which classifies the groupoid of smooth proper curves of genus g, cf. 5.8, 5.14 [DM69] and [Se60]. By forgetting the level structure we obtain a map $\mathscr{M}_g[n] \to \mathscr{M}_g \times_{\mathbb{Z}} \mathbb{F}_p$ carrying the structure of an étale $Gl_{2g}(\mathbb{Z}/n)$-torsor by changing the level structure accordingly. The coarse moduli space M_g of section 7.1.1 is the coarse moduli scheme belonging to $\mathscr{M}_g$, cf. 5.6 [GIT94]. It consists of the quotient in the sense of schemes of the $Gl_{2g}(\mathbb{Z}/n)$-action on $\mathscr{M}_g[n]$.

The natural action of Frobenius on $\mathscr{M}_g[n](S)$ comprises in $X \mapsto X(1)$. Here the level structure on $X(1)$ is the one induced by π_1 applied to the relative Frobenius map $F_S : X \to X(1)$, cf. the example in Section 7.2.3 .

7.3.2 F-cohomological rigidity for curves

We return to the proof of Theorem 5.1.3. In Section 5.3 we dealt with a class $\mathscr{V}$ of geometrically connected varieties over separable extensions of the field K. Here, $\mathscr{V}$ consists of proper hyperbolic curves that are non-isotrivial. We need to verify the properties (i)-(iii) of Section 5.3 for this choice of $\mathscr{V}$. Proposition A.4.1 yields (iii). By the next lemma $\mathscr{V}$ is stable under finite étale covers, hence (i).

Lemma 7.3.1. *Let $Y \to X$ be an étale cover of proper hyperbolic curves. Then*

$$X \text{ is isotrivial} \iff Y \text{ is isotrivial} .$$

Proof: If X is isotrivial then so is Y by X 1.8 [SGA 1]. Assume conversely that Y is isotrival. By "$\Rightarrow$" we may assume that Y/X is Galois with group G. After a finite extension of the base field all automorphism of Y from G are already defined over a finite field as the automorphism scheme of a hyperbolic curve is unramified. The respective quotient yields a model of $X_{\overline{K}}$ over a finite field. $\square$

Now we prove that proper non-isotrivial hyperbolic curves over K are F-cohomologically rigid (property (ii) of Section 5.3).

Lemma 7.3.2. *Let X/K be a proper hyperbolic curve endowed with a level n structure corresponding to $\xi_X : \mathrm{Spec}(K) \to \mathscr{M}_g[n]$. Then*

$$X \textit{ is isotrivial} \iff \xi_X\big(\mathrm{Spec}(K)\big) \textit{ is a closed point .}$$

Proof: The map $\mathscr{M}_g[n] \to \mathscr{M}_g$ is finite étale. $\square$

Proposition 7.3.3. *Let X, X' be proper hyperbolic curves over K, and let $n \geq 3$ be invertible in K. Assume that at least one of X, X' is not isotrivial. Then the canonical map*

$$\underline{\mathrm{Isom}}_{K,F_K^{-1}}(X, X') \hookrightarrow \underline{\mathrm{Isom}}_{G_K}\big(\pi_1(X_{\overline{K}})^{\mathrm{ab}}/n, \pi_1(X'_{\overline{K}})^{\mathrm{ab}}/n\big)$$

of étale sheaves of sets induced by π_1 is injective.

Proof: We work étale locally on K, endow X with a level n structure $\varphi : (\mathbb{Z}/n)^{2g} \cong \pi_1(X_{\overline{K}})^{\mathrm{ab}}/n$ and obtain the characteristic map $\xi_X \in \mathscr{M}_g[n](K)$. The induced level structure on $X(m)$ is called $\varphi(m)$.

Let $g = (F_K)^{-m'} \circ f \circ (F_K)^m$ for some $m, m' \in \mathbb{N}$ and isomorphism $f : X(m) \cong X(m')$ be a preimage of the identity. Then f is an isomorphism of curves with level structure as the following diagram of isomorphisms commutes.

$$\begin{array}{ccccccc}
(\mathbb{Z}/n)^{2g} & \xrightarrow{\varphi} & \pi_1(X_{\overline{K}})^{\mathrm{ab}}/n & \xrightarrow{\pi_1(g_{\overline{K}})^{\mathrm{ab}}/n = id} & \pi_1(X_{\overline{K}})^{\mathrm{ab}}/n & \xleftarrow{\varphi} & (\mathbb{Z}/n)^{2g} \\
& \searrow^{\varphi(m)} & \downarrow{\scriptstyle (F_K)^m} & & \downarrow{\scriptstyle (F_K)^{m'}} & \swarrow_{\varphi(m')} & \\
& & \pi_1(X(m)_{\overline{K}})^{\mathrm{ab}}/n & \xrightarrow[\pi_1(f_{\overline{K}})^{\mathrm{ab}}/n]{} & \pi_1(X(m')_{\overline{K}})^{\mathrm{ab}}/n & &
\end{array}$$

Hence $\xi_X \circ F^m = \xi_X \circ F^{m'}$. As X is not isotrivial $m = m'$ by Lemma 7.3.2 and we may further assume that $m = 0$. The induced automorphism f^* on the jacobian Jac_X has finite order as X is hyperbolic and acts trivial on n-torsion points as $\mathrm{Jac}_X[n] \cong \pi_1(X_{\overline{K}})^{\mathrm{ab}}/n$. By [Se60] f^* is the identity. Thus $f(P) - P \sim f(Q) - Q$ for all $P, Q \in X(\overline{K})$.

The Lefschetz number of f is $\mathrm{tr}(f^*, \mathrm{H}^*_{et}) = 2 - 2g < 0$. Consequently f has a fixed point. But then $f(P) - P \sim 0$ for all P and f is the identity as X is not $\mathbb{P}^1_K$. $\square$

Corollary 7.3.4. *A proper hyperbolic curve that is non-isotrivial satisfies F-cohomological rigidity for all coefficients $\mathbb{Z}/n$ with $n \geq 3$ prime to the characteristic of the ground field.*

Proof: $\mathrm{H}^1\big(X_{\overline{K}}, \mathbb{Z}/n\big) = \mathrm{Hom}\big(\pi_1(X_{\overline{K}})^{\mathrm{ab}}/n, \mathbb{Z}/n\big)$. $\square$

7.3.3 The retraction for curves

To complete the proof of Theorem 5.1.3 we need to construct a partial retraction λ for π_1 as in Lemma 5.3.1.

Let X, X' be as in the Theorem 5.1.3. Diagram (5.2.2) from the proof of Lemma 5.2.2 has a sheafified version due to Lemma 4.1.3

$$\begin{array}{ccc} \underline{\mathrm{Isom}}_{K,F_K^{-1}}\big(X,X'\big) & \xrightarrow{\pi_1} & \underline{\mathrm{Isom}}_{G_K}\big(\pi_1(X_{\overline{K}}),\pi_1(X'_{\overline{K}})\big) \\ \big\downarrow & & \big\downarrow \mathrm{H}^* \\ \underline{\mathrm{Isom}}_{G_K}\big(\mathrm{H}^*(X'_{\overline{K}}),\mathrm{H}^*(X_{\overline{K}})\big) & = & \underline{\mathrm{Isom}}_{G_K}\big(\mathrm{H}^*(\pi_1(X'_{\overline{K}})),\mathrm{H}^*(\pi_1(X_{\overline{K}}))\big) \end{array}$$

wherein all coefficients for cohomology are constant $\mathbb{Z}/n$ for $n \geq 3$ prime to the characteristic. Hence the existence of $\lambda = \lambda_n$ for this n is equivalent to the map H^* of étale sheaves of sets on $\mathrm{Spec}(K)_{et}$ to factorise through the subsheaf $\underline{\mathrm{Isom}}_{K,F_K^{-1}}\big(X,X'\big)$. In particular we may work étale locally on K and λ_n is unique if it exists at all. If n divides N and both λ_n and λ_N exist then they coincide in view of uniqueness; hence λ_n is independent of n and we may suppress the index again.

Let $\alpha \in \mathrm{Isom}_{G_K}\big(\pi_1(X_{\overline{K}}),\pi_1(X'_{\overline{K}})\big)$. We fix $n \geq 3$. By the above, we may enlarge K and therefore assume, that both curves possess level n structures. We choose a level n structure on X and transport it to X' via α. Now we extend the data to some base S of finite type over $\mathbb{F}_p$ with function field K such that the curves X, X' are the generic fibres of some smooth proper fibrations $X/S, X'/S$ with geometrically connected fibres. We need to describe the representing maps $\xi_X, \xi_{X'} : S \to \mathscr{M}_g[n]$. After possibly shrinking S we know by Theorem 7.2.3 that for all closed points $s \in S$ the special fibres X_s and X'_s are isomorphic as curves with level structure — but only up to a twist by Frobenius. The effect of the latter on representing maps being composition by the Frobenius map, we conclude that the topological component of ξ_X and $\xi_{X'}$ coincide.

This is the point where the rigidifying effect of the result on topological coincidence of maps develops its strength. From Theorem 1.2.1 we know that ξ_X and $\xi_{X'}$ differ only by a unique power of Frobenius. Hence there is $m \in \mathbb{Z}$ such that $X(m) \cong X'$ or $X \cong X'(-m)$ as S-curves with level structure. Taking generic fibres the resulting $f \in \mathrm{Isom}_{K,F_K^{-1}}(X,X')$ is uniquely determined by this procedure. It does not depend on the chosen level structure for the given n as these are transitively permuted by a $Gl_{2g}(\mathbb{Z}/n)$-action on $\mathscr{M}_g[n]$. Furthermore f acts like α on level n structures.

By Proposition 7.3.3 and the proof of its Corollary 7.3.4 we conclude that the effects of α and f on cohomology with values in $\mathbb{Z}/n$ coincide. We therefore declare $\lambda(\alpha) := f$. This completes the construction of a partial retraction satisfying the prerequisits of Lemma 5.3.1.

Hence, by the lemmata of Section 5.3, λ is inverse to the map π_1 in Theorem 5.1.3 proving bijectivity. This completes the proof of Theorem 5.1.3 and thus the main result of this thesis. $\square$

Appendix A

Algebraic $K(\pi, 1)$ Spaces

For a suitable topological space X with universal covering $\hat{X}/X$ there is a tautological map $\xi_{\hat{X}} : X \to B\pi_1(X)$ to the classifying space of its fundamental group corresponding to the $\pi_1(X)$-torsor $\hat{X}/X$. The space X is an Eilenberg-MacLane space $K(\pi, 1)$, i.e., the fundamental group is the only non-vanishing homotopy group, if and only if $\xi_{\hat{X}}$ is a homotopy equivalence. By Whitehead and Hurewicz this is equivalent to the property that $\xi_{\hat{X}}$ induces isomorphisms

$$\mathrm{H}^*(\pi_1(X), A) = \mathrm{H}^*(B\pi_1(X), \mathscr{A}) \to \mathrm{H}^*(X, \mathscr{A})$$

for all $\pi_1(X)$-modules A and the associated locally constant system $\mathscr{A}$ on X, resp. on $B\pi_1(X)$.

This appendix analyses analogues of the above for étale cohomology. The main result states that curves over algebraically closed fields other than the projective line are algebraic $K(\pi, 1)$ spaces.

A.1 Definition and higher direct images

Definition A.1.1. *For a profinite group Γ let $\mathscr{B}\Gamma$ denote the site of continuous Γ-sets endowed with the canonical topology, cf. I (1.3.3) [Tm94]. The category of sheaves of abelian groups on $\mathscr{B}\Gamma$ is denoted by $\widetilde{\mathscr{B}\Gamma}$, the subcategory of sheaves of locally constant torsion sheaves by $\left(\widetilde{\mathscr{B}\Gamma}\right)_f$, and for a prime number ℓ the subcategory of locally constant ℓ-primary torsion sheaves by $\left(\widetilde{\mathscr{B}\Gamma}\right)_\ell$.*

Let X be a connected scheme, $\overline{x} \in X$ a geometric point, and $\pi = \pi_1(X, \overline{x})$ its étale fundamental group. Then there are natural continuous maps of sites

$$\begin{array}{ccc} X_{et} & \xrightarrow{\gamma} & \mathscr{B}\pi \\ & \searrow^{\gamma_\ell} & \downarrow \\ & & \mathscr{B}\pi^\ell \end{array}$$

that map a transitive π (resp. π^ℓ) set to the corresponding connected étale cover. There are adjoint maps $\mathrm{id} \to \mathrm{R}\,\gamma_* \gamma^*$ (resp. $\mathrm{id} \to \mathrm{R}\,\gamma_{\ell,*}\gamma_\ell^*$) that induce maps in cohomology

$$c^* : \mathrm{H}^*(\pi, A) \to \mathrm{H}^*(X_{et}, \gamma^* A)$$

$$c_\ell^* : \mathrm{H}^*(\pi^\ell, B) \to \mathrm{H}^*(X_{et}, \gamma_\ell^* B)$$

for $A \in \big(\widetilde{\mathscr{B}\pi}\big)_f$ and $B \in \big(\widetilde{\mathscr{B}\pi^\ell}\big)_\ell$. We suppress the index $\acute{e}t$ in the sequel.

Definition A.1.2. *(1) X is called an* **algebraic** $K(\pi,1)$ **space** *if c^* is an isomorphism for all A.*

(2) X is called a **pro-ℓ** $K(\pi,1)$ **space** *if c_ℓ^* is an isomorphism for all B.*

Example. The spectrum $\mathrm{Spec}(K)$ of a field is an algebraic $K(\pi,1)$ space as obviously $\gamma : \mathrm{Spec}(K)_{et} \to \mathscr{B}G_K$ is an equivalence of sites.

The site $\mathscr{B}\pi$ (resp. $\mathscr{B}\pi^\ell$) has, up to isomorphism, a unique point, i.e., a unique exact conservative functor from abelian sheaves to abelian groups. This point can be constructed by means of a universal covering $\hat{X} \to X$ (resp. pro-ℓ universal covering $\hat{X}_\ell \to X$) corresponding to the set π (resp. π^ℓ) with left action under itself by translation. Hence the higher direct images of γ (resp. γ_ℓ) are governed by a single stalk which by II (5.5) [Tm94] is given by the following formula. For $A \in \big(\widetilde{\mathscr{B}\pi}\big)_f$, resp. $B \in \big(\widetilde{\mathscr{B}\pi^\ell}\big)_\ell$,

$$\big(\mathrm{R}^q\,\gamma_*\gamma^* A\big)_{\hat{X}} = \varinjlim_{Y/X} \mathrm{H}^q(Y, \gamma^* A|_Y) \tag{A.1.1}$$

$$\big(\mathrm{R}^q\,\gamma_{\ell,*}\gamma_\ell^* B\big)_{\hat{X}_\ell} = \varinjlim_{Y/X} \mathrm{H}^q(Y, \gamma_\ell^* B|_Y) \tag{A.1.2}$$

where Y/X runs over all finite étale (resp. pro-ℓ finite étale) subcovers of the universal covering.

Lemma A.1.3. *(1) $\mathrm{R}^q\,\gamma_*\gamma^*$ vanishes on $\big(\widetilde{\mathscr{B}\pi}\big)_f$ if and only if $\mathrm{R}^q\,\gamma_*\gamma^*(\mathbb{Z}/\ell)$ vanishes for all prime numbers ℓ.*

(2) $\mathrm{R}^q\,\gamma_{\ell,*}\gamma_\ell^*$ vanishes on $\big(\widetilde{\mathscr{B}\pi^\ell}\big)_\ell$ if and only if $\mathrm{R}^q\,\gamma_{\ell,*}\gamma_\ell^*(\mathbb{Z}/\ell)$ vanishes.

Proof: By $\varinjlim$-arguments it suffices to deal with finite sheaves. In view of formulas (A.1.1) and (A.1.2) we may apply devissage arguments to reduce to suchlike "simple" sheaves, that any sheaf under consideration has *locally* a composition series with factors that are restrictions of these. That locally the sheaves become constant proves the lemma. □

Corollary A.1.4. *The first direct image $\mathrm{R}^1\,\gamma_*\gamma^*$ vanishes on $\big(\widetilde{\mathscr{B}\pi}\big)_f$, and $\mathrm{R}^1\,\gamma_{\ell,*}\gamma_\ell^*$ vanishes on $\big(\widetilde{\mathscr{B}\pi^\ell}\big)_\ell$. In particular c^1, c_ℓ^1 are always bijective and c^2, c_ℓ^2 are always injective.*

Proof: This follows from the Leray spectral sequence and the vanishing of

$$\big(\mathrm{R}^1\,\gamma_*\gamma^*\mathbb{Z}/\ell\big)_{\hat{X}} = \varinjlim_{Y/X} \mathrm{H}^1(Y, \mathbb{Z}/\ell) = 0\ ,$$

$$\big(\mathrm{R}^1\,\gamma_{\ell,*}\gamma_\ell^*\mathbb{Z}/\ell\big)_{\hat{X}_\ell} = \varinjlim_{Y/X} \mathrm{H}^1(Y, \mathbb{Z}/\ell) = 0\ .$$

Indeed, we may interpret elements of H^1 as étale torsors. Then the cohomology classes kill themselves. □

A.2 Hochschild–Serre–Shapiro

In this section we prove that being an algebraic $K(\pi,1)$ space passes to étale covers and vice versa. The property descends along Galois covers by the Hochshild–Serre spectral sequence and ascends arbitrarily via Shapiro's Lemma.

Let $Y \to X$ be a pro-étale Galois cover corresponding to a normal subgroup H with quotient group $G = \pi/H$. Then γ_X, γ_Y induce maps between Hochschild–Serre spectral sequences for $A \in \big(\widetilde{\mathscr{B}\pi}\big)_f$

$$\begin{array}{ccc} \mathrm{H}^p\big(G, \mathrm{H}^q(H, A|_H)\big) & \Longrightarrow & \mathrm{H}^{p+q}\big(\pi, A\big) \\ \downarrow{\scriptstyle c_Y} & & \downarrow{\scriptstyle c_X} \\ \mathrm{H}^p\big(G, \mathrm{H}^q(Y, \gamma_X^* A|_Y)\big) & \Longrightarrow & \mathrm{H}^{p+q}\big(X, \gamma_X^* A\big) \end{array}$$

as canonically $(\gamma_X^* A)|_Y = \gamma_Y^*(A|_H)$. If, moreover, G is a pro-ℓ group, a similar diagram exists for the pro-ℓ version.

Corollary A.2.1. *(1) If $Y \to X$ is a connected pro-étale Galois cover such that Y is an algebraic $K(\pi,1)$ space, then X is an algebraic $K(\pi,1)$ space.*

(2) If $Y \to X$ is a connected pro-étale Galois cover with pro-ℓ Galois group such that Y is a pro-ℓ $K(\pi,1)$ space, then X is a pro-ℓ $K(\pi,1)$ space. ☐

Let $f : Y \to X$ be finite étale corresponding to an open subgroup $H < \pi$. For $A \in \big(\widetilde{\mathscr{B}H}\big)_f$ coinduced module $\mathrm{Hom}_H^{\mathrm{cont}}(\pi, A)$ is denoted by $M_H^\pi(A)$. Since $\gamma_X^*(M_H^\pi(A)) = f_*\gamma_Y^* A$, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^*\big(H, A\big) & \xrightarrow{\;c_Y\;} & \mathrm{H}^*\big(Y, \gamma_Y^* A\big) \\ \downarrow{\scriptstyle \mathrm{cor}} & & \downarrow{\scriptstyle f_*} \\ \mathrm{H}^*\big(\pi, M_H^\pi(A)\big) & \xrightarrow{\;c_X\;} & \mathrm{H}^*\big(X, \gamma_X^*(M_H^\pi(A))\big) \end{array} \tag{A.2.3}$$

whose vertical maps are isomorphisms by Shapiro's Lemma and the vanishing of the higher direct images for finite maps in étale cohomology.

Corollary A.2.2. *(1) If $Y \to X$ is a connected finite étale cover such that X is an algebraic $K(\pi,1)$ space, then Y is an algebraic $K(\pi,1)$ space.*

(2) If $Y \to X$ is a connected finite étale cover with ℓ-primary Galois closure such that X is a pro-ℓ $K(\pi,1)$ space, then Y is a pro-ℓ $K(\pi,1)$ space.

Proof: (1) is imediate from the diagram (A.2.3). (2) By the assumption on the Galois closure H^ℓ is a subgroup of π^ℓ. Then we use a pro-ℓ version of diagram (A.2.3). ☐

Proposition A.2.3.

(1) Let $Y \to X$ be a connected pro-étale cover. Then

X is an algebraic $K(\pi,1)$ space $\iff$ Y is an algebraic $K(\pi,1)$ space.

(2) If moreover $Y \to X$ has a pro-ℓ Galois closure then

X is a pro-ℓ $K(\pi,1)$ space $\iff$ Y is a pro-ℓ $K(\pi,1)$ space.

Proof: This follows from $\varinjlim$-arguments and the combination of Corollary A.2.1 and Corollary A.2.2. ☐

A.3 Criteria for being algebraic $K(\pi, 1)$

For the following compare with I 2.6 [Se97].

Proposition A.3.1. *The following are equivalent:*

(a) X *is an algebraic* $K(\pi,1)$ *space.*

(b) *For all* $A \in \big(\widetilde{\mathscr{B}\pi}\big)_f$ *the map* $c^* : \mathrm{H}^*(\pi, A) \to \mathrm{H}^*(X, \gamma^* A)$ *is surjective.*

(c) $\mathrm{H}^*\big(X, \gamma^*(\cdot)\big)$ *is elementwise effacable as a functor on* $\big(\widetilde{\mathscr{B}\pi}\big)_f$.

(d) *The restriction* $\mathrm{H}^*(X, \gamma^* A) \to \varinjlim_{Y/X} \mathrm{H}^*(Y, \gamma^* A|_Y)$ *where* Y/X *runs over all finite étale covers is the zero map for* $q > 0$ *and all* $A \in \big(\widetilde{\mathscr{B}\pi}\big)_f$.

(e) $\mathrm{R}^q\,\gamma_*\gamma^*$ *vanishes on* $\big(\widetilde{\mathscr{B}\pi}\big)_f$ *for* $q > 0$.

(f) *The adjoint map* $\mathrm{id} \to \mathrm{R}\,\gamma_*\gamma^*$ *is a quasi-isomorphism on* $\big(\widetilde{\mathscr{B}\pi}\big)_f$.

Proof: The implications (e) ⇒ (f) ⇒ (a) ⇒ (b) are obvious. For (b) ⇒ (c) observe that by using coinduced modules we can efface the whole cohomology group. By $\varinjlim$-arguments, a finite, locally constant submodule suffices to efface a single element. For (c) ⇒ (a), we invoke Grothendiecks theorem that elementwise effacable cohomological functors are universal.

For (a) ⇒ (d), note that elements of $\mathrm{H}^q(\pi, A)$ for $q > 0$ are killed by restriction to a suitable open subgroup and the map of (d) factorises through any such restriction. For (d) ⇒ (c), observe that if an element of $\mathrm{H}^q(X, \gamma^* A)$ is killed in the limit then also already on some finite level $\mathrm{H}^q(Y, \gamma^* A|_Y)$. The embedding in the respective coinduced module $M_H^\pi(A|_H)$ with H corresponding to Y/X effaces the element.

For (a) & (d) ⇒ (e), we use the formula for the stalk (A.1.1). The stalk limit is generated by the images under restriction of all $\mathrm{H}^q(Y, \gamma^* A|_Y)$ which vanish by (d) as all Y are algebraic $K(\pi,1)$ spaces by (a) and Proposition A.2.3. □

Corollary A.3.2. *The following are equivalent:*

(a) X *is an pro-*ℓ $K(\pi,1)$ *space.*

(b) *For all* $B \in \big(\widetilde{\mathscr{B}\pi^\ell}\big)_\ell$ *the map* $c_\ell^* : \mathrm{H}^*(\pi^\ell, B) \to \mathrm{H}^*(X, \gamma_\ell^* B)$ *is surjective.*

(c) $\mathrm{H}^*\big(X, \gamma_\ell^*(\cdot)\big)$ *is elementwise effacable as a functor on* $\big(\widetilde{\mathscr{B}\pi^\ell}\big)_\ell$.

(d) *The restriction* $\mathrm{H}^*(X, \gamma_\ell^* B) \to \varinjlim_{Y/X} \mathrm{H}^*(Y, \gamma_\ell^* B|_Y)$ *where* Y/X *runs over all finite étale Galois-covers with* ℓ*-primary Galois group is the zero map for* $q > 0$ *and all* $B \in \big(\widetilde{\mathscr{B}\pi^\ell}\big)_\ell$.

(e) $\mathrm{R}^q\,\gamma_{\ell,*}\gamma_\ell^*$ *vanishes on* $\big(\widetilde{\mathscr{B}\pi^\ell}\big)_\ell$ *for* $q > 0$.

(f) *The adjoint map* $\mathrm{id} \to \mathrm{R}\,\gamma_{\ell,*}\gamma_\ell^*$ *is a quasi-isomorphism on* $\big(\widetilde{\mathscr{B}\pi^\ell}\big)_\ell$.

Proof: Completely analogous to Proposition A.3.1. □

Definition A.3.3. *Let Γ be a pro-finite group. We say that Γ is* **ℓ-good** *if the inflation map $\mathrm{H}^*(\Gamma^\ell, B) \to \mathrm{H}^*(\Gamma, B)$ is an isomorphism for all $B \in \big(\widetilde{\mathscr{B}\Gamma}\big)_\ell$.*

Proposition A.3.4. *The following are equivalent:*

(a) X is an algebraic $K(\pi, 1)$ space and all open subgroups H of π are ℓ-good for all prime numbers ℓ.

(b) For all finite étale covers $Y \to X$ the scheme Y is a pro-ℓ $K(\pi, 1)$ space for all prime numbers ℓ.

Proof: (a) $\Rightarrow$ (b): By Proposition A.2.3 it suffices to treat X. For $B \in \big(\widetilde{\mathscr{B}\pi^\ell}\big)_\ell \subset \big(\widetilde{\mathscr{B}\pi}\big)_f$ the map

$$\mathrm{H}^*(\pi^\ell, B) \to \mathrm{H}^*(\pi, B) \to \mathrm{H}^*(X, \gamma_\ell^* B)$$

is a composition of isomorphisms. Therefore (b) holds.

(b) $\Rightarrow$ (a): First we prove that X is an algebraic $K(\pi, 1)$ space. By Proposition A.3.1 (e) and Lemma A.1.3, we need to show that $\varinjlim_{Y/X} \mathrm{H}^q(Y, \mathbb{Z}/\ell)$ vanishes for $q > 0$. This holds due to Corollary A.3.2 as the elements of $\mathrm{H}^q(Y, \mathbb{Z}/\ell)$ vanish already in finite ℓ-primary étale Galois extensions of Y.

To prove the assertion on the property ℓ-good, it suffices to treat π itself. For $B \in \big(\widetilde{\mathscr{B}\pi^\ell}\big)_\ell$ two of the three maps in the diagram

$$\begin{array}{ccc} \mathrm{H}^*(\pi^\ell, B) & \xrightarrow{\sim} & \mathrm{H}^*(X, \gamma_\ell^* B) \\ \downarrow & \nearrow_{\sim} & \\ \mathrm{H}^*(\pi, B) & & \end{array}$$

are isomorphisms and so is the third. Thus π is ℓ-good. $\square$

A.4 Cohomology of curves

The projective line is not an algebraic $K(\pi, 1)$ space as $\mathbb{P}^1_K$ is simply connected but $\mathrm{H}^2\big(\mathbb{P}^1_K, \mathbb{Z}/n(1)\big) \cong \mathbb{Z}/n$ for n prime to the characteristic. However, for all other curves we have the following result.

Proposition A.4.1. *Let K be an algebraically closed field. Let X/K be a smooth, connected curve which is not $\mathbb{P}^1_K$. Let $\pi = \pi_1(X)$ be its fundamental group. Then the following holds:*

(1) X is an algebraic $K(\pi, 1)$ space.

(2) π is an ℓ-good group for all prime numbers ℓ.

(3) X is a pro-ℓ $K(\pi, 1)$ space for all prime numbers ℓ.

Proof: (1)&(2) follow from (3) by Proposition A.3.4. Lemma A.1.3 implies that to prove (3) it suffices to show the vanishing of $\varinjlim_{Y/X} \mathrm{H}^q(Y, \mathbb{Z}/\ell)$ for all $q > 0$ where Y/X runs over all ℓ-primary étale Galois covers. This holds in general for $q = 1$. Since $\mathrm{cd}_\ell(Y) \leq 2$ only the case of $q = 2$ remains open.

For affine curves or if ℓ equals the characteristic of K even $\mathrm{cd}_\ell(Y) \leq 1$ holds. Thus we may assume that X/K is proper (as well as all its covers Y/K) and ℓ is different from the characteristic. By Poincare duality, $\mathrm{H}^2(Y, \mathbb{Z}/\ell)$ is generated by cup products of classes of degree 1 which vanish in the above limit, hence also the elements of H^2 are killed. $\square$

Appendix B

Automorphisms and Inverting Frobenius

Let X be a variety over a field K of positive characteristic. The objective of this appendix is to discuss the difference between $\mathrm{Aut}_K(X)$ and $\mathrm{Aut}_{K,F_K^{-1}}(X)$, cf. Section 4.1.2. Furthermore, we are interested in the interplay of the properties of X/K being isotrivial and being F-cohomologically rigid with the question on automorphism groups.

Definition B.0.1. *Let X be a scheme over S. We say that X/S is* **isotrivial** *if locally in the topology generated by Zariski-coverings and finite surjective maps (cf. 18.12.13 [EGA$_{IV}$]) X/S comes from some $X_0/\mathbb{F}_q$ for "$q \gg p$". In particular, S must be a scheme over $\mathbb{F}_p$.*

From this definition it is imediate that isotriviality of some X/S of finite type and isotriviality of X'/S' which is the base change by an integral dependent affine surjective $S' \to S$ are equivalent. In particular X/S and its twist $X(1)/S$ are simultaneously (not) isotrivial.

B.1 Frobenius modules

In the sequel all schemes will be in characteristic $p > 0$. We fix $q = p^m$, a power of p, and let Φ denote the respective map on schemes over $\mathbb{F}_p$ of raising functions to q^{th}-power.

Definition B.1.1. *Let S be a scheme over $\mathbb{F}_p$. A pair $(\mathscr{M}, \varphi)$ is a* **Frobenius module over** *S if $\mathscr{M}$ is a quasi-coherent module on S together with a $\mathcal{O}_S$-linear map $\varphi : \Phi^*\mathscr{M} \to \mathscr{M}$. Morphisms of Frobenius modules are morphisms of the underlying quasi-coherent module that commute with the φ's. The additive category of all Frobenius modules over S is denoted by $\mathsf{FM}(S)$.*

Over an affine base $\mathrm{Spec}(A)$, enhancing $\mathscr{M} = \widetilde{M}$ to a Frobenius module amounts by adjointness to find a q-linear map $\varphi : M \to M$, i.e., $\varphi(am) = a^q\varphi(m)$. The trivial example consists in the pair $(\mathcal{O}_S, \mathrm{id} : \Phi^*\mathcal{O}_S \to \mathcal{O}_S)$ that will be called the constant Frobenius module of rank 1. Over an affine base $\mathrm{Spec}(A)$ the constant Frobenius module is raising to q^{th}-power on A. Constant modules of higher rank are obtained as direct sums.

Let $(\mathscr{M}, \varphi)$ be a Frobenius module over S. Then the sheaf of abelian groups $V(\mathscr{M})$ on S is defined as the sections m of $\mathscr{M}$ such that $\varphi(1 \otimes m) = m$. If S is a scheme over $\mathbb{F}_q$ then $V(\mathscr{M})$ is a sheaf of $\mathbb{F}_q$-vector spaces. Obviously one has $\mathrm{Hom}\big((\mathcal{O}_S, \mathrm{id}), (\mathscr{M}, \varphi)\big) = V(\mathscr{M})$.

Base change. Let $S' \to S$ be a map of schems over $\mathbb{F}_p$. As it comutes with Φ we may define the base change $f^*(\mathscr{M}, \varphi)$ of a Frobenius module $(\mathscr{M}, \varphi)$ on S as the Frobenius module on S' given by $f^*\mathscr{M}$ and $\Phi^* f^* \mathscr{M} = f^* \Phi^* \mathscr{M} \xrightarrow{f^*\varphi} f^*\mathscr{M}$. For example the base change of the constant Frobenius module of rank 1 is again constant of rank 1.

From now on we will make the assumption that $\mathscr{M}$ is even a vector bundle of finite rank. The resulting subcategory of $\mathsf{FM}(S)$ is denoted by $\mathsf{FVB}(S)$ and preserved under base change. The geometric fibre of such a Frobenius vector bundle $\mathscr{M}$ is a Frobenius module M over a perfect field and as such has a canonical decomposition $M = M^{ss} \oplus M^{np}$ into a semisimple and a nilpotent part. By perfectness image and kernel are again vector spaces and as with linear maps $M^{ss} = \varphi^n(M)$ and $M^{np} = \ker(\varphi^n)$ for $n \gg 0$.

Definition B.1.2. *A Frobenius vector bundle $(\mathscr{M}, \varphi)$ is called* **ordinary** *if φ is an isomorphism.*

This just means that we ask for the nilpotent part to vanish. Now we have the following isotriviality result.

Proposition B.1.3. *Any ordinary Frobenius vector bundle is locally constant in the étale topology. In particular it comes after restriction to an étale covering by base change from the constant Frobenius vector bundle over $\mathbb{F}_p$ of the same rank.*

Proof: Let $(\mathscr{M}, \varphi)$ be an ordinary Frobenius vector bundle on S. Then we have to find an étale covering $\{U \to S\}$ such that the base changes $(\mathscr{M}, \varphi)|_U$ are constant. We may assume that $S = \mathrm{Spec}(A)$ is affine and $\mathscr{M} = \widetilde{M}$ is free of rank r. We choose a basis $e_i \in M$ and write a generic element of M as $x = \sum x_i e_i$ with variables x_i. As $\mathscr{M}$ is ordinary also the $\varphi(e_i)$ form a basis and thus $e_j = \sum c_{ji} \varphi(e_i)$ for some matrix $C = (c_{ij}) \in GL_r(A)$. Solving $\varphi(x) = x$ amounts to solve coordinatewise

$$\sum x_i^q \varphi(e_i) = \sum x_j c_{ji} \varphi(e_i) \ .$$

The universal solution lives in the following ring

$$B' = A[x_1, \ldots, x_r]/(x_i^q - \sum x_j c_{ji}, 1 \le i \le r)$$

which is finite étale over A of rank q^r. Indeed, it is finite flat and the relative differentials are $\Omega_{B'/A} = \bigoplus B' dx_i / (\sum c_{ji} dx_j) = (0)$. Let B be the Galois closure of B'/A

Restricting to $\mathrm{Spec}(B)$ we may now assume that $V(M)$ is a vector space over $\mathbb{F}_q$ of dimension r as the equation $\varphi(x) = x$ has q^r solutions in B. We choose a basis $\{v_i\} \subset V(M)$ over $\mathbb{F}_q$ and consider the respective map $\mathcal{O}_S^r \to \mathscr{M}$. It must be injective as any relation $\sum b_i v_i = 0$ implies that $\sum b_i^q v_i = \varphi(\sum b_i v_i) = 0$. Any minimal relation with $b_i = 1$ for some i therefore must have coefficients in $\mathbb{F}_q$, contradicting $\{v_i\}$ being a basis.

For surjectivity we may argue fibrewise by Nakayama's Lemma. As both vector bundles have the same rank, its fibres are vector spaces of the same finite dimension. Then surjectivity follows from injectivity which in turn still holds by the same argument as before. □

Corollary B.1.4. *There is a canonical bijection*

$$\left\{\begin{array}{c}\textit{isomorphism classes of}\\ \textit{ordinary Frobenius vector}\\ \textit{bundles of rank } r \textit{ over } S\end{array}\right\} = \mathrm{H}^1_{et}\big(S, Gl_r(\mathbb{F}_q)\big) = \mathrm{Hom}_{\mathscr{G}}\big(\pi_1(S), Gl_r(\mathbb{F}_q)\big)$$

such that $(\mathscr{M},\varphi) \mapsto \underline{\mathrm{Isom}}_{\mathbb{F}_q}\big((\mathcal{O}_S^r,\mathrm{id}),(\mathscr{M},\varphi)\big)$ *as* $Gl_r(\mathbb{F}_q)$*-torsor.* □

Cohomology. Let $p : X \to S$ be a map of schemes over $\mathbb{F}_p$. For a Frobenius module $(\mathscr{M},\varphi)$ over X the higher direct images $\mathrm{R}^i\, p_*\mathscr{M}$ are again Frobenius modules (over S) defined as

$$\Phi^*\,\mathrm{R}^i\, p_*\mathscr{M} \xrightarrow{\mathrm{bc}} \mathrm{R}^i\, p_*\Phi^*\mathscr{M} \xrightarrow{\varphi} \mathrm{R}^i\, p_*\mathscr{M}$$

with the help of the base change morphism bc. Let us call these Frobenius modules $\mathrm{R}^i\, p_*(\mathscr{M},\varphi)$.

However, even if $\Phi : S \to S$ is flat the higher direct images need not preserve being vector bundles or ordinarity in general. The latter will depend on the vanishing of $\mathrm{R}^i\, p(m)_*(\mathscr{B}\otimes \mathrm{pr}^*\mathscr{M})$ for the sheaf of local exact differentials $\mathscr{B} = \Phi_{S,*}\mathcal{O}_X/\mathcal{O}_{X(m)}$ and thus happens to be related with ordinarity of X/S.

B.2 A criterion for isotriviality

We keep the notation from the preceeding section: $q = p^m$ and Φ is the q^{th}-power Frobenius on schemes over $\mathbb{F}_p$. In this way, base change of X/S by Φ is the m^{th} Frobenius twist $X(m)$. The pullback by $X(m) \to X$ of an $\mathcal{O}_X$-module $\mathscr{F}$ is denoted by $\mathscr{F}^{(m)}$.

Proposition B.2.1. *Let $p : X \to S$ be a flat map of schemes over $\mathbb{F}_p$ such that Φ is flat on S. Let $\mathscr{L}$ be a relatively over S ample line bundle on X. Let furthermore $f : X \cong X(m)$ be an isomorphism of S-schemes and $\varphi : f^*\mathscr{L}^{(m)} \cong \mathscr{L}$ an isomorphism of line bundles.*

Then X/S is isotrivial. More precisely, X/S together with a projective embedding defined by $\mathscr{L}$ is étale locally on S isomorphic to some constant model $X_0/\mathbb{F}_q$.

Proof: We may assume that $\mathscr{L}$ is very ample relative an affine $S = \mathrm{Spec}(A)$ over $\mathbb{F}_q$. Then the embedding $X \hookrightarrow \mathbb{P}_A\big(\mathrm{H}^0(X,\mathscr{L})\big)$ into the projective space over A is defined by the following map of graded A-modules.

$$A\big[\mathrm{H}^0(X,\mathscr{L})\big] \to \bigoplus_{d\geq 0} \mathrm{H}^0(X,\mathscr{L}^{\otimes d}) \tag{B.2.1}$$

By flat base change

$$\Phi^*\mathrm{H}^0(X,\mathscr{L}^{\otimes d}) \xrightarrow{\mathrm{bc}} \mathrm{H}^0(X, f^*(\mathscr{L}^{(m)})^{\otimes d}) \xrightarrow{\varphi^{\otimes d}} \mathrm{H}^0(X,\mathscr{L}^{\otimes d})$$

is an isomorphism. Hence the graded pieces of (B.2.1) are Frobenius modules over Spec(A). They are even ordinary Frobenius vector bundles for $d \gg 0$ by the theorem of Cohomology and Base Change and the vanishing of cohomology in positive degrees after sufficient twisting. Now we consider the diagram

$$\begin{array}{ccc}\mathbb{F}_q\left[V\big(\mathrm{H}^0(X,\mathscr{L})\big)\right]\otimes_{\mathbb{F}_q} A & \longrightarrow & \bigoplus_{d\geq 0} V\big(\mathrm{H}^0(X,\mathscr{L}^{\otimes d})\big)\otimes_{\mathbb{F}_q} A\\ \downarrow & & \downarrow\\ A\big[\mathrm{H}^0(X,\mathscr{L})\big] & \longrightarrow & \bigoplus_{d\geq 0}\mathrm{H}^0(X,\mathscr{L}^{\otimes d})\ .\end{array}$$

By Proposition B.1.3 and finite generation as A-algebra the vertical maps become isomorphisms up to finite length $A\big[\mathrm{H}^0(X,\mathscr{L})\big]$-modules after restriction to a finite étale cover of $\mathrm{Spec}(A)$. We may therefore asume that they are already isomorphisms (up to finite length) over $\mathrm{Spec}(A)$. The kernel of $\mathbb{F}_q\left[V\big(\mathrm{H}^0(X,\mathscr{L})\big)\right] \to \bigoplus_{d\geq 0} V\big(\mathrm{H}^0(X,\mathscr{L}^{\otimes d})\big)$ defines a closed subscheme $X_0 \hookrightarrow \mathbb{P}_A\big(V(\mathrm{H}^0(X,\mathscr{L}))\big)$. As in the Proj construction finite length modules are invisible, we get an isomorphism $X_0 \times_{\mathbb{F}_q} \mathrm{Spec}(A) \cong X|_{\mathrm{Spec}(A)}$ as projective schemes over $\mathrm{Spec}(A)$. □

Corollary B.2.2. *Let S be a scheme over $\mathbb{F}_p$ such that Φ is flat on S. Let X/S be smooth projective such that either the relative canonical bundle $\omega_{X/S}$ or its inverse is ample. Then the following are equivalent:*

(a) There are $m \neq m'$ such that $X(m)$ and $X(m')$ are isomorphic over S.

(b) X is isotrivial over S.

Proof: Only (a) ⇒ (b) requires proof. We apply Proposition B.2.1 with $\mathscr{L}$ either the canonical bundle or its dual. Pullback of differential forms induces the identification of $\mathscr{L}$ with the pullback of its twist. □

Corollary B.2.3. *Let S be a scheme over $\mathbb{F}_p$ such that Φ is flat on S. Let X/S be a smooth projective curve. Then the following are equivalent:*

(a) There are $m \neq m'$ such that $X(m)$ and $X(m')$ are isomorphic over S.

(b) X is isotrivial over S.

Proof: For genus g different from 1 we may use Corollary B.2.2. For $g = 1$ we first use Lemma 3.2.22 to reduce to the case of elliptic curves, i.e., families of genus 1 with a section and relative jacobian. Then by a translation we may assume that the isomorphism between the two twists respects the relative divisor $e(S)$ which is the image of the identity-section of the group law. Now $\mathcal{O}_X\big(e(S)\big)$ is relatively ample over S and intrinsic. Hence we conclude again by Proposition B.2.1. □

Corollary B.2.4. *Let X be a smooth hyperbolic curve over a field K of characteristic p. Then the following are equivalent.*

(a) There are $m \neq m'$ such that $X(m)$ and $X(m')$ are isomorphic over K.

(b) X is isotrivial over K.

Proof: Let $X \subset \widetilde{X}$ be the smooth completion (over K). Let $\widetilde{Y} \to \widetilde{X}$ be the maximal ℓ-primary metabelian cover that is at most ramified over $\widetilde{X} \setminus X$ for some ℓ different from p. This cover is intrinsic and thus also $\widetilde{Y}(m) \cong \widetilde{Y}(m')$ as ramified covers over $\widetilde{X}(m) \cong \widetilde{X}(m')$.

By Corollary B.2.3 $\widetilde{Y}, \widetilde{X}$ are isotrivial — even in a compatible way. Let $\widetilde{Y}_0 \to \widetilde{X}_0$ be a model over a finite field for $\widetilde{Y} \to \widetilde{X}$. Then the complement of the branch locus is a model for X. Indeed, the branch locus equals the image of the support of the sheaf of relative differentials which behaves well under base change and $\widetilde{Y} \to \widetilde{X}$ is ramified along all of $\widetilde{X} \setminus X$ (here metabelian instead of abelian is essential only for genus 1 with one point at infinity). □

B.3 Automorphisms

Let X be a variety over a field K of positive characteristic. An endomorphism of X in the category of varieties localised at Frobenii consists in a pair (f, n) such that $n \in \mathbb{N}$ and $f : X \to X(n)$. It represents the map $F_K^{-n} f$ and thus $(f, n) = (F_K f, n+1)$. An inverse (g, m) satisfies $g(n) \circ f = F_K^{n+m}$ and $f(m) \circ g = F_K^{n+m}$. Therefore f must be a universal homeomorphism for having an inverse. The converse is true by Lemma 4.1.2 for normal varieties.

The generic rank of a dominant map in the Frobenius localised category is a well defined rational number with denominator a power of p. Hence we obtain a map $d : \mathrm{Aut}_{K,F_K^{-1}}(X) \to p^{\mathbb{Z}} \cong \mathbb{Z}$. For curves this yields an exact sequence of automorphism groups

$$1 \to \varinjlim_n \mathrm{Aut}_K\big(X(n)\big) \to \mathrm{Aut}_{K,F_K^{-1}}(X) \xrightarrow{d} \mathbb{Z} \ .$$

such that $d(f)$ for an automorphism f which by Dedekind-Weber equivalence essentially is an isomorphism $X(m) \to X(m')$ is defined as the difference $m' - m$. Again for arbitrary varieties, let us define $\Delta(X)$ as the positive generator of the image of d or as 0, when the image is trivial. Then $\Delta\big(X(1)\big) = \Delta(X)$ and $\Delta(X_{\overline{K}})$ vanishes if and only if $\Delta(X_L)$ vanishes for all finite separable L/K. Obviously $\Delta(X)$ is a first numerical invariant that measures the deviation of the automorphism group of X/K from the automorphism group after inverting Frobenius: for curves a vanishing $\Delta(X)$ is equivalent to $\varinjlim_n \mathrm{Aut}_K\big(X(n)\big) \cong \mathrm{Aut}_{K,F_K^{-1}}\big(X\big)$.

Proposition B.3.1. *Let X/K be a variety. We consider the following properties:*

(a) X is isotrivial over K.

(a') There is $n \in \mathbb{Z}$ such that $X(n)$ is isotrivial over K.

(b) There are $m \neq m'$ such that $X(m)$ and $X(m')$ are isomorphic over K.

(c) $\Delta(X) \neq 0$.

Then the implications $(a) \iff (a') \implies (b) \implies (c)$ hold.

Moreover, if X/K is smooth, projective with (anti-)ample canonical bundle, then (a') is equivalent to (b). If X/S is even a smooth, projective curve or a smooth, hyperbolic curve, then all four properties are equivalent.

Proof: We only remark that Frobenius twist is preserved under base change and that (b) holds for varieties over $\mathbb{F}_q$ with $q = p^m$ and $m' = 0$. For the last statement we involve the Corollaries B.2.2, B.2.3 and B.2.4. □

Proposition B.3.2. *Let X/K be a smooth, hyperbolic curve with vanishing $\Delta(X)$. Then $\mathrm{Aut}_{K,F_K^{-1}}(X)$ equals $\mathrm{Aut}_K(X)$.*

Proof: $\varinjlim_n \mathrm{Aut}_K\big(X(n)\big) = \underline{\mathrm{Aut}}(X/K)\big(K^{\mathrm{i}}\big)$ where K^{i} is an inseparable closure of K. By an easy modification of 1.11 [DM69] smooth hyperbolic curves have an automorphism scheme that is unramified over K. This implies that all K^{i}-points are already defined over K. □

Now we turn to the discussion of F-cohomological rigidity. First we recall the definition.

Definition B.3.3. *Let $\overline{X}$ be a connected variety over an algebraically closed field $\overline{K}$. We call it* **cohomologically rigid** *if the natural map*

$$\mathrm{Aut}_{\overline{K}}(X) \to \mathrm{Aut}^{\mathrm{opp}}\big(\mathrm{H}^*(\overline{X}, \mathbb{Z}/n)\big)$$

is injective for all $n \gg 0$ prime to the characteristic. We call it F**-cohomologically rigid** *if the natural map*

$$\mathrm{Aut}_{\overline{K}, F_{\overline{K}}^{-1}}(X) \to \mathrm{Aut}^{\mathrm{opp}}\big(\mathrm{H}^*(\overline{X}, \mathbb{Z}/n)\big)$$

is injective for all $n \gg 0$ prime to the characteristic. If n is big enough such that injectivity holds we call $\overline{X}$ **(F-)cohomologically rigid with coefficients in** $\mathbb{Z}/n$.

A geometrically connected variety X/K is called (F-)cohomologically rigid (with coefficients in $\mathbb{Z}/n$ if $X_{\overline{K}}$ is (F-)cohomologically rigid (with coefficients in $\mathbb{Z}/n$).

Proposition B.3.4. *Let $\overline{X}$ be a smooth hyperbolic curve over an algebraically closed field $\overline{K}$. Then the following are equivalent.*

(a) $\overline{X}(n)$ is cohomologically rigid for all $n \in \mathbb{Z}$, and $\Delta(\overline{X}) = 0$.

(b) $\overline{X}$ is F-cohomologically rigid.

Moreover, if (a) and (b) hold then $\mathrm{Aut}_{\overline{K}, F_{\overline{K}}^{-1}}\big(X\big)$ is a finite group and equals $\mathrm{Aut}_{\overline{K}}\big(X(n)\big)$ for $n \gg 0$.

Proof: This is obvious. □

Corollary B.3.5. *Let $\overline{X}$ be a smooth projective hyperbolic curve over an algebraically closed field $\overline{K}$. Then $\overline{X}$ is F-cohomologically rigid if and only if $\overline{X}$ is not isotrivial.*

Proof: By Proposition B.3.4 and Proposition B.3.1 we need to show that smooth projective curves of genus ≥ 2 are cohomologically rigid. This was proven for Proposition 7.3.3 and holds essentially due to [Se60]. □

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Zusammenfassung

Anabelsche Varietäten

Die Fundamentalgruppe einer zusammenhängenden algebraischen Varietät X/K klassifiziert endliche étale Überlagerungen von X. Sie ist eine pro-endliche Gruppe $\pi_1(X, \overline{x})$, welche von der Wahl eines Basispunktes $\overline{x} \in X$ bis auf einen inneren Automorphismus abhängt. Wir bezeichnen mit $\overline{K}$ einen fixierten algebraischen Abschluß des Körpers K, es sei $G_K = \mathrm{Aut}(\overline{K}/K)$ seine absolute Galoisgruppe, und von X nehmen wir an, daß $X_{\overline{K}} = X \times_K \overline{K}$ immer noch zusammenhängend ist. Dann gibt uns IX 6.1 [SGA 1] eine natürliche kurze exakte Sequenz

$$1 \to \pi_1(X_{\overline{K}}, \overline{x}) \to \pi_1(X, \overline{x}) \to G_K \to 1 \ , \qquad (*)$$

aus der wir im wesentlichen durch Konjugation die folgende äußere Galois Darstellung

$$\rho_X : G_K \to \mathrm{Out}\big(\pi_1(X_{\overline{K}})\big)$$

konstruieren. Dabei bezeichnet Out die Gruppe der Automorphismen modulo inneren Automorphismen. Für die Darstellung ρ_X wird die Wahl des Basispunktes irrelevant.

Anabelsche Geometrie behandelt das Problem, geometrische Informationen über X/K aus gruppentheoretischen Eigenschaften von $\pi_1(X, \overline{x})$ oder der zugehörigen Darstellung ρ_X zu dechiffrieren. Genauer gesagt bezeichnet man eine K-Varietät als anabelsch, wenn sie zu einer Kategorie Anab_K von Varietäten über K gehört, so daß die Einschränkung des Fundamentalgruppenfunktors π_1 auf Anab_K eine Einbettung in eine Kategorie von ausschließlich gruppentheoretischer Natur liefert. Die Bezeichnung *anabelsch* spiegelt die landläufige Meinung wider, daß anabelsche Varietäten eine höchst nichtabelsche Fundamentalgruppe besitzen.

Der Begriff anabelscher Varietäten datiert zurück auf einen Brief Grothendiecks an Faltings aus dem Jahr 1983, siehe [Gr83]. Grothendieck beschreibt dort eine Reihe von Vermutungen aus dem Gebiet, welches wir heute *Anabelsche Geometrie* nennen. Sein "Yoga der anabelschen Geometrie" bezeichnet geeignete Varietäten — insbesondere hyperbolische Kurven — über absolut endlich erzeugten Körpern als anabelsch.

Man unterscheidet drei Arten von Grothendieck Vermutungen der anabelschen Geometrie: die Hom-Form, die Isom-Form und die Vermutung über Spaltungen. Die Hom-Form behandelt Kategorien von Varietäten zusammen mit dominanten Abbildungen. Die Isom-Form hingegen befaßt sich ausschließlich mit Isomorphismen. Daher entspricht die Isom-Form im wesentlichen der Rekonstruktion des Isomorphietyps der anabelschen Varietät zusammen mit einer Diskussion des Fundamentalgruppenfunktors auf den Automorphismen. Die Vermutung über Spaltungen ist ein wenig anders gelagert, da hier die

K-rationalen Punkte einer geometrisch zusammenhängenden Varietät mit den Spaltungen der Sequenz $(*)$ in Bijektion gebracht werden sollen. Die vorliegende Arbeit konzentriert sich auf den Fall anabelscher Geometrie in der Isom-Form, und zwar in positiver Charakteristik.

Das Hauptresultat

In positiver Charakteristik kann die naive Isom-Form der Grothendieck Vermutung nicht gelten. Aufgrund der topologischen Invarianz der Fundamentalgruppe induziert der geometrische Frobenius Isomorphismen zwischen Varietäten, welche als solche aber nicht isomorph sind — etwa zwischen elliptische Kurven mit $\jmath$-Invariante nicht aus $\mathbb{F}_p$ und derjenigen elliptischen Kurve zu $\jmath^p$. Als natürlicher Ausweg bietet sich an, das Problem "wegzudefinieren": wir invertieren formal den geometrischen Frobenius F_K zwischen K-Varietäten.

Als Hauptresultat der Arbeit ergibt sich das folgende Theorem:

Theorem 5.1.1. *Es sei K ein endlich erzeugter Körper der Charakteristik $p \geq 0$ mit algebraischem Abschluß $\overline{K}$. Es seien X, X' zwei glatte, hyperbolische, geometrisch zusammenhängende Kurven über K, von denen mindestens eine nicht isotrivial ist. Dann induziert der Funktor π_1 eine natürliche Bijektion*

$$\pi_1 : \mathrm{Isom}_{K,F_K^{-1}}\big(X, X'\big) \xrightarrow{\sim} \mathrm{Isom}_{G_K}\big(\pi_1(X_{\overline{K}}), \pi_1(X'_{\overline{K}})\big)$$

endlicher Mengen.

Hier bedeutet $\mathrm{Isom}_{K,F_K^{-1}}\big(\cdot,\cdot\big)$ die Menge der Isomorphismen in der Kategorie der K-Varietäten, wobei der geometrische Frobenius F_K formal invertiert worden ist.

Natürlich darf nicht unerwähnt bleiben, daß von diesem Resultat bereits Tamagawa den Fall affiner hyperbolischer Kurven in Charakteristik 0 bewiesen hat, siehe [Ta97]. Mochizuki gelang es, die Einschränkung auf affine Kurven zu beseitigen, siehe [Mz99]. Des weiteren hat der Autor in seiner Diplomarbeit bereits das Resultat Tamagawas auf den Fall eines endlich erzeugten Grundkörpers positiver Charakteristik ausgedehnt, siehe [Sx02]. Es bleibt also nur, den im Titel der Arbeit festgehaltenen Fall von projektiven (hyperbolischen) Kurven in positiver Charakteristik zu behandeln. Wir beschreiben im Folgenden, wie dies geschehen soll.

Strategie und Methode

Zunächst gelingt eine formale Reduktion des Beweises von Theorem 5.1.1 darauf, eine Retraktion für die von π_1 auf den Mengen von Isomorphismen induzierte Abbildung konstruieren zu müssen. Die dabei relevanten gruppentheoretischen Eigenschaften — insbesondere das Verschwinden des Zentrums der geometrischen Fundamentalgruppe — können aus plausiblen geometrischen Annahmen gefolgert werden, welche für anabelsche Varietäten gelten sollen. So untersuchen wir zum Beispiel algebraische $K(\pi,1)$-Räume. Für diese entspricht per Definition die Gruppenkohomologie der Fundamentalgruppe zu Torsionskoeffizienten der étalen Kohomologie mit Werten im zugehörigen lokal konstanten System. Damit verlassen wir den Fall von ausschließlich Kurven und verfolgen die Leitidee, möglichst viele der geometrische Strukturen und Eigenschaften auf der gruppentheoretischen Seite wieder zu finden. Dazu gehört Galois Abstieg für Isomorphismen bezüglich Konstantenkörpererweiterung, die Behandlung von Torsoren und die Eigenschaft dominanter Abbildungen, Epimorphismen zu sein.

Um nun zu einem gruppentheoretischen Isomorphismus einen zugehörigen geometrischen zu konstruieren, dehnt man die Kurven zu einer Familie über einer Basis S aus, welche aus eine Varietät über einem endlichen Körper besteht. Ein Isomorphismus ist gefunden, wenn die zugehörigen darstellenden Abbildungen $\xi_i : S \to \mathscr{M}$ in den Modulraum der Kurven übereinstimmen, vorausgesetzt man arbeitet mit einem feinen Modulraum. Mittels Spezialisierung und Tamagawas Theorem der Grothendieck Vermutung der anabelschen Geometrie für hyperbolische Kurven über einem endlichen Körper beschreibt man das Bild von abgeschlossenen Punkten s und kontrolliert so ξ. Jedoch hat man mit diesem naiven Ansatz die folgenden drei Probleme:

(1) Der Modulraum der Kurven ist ein Stack, wir benötigen hingegen ein Schema.

(2) Das Resultat für Kurven über endlichen Körpern ist von absoluter Natur. Daher liefert es den Isomorphietyp der Kurve nur bis auf einen Frobeniustwist. Dieser hängt a priori vom Punkt $s \in S$ ab.

(3) Das Resultat für Kurven über endlichen Körpern ist nur für affine, hyperbolische Kurven bekannt; unsere Kurven sind eigentlich.

Diese Probleme lösen wir wie folgt. (1) Der Gebrauch von Level-Strukturen verschafft uns ein feines Modulschema von glatten projektiven Kurven mit Level-Struktur. Es ist aber ein leichtes, die Level-Struktur mittels der Fundamentalgruppe zu kontrollieren.

(2) Ein Frobeniustwist komponiert die darstellende Abbildung mit einer Potenz des Frobenius, ändert also die topologische Komponente nicht. Wir kennen also immerhin die topologische Komponente der darstellenden Abbildung. In Kapitel 1 (wie schon in [Sx02]) wird bewiesen, daß zwei topologisch identische Abbildungen von einer irreduziblen und reduzierten Varietät über einem endlichen Körper in eine beliebige Varietät sich nur um eine globale Potenz des Frobenius unterscheiden.

(3) Nur affine, hyperbolische Kurven über endlichen Körpern verwenden zu dürfen stellt das unangenehmste Problem dar. Wir müssen degenerieren. Dazu arbeiten wir mit G-Galois Überlagerungen, welche am gewählten Punkt stabil degenerieren und arbeiten G-äquivariant. Ein Van Kampen Theorem erlaubt die Rekonstruktion der zahmen Fundamentalgruppe des glatten Teils von irreduziblen Komponenten der speziellen Faser. Auf diese können wir nun wie oben beschrieben Tamagawas Resultat anwenden. Hierbei ist es nötig, auch die arithmetische Aktion auf der zahmen Fundamentalgruppe zu erkennen. Ohne diese arithmetische Aktion hätte an dieser Stelle auch die kummersche Fundamentalgruppe zusammen mit dem Hauptresultat aus [Saï97] ausgereicht.

Die in dieser Arbeit gewählte Methode verwendet logarithmische Strukturen auf stabilen Kurven. Um das Van Kampen Theorem anwenden zu können, welches in Kapitel 2 in abstrakter Form ausführlich behandelt wird, sind Ergebnisse aus der Abstiegstheorie für logarithmisch étale Überlagerungen nötig. Aus dieser Notwendigkeit heraus entwikkelte sich das zweite Thema dieser Arbeit. In Kapitel 3 wird zunächst alles Nötige aus der Theorie der logarithmischen Schemata wiederholt, um dann für eine ausführliche Behandlung von Abstiegstheorie für logarithmisch étale Überlagerungen nach Art von [SGA 1] zur Verfügung zu stehen. Anschließend wird zum Zweck der Charakterisierung der logarithmischen Spezialisierungsabbildung die Frage nach logarithmisch glatter Reduktion von Kurven und ihren Überlagerungen behandelt. Als interessantes Teilergebnis sei erwähnt, daß eine glatte, projektive, hyperbolische Kurve genau dann logarithmisch gute Reduktion über einem exzellenten, henselschen, diskreten Bewertungsring (ausgestattet mit der Standard log Struktur) mit perfektem Restklassenkörper hat, wenn die wilde Trägheit trivial auf der pro-ℓ Komplettierung der Fundamentalgruppe operiert.