Affine Anabelian Curves in Positive Characteristic

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August 13, 2002

Abstract

An investigation of morphisms that coincide topologically is used to generalize to all characteristics and partly reprove Tamagawa's theorem on the Grothendieck conjecture in anabelian geometry for affine hyperbolic curves. The theorem now deals with π_1^{tame} of curves over a finitely generated field and its effect on the sets of isomorphisms. Universal homeomorphisms are formally inverted.

1 Introduction

Anabelian geometry deals with anabelian categories of schemes for which the étale fundamental group functor encodes many, if not all, algebraic properties. In other words, the functor π_1 should be an equivalence of a geometric category with a group theoretic category.

In 1983 Grothendieck announced a list of anabelian conjectures within a letter to Faltings [Gr1]. His 'Yoga der anabelschen Geometrie' declares suitable varieties — including hyperbolic curves — over fields of absolutely finite type to be anabelian. For a survey and recent results see [Fa] and [Mz].

This paper describes a category FC_k of curves over a field k and a category of pro-finite exterior Galois representations $\mathscr{G}(G_k)$ which is the natural target for the tame fundamental group functor π_1^{t} applied to schemes with k structure. The following theorem of an anabelian nature is proved in Section 5. It generalizes Tamagawa's result from [Ta] which treats finite fields k and those of characteristic 0.

Theorem 3.2. *Let k be a field of absolutely finite type. For affine hyperbolic curves C and C' over k the map*

$$\pi_1^{\mathrm{t}} : \mathrm{Isom}_{\mathrm{FC}_k}(C, C') \to \mathrm{Isom}_{G_k}\left(\overline{\pi}_1^{\mathrm{t}}(C), \overline{\pi}_1^{\mathrm{t}}(C')\right)$$

is bijective unless both curves are isotrivial. In that case the map is dense injective.

The method of proof relies on the idea that if a *fine* moduli space $\mathscr{M}$ is available, then isomorphisms of curves correspond to coincidence of representing maps ξ.

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By specialization and the finite field case π_1^{t} controls the topological component of ξ for an extension of the curve over a base S of finite type over $\mathbb{Z}$ with function field k. Indeed, the finite field case not being rigid, the ξ's of two curves with isomorphic π_1^{t} differ by a 'Frobenius twist'. A priori, this twist may vary among the various closed points of S. This motivates the search for the following rigidifying result:

Proposition 2.3. *Let $S, \mathscr{M}$ be of finite type over $\mathbb{F}_p$, S be irreducible and reduced, and $\xi_1, \xi_2 : S \to \mathscr{M}$ be maps such that $\xi_1^{\mathrm{top}} = \xi_2^{\mathrm{top}}$. Then ξ_1 and ξ_2 differ only by a power of the Frobenius map.*

The case of isotrivial curves needs to be dealt with separately. They behave differently as their π_1^{t} possesses an automorphism of infinite order: a suitable geometric Frobenius: Hence $\hat{\mathbb{Z}} \subset \mathrm{Aut}_{G_k}(\overline{\pi}_1^{\mathrm{t}})$.

2 Topological Coincidence of Maps

For any scheme X and any prime number p, consider $X(\mathbb{F}_p)$ as a subset of the topological space underlying X.

Lemma 2.1. *Let X be irreducible, of finite type over* $\mathrm{Spec}(\mathbb{Z})$, *such that $X_{\mathbb{Q}}$ is nonempty. Then $\bigcup_p X(\mathbb{F}_p)$ is Zariski-dense in X.*

Proof: The generic point of X is in the generic fiber $X_{\mathbb{Q}}$ which is the closure of its closed points. Any of these points has a number field as its residue field and defines a closed subscheme Z of dimension 1. By the Čebotarev theorem, Z is the closure of $\bigcup_p Z(\mathbb{F}_p) \subseteq \bigcup_p X(\mathbb{F}_p)$. □

Lemma 2.2. *Let X/k be irreducible and of finite type. Then any two closed points of X lie on an irreducible curve on X.*

Proof: This is a standard application of Bertini. □

Call the topological component of a map f of schemes f^{top}. In positive characteristic, there is the Frobenius map F which is raising to p^{th} power and has $F^{\mathrm{top}} = id$.

Proposition 2.3. *Let X, Y be of finite type over* $\mathrm{Spec}(\mathbb{Z})$, *$X$ be irreducible and reduced, and $f, g : X \to Y$ be maps such that $f^{\mathrm{top}} = g^{\mathrm{top}}$.*

Then $f = g$ or $X/\mathbb{F}_p$. If $X/\mathbb{F}_p$, then f and g differ only by a power of the Frobenius map. Uniqueness of the exponent is equivalent to $f^{\mathrm{top}} = g^{\mathrm{top}}$ not being constant.

Proof: The assertion on uniqueness is clear as $f = f \circ F^m$ implies that the residue field at the image of the generic point of X is fixed by F^m. From now on, we assume that f^{top} is nonconstant.

In view of uniqueness, we may assume X, Y affine. Now the locus of coincidence $\{f \equiv g\}$ is closed in X and contains $\bigcup_p X(\mathbb{F}_p)$. If $X_{\mathbb{Q}} \neq \emptyset$ we are done by Lemma 2.1 .

From now on, assume $X/\mathbb{F}_p$. For $m \in \mathbb{Z}$ define $X_m = \{f \circ F^m \equiv g\}$ or $\{f \equiv g \circ F^{-m}\}$ depending on the sign of m. Topological coincidence implies that

$$X(\mathbb{F}_q) \subseteq \bigcup_{m \in \mathbb{Z}} X_m(\mathbb{F}_q)$$

since $\mathbb{F}_q$-points are topologically identical if and only if they are $G(\mathbb{F}_q/\mathbb{F}_p)$ conjugate, and this Galois group is generated by Frobenius. If $q = p^r$, then it is sufficient to allow m to vary over representatives of $\mathbb{Z}/r\mathbb{Z}$. To keep things small, we choose representatives with minimal absolute value and thereby conserve symmetry:

$$X(\mathbb{F}_{p^r}) \subseteq \bigcup_{-r/2<m\leq r/2} X_m(\mathbb{F}_{p^r}) \ . \tag{2.1}$$

The case of $\dim X = 1$. First consider the following lemma:

Lemma 2.4. *If $\dim X = 1$, there is a constant c such that for all m with $X \neq X_m$ the bound $\#X_m(\mathbb{F}_q) \leq cp^{|m|}$ holds.*

Proof: Choose closed immersions $X \subseteq \mathbb{A}^d, Y \subseteq \mathbb{A}^n$ and consider the graph Γ of (f, g)

$$X \xleftarrow[\sim]{\mathrm{pr}_1} \Gamma \subseteq X \times Y \times Y \subseteq \mathbb{A}^{d+2n} \subseteq \mathbb{P}^{d+2n} \ .$$

Let $(x_1, \dots, x_n, y_1, \dots, y_n)$ be coordinates of the factor $\mathbb{A}^{2n}$. For $m \in \mathbb{Z}$ let R_m be the vanishing locus in $\mathbb{P}^{d+2n}$ of those n sections of $\mathcal{O}(p^{|m|})$ described by $x_i^{p^m} - y_i$ or $y_i^{p^{-m}} - x_i$ (depending on the sign of m) on the affine part $\mathbb{A}^{d+2n}$. This R_m is the closure of the product of $\mathbb{A}^d$ with the graph of F^m, and $\mathrm{pr}_1(\Gamma \cap R_m) = X_m$.

If $X_m \neq X$, then a hypersurface H_m of degree $p^{|m|}$ defined by a single suitably chosen such section suffices to cut down Γ to dimension 0. An easy intersection theoretic estimate in $\mathbb{P}^{d+2n}$ with the closure $\overline{\Gamma}$ of Γ gives

$$\#X_m(\mathbb{F}_q) \leq \deg(\overline{\Gamma} \cap H_m) = \deg(\overline{\Gamma})\deg(H_m) = \deg(\overline{\Gamma})p^{|m|} \ . \qquad \square$$

If $X_m = X$ for no $m \in \mathbb{Z}$, then by Lemma 2.4 and (2.1)

$$\#X(\mathbb{F}_{p^r}) \leq \sum_{-r/2<m\leq r/2} \#X_m(\mathbb{F}_{p^r}) \leq crp^{r/2} \ ,$$

for some constant c. This contradicts the 'Weil conjectures' (vary $\mathbb{F}_{p^r}$ within the finite fields containing the field of constants of the smooth part of X, then $\#X(\mathbb{F}_{p^r})$ has order of magnitude p^r). In fact, we need only the case of algebraic curves, which is known by a theorem of Hasse–Weil.

The case of $\dim X > 1$. Take $C_1, C_2 \subseteq X$ irreducible, horizontal curves, i.e., such that $f|_{C_i}$ is not constant. By the one-dimensional case there are unique $m_i \in \mathbb{Z}$ such that $C_i \subseteq X_{m_i}$. For points $x_i \in C_i$, choose an irreducible curve $C \subset X$ passing through x_1, x_2. If $f(x_1) \neq f(x_2)$, then $C \subseteq X_m$ for a unique $m \in \mathbb{Z}$.

If a closed point y lies in $X_r \cap X_{r'}$ then $\deg\big(f(y)\big) \mid r - r'$. Consequently,

$$\deg\big(f(x_i)\big) \mid m_i - m \ .$$

Thus

$$\gcd\Big(\deg(f(x_1)), \deg(f(x_2))\Big) \mid m_1 - m_2 \ .$$

Varying x_1, x_2 shows that $m_1 - m_2$ has arbitrary large divisors, hence $m_1 = m_2$. Finally, X is the closure of the union of such horizontal curves and so $X = X_m$ for some m, as desired. $\square$

3 Preliminaries on π_1^{t}, Statement of Theorems

Definition 3.1. *Let S be a scheme. A* **smooth curve** *over S is a proper, smooth map $p : X \to S$ of finite presentation with geometrically irreducible fibers of dimension 1 together with a relative effective étale divisor D.*

Denote by C the complement $X-D$ with its induced scheme structure over S. Consider C itself or (X,D) as a short notation for the smooth curve. The geometric genus g of the fiber is a locally constant function on S. The smooth curve is called affine, if and only if $\deg(D) > 0$. It is called hyperbolic, if and only if its Euler characteristic $\chi_C = 2-2g-\deg(D)$ is negative.

Tame fundamental group. Let X be a normal scheme and $D \subset X$ a divisor with normal crossing. The pair (X,D) is associated a tame fundamental group $\pi_1^{\mathrm{t}}(X,D)$ such that the Galois category $\mathrm{Rev}_{X,D}^{tame}$ of normal covers which are at most tamely ramified along D is equivalent to the category of finite continuous $\pi_1^{\mathrm{t}}(X,D)$-sets, cf. [SGA 1]. Neglecting base points results in a functor with values in $\mathscr{G}$, the category of pro-finite groups with exterior continuous morphisms, i.e., equivalence classes of maps up to composition with inner automorphisms.

Exterior Galois representation. Let k always denote a field, $\bar{k}$ its algebraic closure. Basechange $X \times_k \bar{k}$ is abbreviated by $X_{\bar{k}}$, and $G_k = \mathrm{Aut}(\bar{k}/k)$ is the absolute Galois group of k. For a smooth curve $C = (X,D)$ over k the group $\overline{\pi}_1^{\mathrm{t}}(C) := \pi_1^{\mathrm{t}}(X_{\bar{k}}, D_{\bar{k}})$ carries a right action of G_k in $\mathscr{G}$ by functorial transport of the right G_k-action on the scheme. Using inverses we transform it to a left action which we denote $\rho_C^{\mathrm{t}} : G_k \to \mathrm{Aut}_{\mathscr{G}}(\overline{\pi}_1^{\mathrm{t}}(C))$. Clearly we obtain a functor π_1^{t} from smooth k-curves with values in $\mathscr{G}(G_k)$, the category of G_k-representations in $\mathscr{G}$, i.e., pairs (V,ρ) where $V \in \mathscr{G}$ and $\rho : G_k \to \mathrm{Aut}_{\mathscr{G}}(V) =: \mathrm{Out}(V)$.

$\mathbf{G}_k$ extensions. It is known [SGA 1] that the tame fundamental group of a smooth curve forms naturally an extension

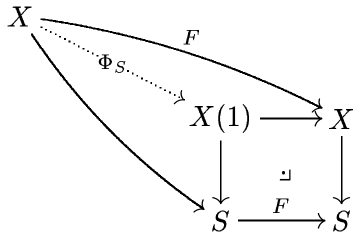
$$1 \to \overline{\pi}_1^{\mathrm{t}}(C) \to \pi_1^{\mathrm{t}}(C) \to G_k \to 1 \ .$$

Maps of k-curves produce morphisms between these short exact sequences with identity on the G_k-part. The base points being neglected only $\overline{\pi}_1^{\mathrm{t}}(C)$-conjugacy classes of maps are well defined. We obtain a functor with values in $\mathrm{EXT}[G_k]$, which denotes this category of G_k-extensions and classes of maps. Acting on the kernel by conjugation passes to G_k if considered as an action in $\mathscr{G}$, thus a functorial construction $R : \mathrm{EXT}[G_k] \to \mathscr{G}(G_k)$. We recover ρ_C^{t} from the above extension. Moreover, R is an equivalence when restricted to isomorphisms, extensions with center free kernel, and G_k-representations on center free groups. The extension is recovered by pullback: $G_k \times_{\mathrm{Out}(V)} \mathrm{Aut}(V)$.

Fact. If the curve is hyperbolic then $\overline{\pi}_1^{\mathrm{t}}(C)$ is center free, cf. [Fa]. The same is valid for all open subgroups as these are $\overline{\pi}_1^{\mathrm{t}}$ of covers which are hyperbolic themselves.

Topological invariance. It is known [SGA 4, VIII 1.1] that π_1 applied to universal homeomorphisms yields isomorphisms. An easy descent argument for tameness ensures the same behaviour for π_1^{t} of curves. The tame fundamental group is therefore not affected by pure inseparable covers. The natural conclusion suggests to formally invert the class of universal homeomorphisms, a task already foreseen by Grothendieck in his 'esquisse', cf. [Gr2, footnote 3].

Frobenius [SGA 5, XV §1]. Fix a prime number p. The Frobenius map F commutes with all maps between schemes of characteristic p, that is in $\mathrm{Sch}_{\mathbb{F}_p}$. If $S \in \mathrm{Sch}_{\mathbb{F}_p}$ then the diagram



defines a functor 'Frobenius twist' $\cdot(1) : Sch_S \to Sch_S$ and a natural transformation $\Phi_S : id_{Sch_S} \to \cdot(1)$, the 'geometric S-Frobenius'. They behave well under base change: $\big(X \times_S T\big)(1) = X(1) \times_S T$ and $\Phi_T = \Phi_S \times_S T$. The m^{th} iterated twist will be denoted by $X(m)$.

In general, X and its twist $X(1)$ are not isomorphic, e.g., the twist of $\mathbb{P}^1_k - \{0, 1, \lambda, \infty\}$ is still genus 0 but punctured in $0, 1, \lambda^p, \infty$.

$\mathrm{FC}_{\mathbf{k}}$. Let k be a field of positive characteristic. Consider the category of smooth k-curves and dominant maps. Its localization at universal homeomorphisms is easily seen to be equivalent to its localization at geometric k-Frobenius maps between curves. By Dedekind–Weber equivalence, this localization can be constructed by considering the perfection, i.e., the pure inseparable closure, of the function field together with the unique prolongation of the set of infinite places and maps respecting these places. Denote the resulting category FC_k.

By topological invariance the tame fundamental group functor factorizes as

$$\pi_1^{\mathrm{t}} : \mathrm{FC}_k \to \mathscr{G}(G_k), \quad C \mapsto \rho_C^{\mathrm{t}} \ .$$

Results. A k-curve C is called isotrivial if $C_{\bar{k}}$ is defined over a finite field. A field k is said to be absolutely of finite type if it is finitely generated over its prime field. The following theorems generalize the main result of A. Tamagawa from [Ta]. His result treats curves over base fields which are either of characteristic 0 or finite. Proofs will be given in the last section.

Theorem 3.2. *Let k be absolutely of finite type, C and C' be affine hyperbolic curves over k, such that at least one of them is not isotrivial. Then*

$$\pi_1^{\mathrm{t}} : \mathrm{Isom}_{\mathrm{FC}_k}(C, C') \xrightarrow{\sim} \mathrm{Isom}_{G_k}\big(\overline{\pi}_1^{\mathrm{t}}(C), \overline{\pi}_1^{\mathrm{t}}(C')\big)$$

is a bijection.

Theorem 3.3. *Let k, C, C' be as above but C, C' both isotrivial. Then the map*

$$\pi_1^{\mathrm{t}} : \mathrm{Isom}_{\mathrm{FC}_k}(C, C') \hookrightarrow \mathrm{Isom}_{G_k}\big(\overline{\pi}_1^{\mathrm{t}}(C), \overline{\pi}_1^{\mathrm{t}}(C')\big)$$

is injective with dense image.

In particular, affine hyperbolic curves C, C' have isomorphic tame fundamental groups (with k-structure) if and only if there are $m, m' \in \mathbb{N}$ such that $C(m) \cong C'(m')$ (with k-structure).

The statement of Theorem 3.3 will become clear as one finds an action of $\mathbb{Z}$, respectively $\hat{\mathbb{Z}}$, on the Isom-sets compatible with the natural inclusion such that the induced map on the quotients is bijective. This holds essentially due to the following:

Theorem 3.4 (Tamagawa, [Ta, 0.5]). *Let C, C' be hyperbolic curves over finite fields. Then the following map is a natural bijection:*

$$\pi_1^{\mathrm{t}} : \mathrm{Isom}_{\mathrm{Sch}}\left(C, C'\right) \xrightarrow{\sim} \mathrm{Isom}_{\mathscr{G}}\left(\pi_1^{\mathrm{t}}(C), \pi_1^{\mathrm{t}}(C')\right) .$$

Remarks. (1) The condition 'affine' in the theorem could be dropped if a characterization of projective hyperbolic curves over finite fields by their π_1^{t} were available.

(2) The method of proof relies on specialization and the finite field case like Tamagawa's proof does, but gives a unified treatment for arbitrary characteristic thus also reproving the previously known.

4 Prerequisites for the Proof

Lemma 4.1 (sheaf). *Let C, C' be affine hyperbolic k-curves, and $\rho, \rho' \in \mathscr{G}(G_k)$, i.e., $\rho : G_k \to \mathrm{Out}(V), \rho' : G_k \to \mathrm{Out}(V')$. Then*

(1) $\underline{\mathrm{Isom}}_{\mathrm{FC}_k}(C, C') : l/k \mapsto \mathrm{Isom}_{\mathrm{FC}_l}(C, C')$,

(2) $\underline{\mathrm{Isom}}_{G_k}(\rho, \rho') : l/k \mapsto \mathrm{Isom}_{G_l}(V, V')$

are étale sheaves of sets on $\mathrm{Spec}(k)_{\text{ét}}$. *Moreover,*

$$\pi_1^{\mathrm{t}} : \underline{\mathrm{Isom}}_{\mathrm{FC}_k}(C, C') \to \underline{\mathrm{Isom}}_{G_k}\left(\rho_C^{\mathrm{t}}, \rho_{C'}^{t}\right)$$

is a morphism of étale sheaves that behaves natural with respect to composition.

Proof: (1) Galois descent, having localized does not matter. (2) Obvious in the Galois case. Let l/k' be Galois, then $\underline{\mathrm{Isom}}_{G_k}(\rho, \rho')(k)$ are the invariants of the $G(l/k)$-action on $\underline{\mathrm{Isom}}_{G_k}(\rho, \rho')(l)$ by conjugation.

The last statement is again obvious because π_1^{t} is a functor and both Galois actions have geometric origin by conjugation with isomorphisms as schemes. □

Étale G-torsors. Let G be a finite group, and X a scheme endowed with a geometric point x. Then almost by definition, $\mathrm{Hom}(\pi_1(X, x), G)$ is the set of pointed G-torsors (E, e) on (X, x) up to isomorphism. Shifting the pointing $e \mapsto g.e$ within the fiber corresponds to composition with the inner automorphism $g(\cdot)g^{-1}$. Hence

$$\mathrm{Hom}_{\mathscr{G}}(\pi_1(X), G) = \left\{ \begin{array}{c} \text{isomorphy classes of} \\ G\text{-torsor } E \to X\} \end{array} \right\} .$$

Surjectivity is equivalent to connectedness of the torsor.

Geometrically connected tame G-torsors on a curve C/k are described by $\psi : \pi_1^{\mathrm{t}}(C) \to G$, such that $\bar{\psi} = \psi|_{\overline{\pi}_1^{\mathrm{t}}(C)}$ is surjective. An easy diagram chase shows that $\bar{H} = \ker(\bar{\psi})$ carries a commuting outer action of G and G_k.

Definition 4.2. *Let $V \in \mathscr{G}(G_k)$ be center free. A* **center free, geometrically connected G-torsor on** *V is a center free $W \in \mathscr{G}(G \times G_k)$ together with an isomorphism $G \times_{\mathrm{Out}(W)} \mathrm{Aut}(W) \cong V$ in $\mathscr{G}(G_k)$, where the G_k-action on the left group is by functoriality of the construction with respect to G compatible isomorphisms.*

Lemma 4.3. *Let C/k be a hyperbolic curve. Then we have a bijection of isomorphism classes:*

$$\left\{ \begin{array}{c} \textit{isomorphy classes of} \\ \textit{tame, geom. connected} \\ G\textit{-torsors on } C/k \end{array} \right\} \stackrel{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{isomorphy classes of} \\ \textit{geom. connected, center} \\ \textit{free } G\textit{-torsors on } \rho_C^{\mathrm{t}} \end{array} \right\} .$$

Proof: $E \to C$ is mapped to $\overline{\pi}_1^{\mathrm{t}}(E)$, $W \in \mathscr{G}(G \times G_k)$ is mapped to

$$\mathrm{pr}_1 : (G \times G_k) \times_{\mathrm{Out}(W)} \mathrm{Aut}(W) \to G .$$

These are mutually inverse. □

Lemma 4.4 (descent). *Let C, C' be hyperbolic k-curves, $F' \to C'$ a G-torsor, $E' \to C'$ a geometrically connected tame G-torsor and $\bar{H}'$ the corresponding G-torsor for $\rho_{C'}^{\mathrm{t}}$. Let W' be a center free geometrically connected G-torsor on $V' \in \mathscr{G}(G_k)$ and $V \in \mathscr{G}(G_k)$. Then we have natural bijections* (1), (2) *and a commutative diagram* (3)*:*

$$(1) \qquad \Big(\coprod_{G\text{-torsors } F/C} \mathrm{Hom}_{G,k}(F, F') \Big)_{/\cong} \xrightarrow{\sim} \mathrm{Hom}_k(C, C'),$$

$$(2) \qquad \Big(\coprod_{G\text{-torsors } W/V} \mathrm{Isom}_{G \times G_k}(W, W') \Big)_{/\cong} \xrightarrow{\sim} \mathrm{Isom}_{G_k}(V, V'),$$

$$(3) \qquad \begin{array}{ccc} \Big(\coprod_{E/C} \mathrm{Isom}_{G,\mathrm{FC}_k}(E, E') \Big)_{/\cong} & \xrightarrow{\pi_1^{\mathrm{t}}} & \Big(\coprod_{\bar{H}/\rho_C^{\mathrm{t}}} \mathrm{Isom}_{G \times G_k}(\bar{H}, \bar{H}') \Big)_{/\cong} \\ {=}\downarrow & & \downarrow{=} \\ \mathrm{Isom}_{\mathrm{FC}_k}(C, C') & \xrightarrow{\pi_1^{\mathrm{t}}} & \mathrm{Isom}_{G_k}\big(\overline{\pi}_1^{\mathrm{t}}(C), \overline{\pi}_1^{\mathrm{t}}(C')\big) . \end{array}$$

Here E (resp. $\bar{H}$) ranges over G-torsors on C (resp. ρ_C^{t}), and $/\cong$ means up to equivalence induced by isomorphisms of the 'variable' G-torsors.

Proof: (1) G-torsors are G-quotient maps and allow pullback construction. (2) There is a map, as the 'base' V is recovered canonically by $G \times_{\mathrm{Out}(W)} \mathrm{Aut}(W)$. It is surjective by structure transport and obviously injective. (3) Trivial. □

Specialization. We restate here for the convenience of the reader Tamagawa's theorem of reconstruction of the specialization map which is formulated entirely in group-theoretical terms. Base changes are abbreviated as $X_T = X \times_S T$.

Theorem 4.5 (reconstruction of sp). *Let $K \supset R \twoheadrightarrow k$ be a henselian discrete valuation ring and let $\mathscr{C}/R, \mathscr{C}'/R$ be smooth hyperbolic curves. Then the kernel of the specialization map $\mathrm{sp} : \pi_1^{\mathrm{t}}(\mathscr{C}_K) \twoheadrightarrow \pi_1^{\mathrm{t}}(\mathscr{C}_k)$ consists of the intersection of those open $H \subset \pi_1^{\mathrm{t}}(\mathscr{C}_K)$ which satisfy the following:*

(i) *The image of H in G_K contains the inertia group I of R.*

(ii) *The image of I in $\mathrm{Out}(\bar{H})$ is trivial. Here $\bar{H} = H \cap \overline{\pi}_1^{\mathrm{t}}(\mathscr{C}_K)$.*

Moreover, there is a natural map Sp

$$\mathrm{Isom}_{G_K}\left(\overline{\pi}_1^{\mathrm{t}}(\mathscr{C}_K), \overline{\pi}_1^{\mathrm{t}}(\mathscr{C}'_K)\right) \to \mathrm{Isom}_{G_k}\left(\overline{\pi}_1^{\mathrm{t}}(\mathscr{C}_k), \overline{\pi}_1^{\mathrm{t}}(\mathscr{C}'_k)\right) .$$

Proof: [Ta, 5.7], uses known criteria for good reduction of a proper curve X via its Jacobian, minimal semistable models of (X, D) and the combinatorics of the dual graphs for a $\mathbb{Z}/l\mathbb{Z}$-cover ramified along all of D. □

Level structure. For a pro-finite group P let P^{ab}/n denote its maximal abelian quotient with exponent $n \in \mathbb{N}$. Let X/k be a proper smooth curve over k of genus g and $1/n \in k^*$. Then $\pi_1(X_{\bar{k}})^{\mathrm{ab}}/n$ is a G_k-module étale locally isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{2g}$ with trivial action. A choice of an isomorphism $\phi : (\mathbb{Z}/n\mathbb{Z})^{2g} \cong \pi_1(X_{\bar{k}})^{\mathrm{ab}}/n$ as G_k-modules is the same as equipping the curve with a level n structure. If $g \geq 2, n \geq 3$ there exists a fine moduli scheme $\mathscr{M}_g[n]$ representing such pairs (X, ϕ), cf. [DM, 5.8,5.14].

Lemma 4.6. *Let k be absolutely of finite type and $C = (X, D)$ be a smooth hyperbolic curve over k. Then ρ_C^{t} encodes $\pi_1(X_{\bar{k}})^{\mathrm{ab}}/n$ as a G_k-module, i.e., for C, C' there is a canonical map*

$$\mathrm{Isom}_{G_k}\left(\overline{\pi}_1^{\mathrm{t}}(C), \overline{\pi}_1^{\mathrm{t}}(C')\right) \to \mathrm{Isom}_{G_k}\left(\pi_1(X_{\bar{k}})^{\mathrm{ab}}/n, \pi_1(X'_{\bar{k}})^{\mathrm{ab}}/n\right) .$$

Proof: There is an exact sequence of G_k-modules:

$$0 \to \mathbb{Z}/n(1) \to \mathbb{Z}/n(1)[D(\bar{k})] \to \overline{\pi}_1^{\mathrm{t}}(C)^{\mathrm{ab}}/n \to \pi_1(X_{\bar{k}})^{\mathrm{ab}}/n \to 0$$

Specialization at places of k modifies this sequence by restriction of its Galois action. We use Theorem 4.5 until k is finite. Now Frobenius weights distinguish $\pi_1(X_{\bar{k}})^{\mathrm{ab}}/n$ as a quotient of $\overline{\pi}_1^{\mathrm{t}}(C)^{\mathrm{ab}}/n$. □

Serre rigidity. As in the sheaf-lemma Isom's of the G_k-modules $\overline{\pi}_1^{\mathrm{ab}}/n$ form a sheaf.

Proposition 4.7. *Let $X/k, X'/k$ be nonisotrivial, proper, and smooth curves of genus ≥ 2, and $n \geq 3$ invertible in k. Then the canonical map*

$$\underline{\mathrm{Isom}}_{\mathrm{FC}_k}(X, X') \hookrightarrow \underline{\mathrm{Isom}}_{G_k}\left(\pi_1(X_{\bar{k}})^{\mathrm{ab}}/n, \pi_1(X'_{\bar{k}})^{\mathrm{ab}}/n\right)$$

of étale sheaves of sets induced by π_1 is injective.

Proof: We work étale locally on k, endow X with a level n structure and obtain the characteristic map $\xi_X \in \mathscr{M}_g[n](k)$. If f is a preimage of the identity then the corresponding $X(m) \cong X(m')$ for some $m, m' \in \mathbb{Z}$ is an isomorphism of curves with level n structure, hence $\xi_X \circ F^m = \xi_X \circ F^{m'}$. As X is not isotrivial $m = m'$.

The induced automorphism f^* on the Jacobian has finite order as $X(m)$ is hyperbolic and must be identity as it acts trivial on n-torsion points [Se]. In other words $f(P) - P \sim f(Q) - Q$ for all $P, Q \in X(\bar{k})$.

The Lefschetz number is $\Lambda(f) = 2 - \mathrm{tr}(f^*|H^1_{\text{ét}}) = 2 - 2g < 0$. Thus f has a fixed point. But then $f(P) - P \sim 0$ for all P and $f = \mathrm{id}$ as X is not $\mathbb{P}^1$. □

5 The Proof

We are going to prove now Theorem 3.2.

Proof: Let $C = (X, D), C' = (X', D')$ be affine hyperbolic k-curves and let α be an element of $\mathrm{Isom}_{G_k}\left(\overline{\pi}_1^{\mathrm{t}}(C), \overline{\pi}_1^{\mathrm{t}}(C')\right)$. By Lemma 4.4 we may prove the theorem for suitable tame covers, hence assume genus $g \geq 2$. (Being isotrivial holds or fails simultaneously for the curve and its cover). By Lemma 4.1 we may enlarge k sufficiently such that the curves in question have potentially level n structures for some $n \geq 3$.

Construction of the inverse. Choose a level structure on X and transport it to X' via α and Lemma 4.6. Extend the data to some base S of finite type over $\mathbb{Z}$ with function field k and apply the following:

Proposition 5.1. *Let S be irreducible, reduced, and of finite type over $\mathbb{Z}$ with function field k. Consider affine hyperbolic curves $\mathscr{C} = (\mathscr{X}, \mathscr{D}), \mathscr{C}' = (\mathscr{X}', \mathscr{D}')$ over S of genus $g \geq 2$ equipped with a level n structure and generic fiber $C/k, C'/k$.*

If $\alpha : \overline{\pi}_1^{\mathrm{t}}(C) \cong \overline{\pi}_1^{\mathrm{t}}(C')$ as exterior G_k-modules such that α respects level n structures then the characteristic maps $\xi_{\mathscr{X}}, \xi_{\mathscr{X}'} : S \to \mathscr{M}_g[n]$ representing $\mathscr{X}, \mathscr{X}'$ coincide topologically.

Proof: We do induction on $\dim(S)$. We may assume S normal. For topological coincidence it suffices to control closed points. The Theorem 4.5 of reconstruction of specialization applied to the pullback of $\mathscr{C}, \mathscr{C}'$ over the henselization $\mathrm{Spec}\, \mathcal{O}_{S,s}^h$ for all s of codimension 1 does the induction step.

If $\dim(S) = 0$ the base is a finite field k. We apply Theorem 3.4 and produce an isomorphism of schemes $f : C \cong C'$ with $\pi_1^{\mathrm{t}}(f) = \alpha$ and ensure k compatibility of f by performing a suitable Frobenius-twist on say C. Correspondingly, the characteristic map is composed with a power of Frobenius and does not change topologically. But now f is an isomorphism of k-curves with level structures as $\alpha = \pi_1^{\mathrm{t}}(f)$ respects them, hence the characteristic maps coincide. □

Now we know that $\xi_{\mathscr{X}}$ and $\xi_{\mathscr{X}'}$ coincide topologically. This is the point where the rigidifying effect of the result on topological coincidence of maps develops its strength. From Proposition 2.3, we know that $\xi_{\mathscr{X}}$ and $\xi_{\mathscr{X}'}$ differ only by a unique power of Frobenius, hence there is $m \in \mathbb{Z}$ such that $\mathscr{X}(m) \cong \mathscr{X}'$ or $\mathscr{X} \cong \mathscr{X}'(-m)$ as S-curves with level structure.

This produces an isomorphism $\tilde{\lambda}(\alpha) \in \mathrm{Isom}_{\mathrm{FC}_k}(X, X')$ which respects the effect of α on level n structures and a natural map $\tilde{\lambda}$ in a commutative diagram:

$$\begin{array}{ccc} \mathrm{Isom}_{\mathrm{FC}_k}(C, C') & \xrightarrow{\pi_1^{\mathrm{t}}} & \mathrm{Isom}_{G_k}\left(\overline{\pi}_1^{\mathrm{t}}(C), \overline{\pi}_1^{\mathrm{t}}(C')\right) \\ & \overset{\lambda}{\dashleftarrow} & \\ \bigcap & \overset{\tilde{\lambda}}{\swarrow} & \downarrow \\ \mathrm{Isom}_{\mathrm{FC}_k}(X, X') & \hookrightarrow & \mathrm{Isom}_{G_k}\left(\pi_1(X_{\bar{k}})^{\mathrm{ab}}/n, \pi_1(X'_{\bar{k}})^{\mathrm{ab}}/n\right) \end{array} .$$

Indeed, naturality is a consequence of injectivity in the bottom line, i.e., Serre rigidity from Proposition 4.7 .

To detect that $\tilde{\lambda}$ even factors as λ we apply Lemma 4.4 to a geometrically connected (eventually enlarge k again) tame G-torsor which is ramified along the whole of D. The construction of $\tilde{\lambda}$ is compatible and produces a map of G-torsors therefore respecting the support of ramification.

Obviously λ is a left inverse to π_1^{t}. To see that $\pi_1^{\mathrm{t}} \circ \lambda = \mathrm{id}$ we observe that the family of $\lambda(\alpha|_{\alpha^{-1}(H')})$ where H' varies over the open subgroups of $\pi_1^{\mathrm{t}}(C')$ defines a natural transformation $\alpha \to \lambda(\alpha)^*$, compatible with k structures, of functors

$$\alpha, \lambda(\alpha)^* : \mathrm{Rev}_{C'}^{tame} \to \mathrm{Rev}_{C}^{tame}$$

where $\lambda(\alpha)^*$ is pullback. But natural isomorphic functors are identical on π_1^{t} in $\mathscr{G}(G_k)$. □

Isotrivial curves. Only a sketch of proof for the case of isotrivial curves will be given.

Step 1. If $k = \mathbb{F}_q, q = p^f$ then we need to quotient out the faithfull compatible action on both Isom-sets of $\langle F^f \rangle \subset G_{\mathbb{F}_q}$ to reduce to Theorem 3.4.

Step 2. If $C \cong C_0 \times_{\mathbb{F}_q} k$ such that $\mathbb{F}_q \subset k$ is relatively algebraically closed, then base change $\times_{\mathbb{F}_q} k$ is an isomorphism on rational points of the Isom-scheme, which is finite unramified, and the G_k action on $\pi_1^{\mathrm{t}}(C) \cong \pi_1^{\mathrm{t}}(C_0)$ factors through $G_{\mathbb{F}_q}$. This reduces to step 1.

Step 3. We use Galois descent with some care for the density assertion. Essentially Galois action and Frobenius commute because of the notion of inseparable degree for morphisms in FC_k.

Acknowledgements This article contains a condensed version of the author's 'Diplomarbeit', Bonn (2000), carried out under the guidance of his advisor Prof. Florian Pop. He deserves my sincere gratitude for invaluable support, encouragement, and for introducing me to the anabelian world. Moreover, I thank Prof. A. Tamagawa for useful discussions at the MSRI and all the others who showed interest in my work. Gratefully, I enjoyed the MSRI's hospitality and excellent working conditions during the Galois Program in 1999 allowing substantial work for this article to be done.

I am indebted to the referee for various remarks and comments. Similarly, to Eike Lau and Sabine Lange who read earlier versions of the manuscript, I owe valuable suggestions that helped improve the article.

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