A monodromy criterion for extending curves

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Abstract — A family of proper smooth curves of genus ≥ 2, parametrised by an open dense subset U of a normal variety S, extends to S if the natural map $\pi_1 U \to \pi_1 S$ is an isomorphism. The criterion of this note is actually more precise.

1 The monodromy criterion

(1.1) All schemes are locally noetherian. For a scheme S we denote by $\mathscr{M}_{g,n}(S)$ the groupoid of smooth, proper S-curves with geometrically connected fibres of genus g and n disjoint ordered sections, see [Kn83]. If $2-2g-n$ is negative, $\mathscr{M}_{g,n}$ is a smooth Deligne-Mumford stack over $\mathrm{Spec}(\mathbb{Z})$. The main result of this note is the following theorem.

Theorem 1.2 (monodromy criterion I). *Let $2-2g-n$ be negative. For a dense open subscheme U of a normal, excellent scheme S the following holds.*

(1) The restriction functor $\mathscr{M}_{g,n}(S) \to \mathscr{M}_{g,n}(U)$ is fully faithful.

(2) A U-curve $C \in \mathscr{M}_{g,n}(U)$ extends to an S-curve in $\mathscr{M}_{g,n}(S)$ if and only if for all $N \in \mathbb{N}$ and geometric points $u \in U \otimes \mathbb{Z}[\frac{1}{N}]$ the following commutative diagram exists, where the solid arrows are induced by the natural maps $U \to \mathscr{M}_{g,n}$ and $U \subset S$:

$$\begin{tikzcd}
\pi_1(U \otimes \mathbb{Z}[\frac{1}{N}], u) \arrow[rr] \arrow[dr] & & \pi_1(\mathscr{M}_{g,n} \otimes \mathbb{Z}[\frac{1}{N}], C_u) \\
& \pi_1(S \otimes \mathbb{Z}[\frac{1}{N}], u) \arrow[ur, dotted] &
\end{tikzcd}$$

(1.3) Remarks. (a) The base point C_u of $\mathscr{M}_{g,n}$ is the isomorphy class of the fibre C_u of C over u. The fundamental group of the stack $\mathscr{M}_{g,n}$ is the pro-finite group classifying finite étale covers of $\mathscr{M}_{g,n}$.

(b) We may and will assume that S and consequently U are connected. Furthermore we may and will ignore base points. The map $\pi_1(U \otimes \mathbb{Z}[\frac{1}{N}]) \to \pi_1(S \otimes \mathbb{Z}[\frac{1}{N}])$ is surjective for all N, such that $U \otimes \mathbb{Z}[\frac{1}{N}]$ is not empty.

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(c) We will discuss later that Theorem 1.2 follows from the work of Moret-Bailly and Oda-Tamagawa in case S is regular. Thus the new contribution of the present note consists in the case of a singular, but normal base S.

(1.4) For $C, C' \in \mathscr{M}_{g,n}(S)$ a modification to the pointed case of [DM69] Thm 1.11 shows that the S-scheme $\mathrm{Isom}(C/S, C'/S)$ is finite (unramified) over S, i.e., the stack $\mathscr{M}_{g,n}$ is a separated Deligne–Mumford stack, see [Kn83]. Hence a U-valued point for a dense open U of a normal scheme S extends uniquely to an S-valued point. This proves part (1) of Theorem 1.2.

We may and will replace S by $S \otimes \mathbb{Z}[\frac{1}{N}]$ for various $N \in \mathbb{N}$ and glue afterwards using Theorem 1.2 (1). In particular we may assume that some prime $\ell \in \mathbb{N}$ is invertible in S.

Corollary 1.5. *Let $2 - 2g - n$ be negative and let k be a field. Let U be a dense open subscheme of a normal, connected, excellent scheme S over* $\mathrm{Spec}(k)$.

Then a U-curve $C \in \mathscr{M}_{g,n}(U)$ extends to an S-curve in $\mathscr{M}_{g,n}(S)$ if and only if the following commutative diagram exists, where the solid arrows are induced by the natural maps $U \to \mathscr{M}_{g,n} \otimes k$ and $U \subset S$:

$$\begin{tikzcd} \pi_1 U \arrow[rr] \arrow[dr] & & \pi_1(\mathscr{M}_{g,n} \otimes k) \\ & \pi_1 S \arrow[ur, dotted] & \end{tikzcd}$$

□

Corollary 1.6. *In the context and notation of Corollary 1.5: if $\pi_1 U \to \pi_1 S$ is an isomorphism, then the restriction functor $\mathscr{M}_{g,n}(S) \to \mathscr{M}_{g,n}(U)$ is an equivalence.* □

(1.7) Remark. The motivation behind Theorem 1.2 stems from Grothendieck's idea that $\mathscr{M}_{g,n}$ behaves 'anabelian', meaning that the pro-finite étale fundamental group should control the geometry of $\mathscr{M}_{g,n}$. In the analytic context, the orbifold space $\mathscr{M}_{g,n}(\mathbb{C})$ is a $\mathrm{K}(\pi, 1)$ space for the Teichmüller mapping class group. Hence for varieties S over $\mathbb{C}$ our Theorem 1.2 is trivial up to homotopy by topological methods. But still one needs to prove that some homotopy classes of continuous maps from $S(\mathbb{C})$ to $\mathscr{M}_{g,n}(\mathbb{C})$ contain a unique algebraic map extending a given map on the dense open $U(\mathbb{C})$.

Nevertheless, the difference between the analytic and algebraic situation lies in the pro-finite completion of the fundamental group. The good $\mathrm{K}(\pi, 1)$ properties only prevail in the pro-finite setting if the mapping class group is a good group in the sense of Serre. And this is not known.

2 The outer monodromy representation

(2.1) In what follows, $\mathbb{L}$ is a set of prime numbers invertible on S. Let C/S be an S-curve in $\mathscr{M}_{g,n}(S)$. Let C^0 be the notation for the open complement of its sections, and C^0_s is the fibre of C^0/S over the geometric point $s \in S$. Let c be a geometric point of C^0_s. The group $\pi^{\mathbb{L}}_{s,c} = \pi^{\mathbb{L}}_1(C^0_s, c)$ is the pro-$\mathbb{L}$ completion of $\pi_1(C^0_s, c)$, i.e., the maximal continuous pro-finite quotient that is a pro-$\mathbb{L}$ group (a pro-finite limit of groups of order a product of powers of primes from $\mathbb{L}$). The group $\pi^{\mathbb{L}}_{s,c}$ is noncanonically isomorphic to the pro-$\mathbb{L}$ completion $\pi^{\mathbb{L}}$ of the topological fundamental group of an oriented topological surface of genus g with n cusps.

Lemma 2.2. *(1) The group $\pi^{\mathbb{L}}$ is finitely generated as a pro-finite group. (2) The groups $\mathrm{Aut}(\pi^{\mathbb{L}})$ and $\mathrm{Out}(\pi^{\mathbb{L}})$ are canonically pro-finite groups. (3) If $2-2g-n$ is negative, then the center of $\pi^{\mathbb{L}}$ is trivial.*

Proof: (1) See [SGA 1] XIII 2.12. (2) A finitely generated pro-finite group is a pro-finite limit of characteristic finite quotients. (3) See [An74] Prop 8 and Prop 18. □

(2.3) Let K denote the kernel of the map $\pi_1(C^0,c) \to \pi_1(S,s)$ and N be its smallest normal closed subgroup such that K/N is a pro-$\mathbb{L}$ group. Then N is characteristic in K and thus also a normal subgroup of $\pi_1(C^0,c)$. We set $\pi_1^{(\mathbb{L})}(C^0,c)$ for the quotient $\pi_1(C^0,c)/N$. The sequence

$$\pi^{\mathbb{L}}_{s,c} \to \pi_1^{(\mathbb{L})}(C^0,c) \to \pi_1(S,s) \to 1 \tag{$*$}$$

is exact by [SGA 1] XIII 4.1/4.4. Moreover, if C^0/S admits a section σ, then the sequence $(*)$ is even split exact by [SGA 1] XIII 4.3/4.4, the splitting being induced by the section.

$$\begin{tikzcd} 1 \arrow[r] & \pi^{\mathbb{L}}_{s,\sigma(s)} \arrow[r] & \pi_1^{(\mathbb{L})}(C^0,\sigma(s)) \arrow[r] & \pi_1(S,s) \arrow[l, dotted, bend right, "\sigma"'] \arrow[r] & 1 \end{tikzcd} \tag{$**$}$$

Up to a choice of an isomorphism $\pi^{\mathbb{L}} \cong \pi^{\mathbb{L}}_{s,\sigma(s)}$ the corresponding outer pro-$\mathbb{L}$ representation

$$\rho_{C/S} : \pi_1(S,s) \to \mathrm{Out}\big(\pi^{\mathbb{L}}\big)$$

induced by conjugation is independent of the section and obeys functoriality with respect to maps $f : S' \to S$ and base change $f^*C = C \times_S S'$, i.e., $\rho_{f^*C/S'} = \rho_{C/S} \circ \pi_1 f$, when regarded as an outer homomorphims (up to composition with an inner automorphism).

(2.4) Changing the isomorphism $\pi^{\mathbb{L}} \cong \pi^{\mathbb{L}}_{s,\sigma(s)}$ composes $\rho_{C/S}$ by an inner automorphism of $\mathrm{Out}(\pi^{\mathbb{L}})$, thus leaving the class as an outer homomorphism unchanged. In the sequel outer representations will be regarded as outer homomorphisms. Diagrams of outer homomorphisms commute if they 'commute up to inner automorphisms' for representatives.

(2.5) In fact, the canonical outer pro-$\mathbb{L}$ representation does also exist in absence of a section by the following proposition; see [MT98] for a characteristic 0 analogue.

Proposition 2.6. *Let S be a connected scheme and let C/S be an S-curve in $\mathscr{M}_{g,n}(S)$. Let $\mathbb{L}$ be a set of prime numbers which are invertible on S.*

(1) There is a unique outer pro-$\mathbb{L}$ representation $\rho : \pi_1 S \to \mathrm{Out}\big(\pi^{\mathbb{L}}\big)$, such that for all $f : S' \to S$ where f^*C/S' admits a section the following diagram commutes.

$$\begin{tikzcd} \pi_1 S' \arrow[rr, "\rho_{f^*C/S'}"] \arrow[dr, "\pi_1 f"'] & & \mathrm{Out}\big(\pi^{\mathbb{L}}\big) \\ & \pi_1 S \arrow[ur, "\rho"'] & \end{tikzcd}$$

(2) For all $f : S' \to S$ with S', S connected the outer pro-$\mathbb{L}$ representations ρ (resp. ρ') from (1) applied to C/S (resp. f^*C/S') satisfy $\rho' = \rho \circ \pi_1 f$.

Proof: (2) follows from the uniqueness in (1) which follows from $(*)$ and the diagonal section of $C^0 \times_S C^0$. As any base change which allows a section of C^0/S factors through $\mathrm{pr} : C^0 \to S$, it suffices for (1) to prove that $\rho' = \rho_{\mathrm{pr}^*C/C^0}$ factors through $\pi_1 C^0 \twoheadrightarrow \pi_1 S$.

In other words, we need to descend along $\mathrm{pr} : C^0 \to S$ the pro-étale $\mathrm{Out}(\pi^{\mathbb{L}})$-torsor to which the outer map ρ' corresponds. The criterion that guarantees this descent was given already in terms of fundamental groups in [SGA 1] IX 5.6. Unfortunately one has to be careful and not mix outer homomorphisms with actual homomorphisms when applying [SGA 1] IX 5.6. So for a moment we chose various base points and an identification $\pi^{\mathbb{L}} = \pi^{\mathbb{L}}_{s,c}$ so that ρ' is represented by an actual group homomorphism that we denote by ρ' as well.

For better distinction between base and total space of the curve we set $S' = C^0$ with base point $s' = c$ and $S'' = S' \times_S S'$ with base point $s'' = (s', s')$. Note that both projections $\mathrm{pr}_i : S'' \to S'$ for $i = 1, 2$ respect these base points. The respective base changes of C^0/S are denoted C'^0/S' and C''^0/S'' with base points $c' = (c, c)$ and $c'' = (c, c, c)$ on the diagonals. Note that both $\mathrm{pr}_i^* C'^0$ for $i = 1, 2$ coincide canonically with the base change of C^0/S under the projection $S'' \to S$; hence there is only one curve over S'' under consideration. Finally, let $\widetilde{\mathrm{pr}}_i : C''^0 \to C'^0$ be the corresponding projection of total spaces lying over the map pr_i on the bases.

By [SGA 1] IX 5.6 in order to complete the proof we only have to show that both pro-$\mathbb{L}$ representations $\rho' \circ \pi_1(\mathrm{pr}_i) : \pi_1(S'', s'') \to \mathrm{Out}(\pi^{\mathbb{L}})$ for $i = 1, 2$ coincide (as group homomorphisms, not only outer homomorphisms). But $\rho' \circ \pi_1(\mathrm{pr}_1) = \rho' \circ \pi_1(\mathrm{pr}_2)$ follows by diagram chase from the following diagram with $i = 1, 2$.

$$\begin{tikzcd}
1 \arrow[r] & \pi^{\mathbb{L}} \arrow[r] \arrow[d, equal] & \pi_1^{(\mathbb{L})}(C''^0, c'') \arrow[r] \arrow[d, "\pi_1(\widetilde{\mathrm{pr}}_i)"] & \pi_1(S'', s'') \arrow[r] \arrow[d, "\pi_1(\mathrm{pr}_i)"] & 1 \\
1 \arrow[r] & \pi^{\mathbb{L}} \arrow[r] & \pi_1^{(\mathbb{L})}(C'^0, c') \arrow[r] & \pi_1(S', s') \arrow[r] & 1
\end{tikzcd}$$

Here it is crucial that the fibres of C'^0/S' and C''^0/S'' over the base points s' and s'' can be canonically identified with C_s^0 independently of $i = 1, 2$, which in essence allows to identify the respective pro-$\mathbb{L}$ completed fundamental groups of the fibres with $\pi^{\mathbb{L}}_{s,c}$ and ulimately with $\pi^{\mathbb{L}}$. Hence the identity appears as the left vertical arrow in the above diagram.

Changing the basepoint s or changing the isomorphism $\pi^{\mathbb{L}} \cong \pi^{\mathbb{L}}_{s,c}$ affects the outer representation ρ by composing with an inner automorphism of $\mathrm{Out}(\pi^{\mathbb{L}})$. This concludes the proof of the proposition. □

Proposition 2.7. *Let $2 - 2g - n$ be negative. Let S be a connected scheme and let C/S be an S-curve in $\mathscr{M}_{g,n}(S)$. Let $\mathbb{L}$ be a set of prime numbers which are invertible on S. The following diagram has exact rows.*

$$\begin{tikzcd}
1 \arrow[r] & \pi^{\mathbb{L}}_{s,c} \arrow[r] \arrow[d, "\cong"] & \pi_1^{(\mathbb{L})}(C^0, c) \arrow[r] \arrow[d, "\tilde{\rho}"] & \pi_1(S, s) \arrow[r] \arrow[d, "\rho"] & 1 \\
1 \arrow[r] & \mathrm{Inn}(\pi^{\mathbb{L}}) \arrow[r] & \mathrm{Aut}(\pi^{\mathbb{L}}) \arrow[r] & \mathrm{Out}(\pi^{\mathbb{L}}) \arrow[r] & 1
\end{tikzcd}$$

The extension of groups in the upper row is induced via ρ by the extension of the lower row.

Proof: It suffices to prove that $\pi^{\mathbb{L}}_{s,c} \to \pi_1^{(\mathbb{L})}(C^0,c)$ is injective. For then the outer representation ρ of Proposition 2.6 equals the conjugation action by lifts under a chosen identification $\pi^{\mathbb{L}} \cong \pi^{\mathbb{L}}_{s,c}$. In particular the map $\tilde{\rho}$ is induced by conjugation.

Let K be as above the kernel of $\pi_1(C^0,c) \to \pi_1(S,s)$. Because $\rho_{\mathrm{pr}^*C/C^0}$ factors through $\pi_1(C^0,c) \to \pi_1(S,s)$ by Proposition 2.6, the lift $\tilde{\rho}\circ\Delta : \pi_1(C^0,c) \to \mathrm{Aut}(\pi^{\mathbb{L}}_{s,c})$ invoking the diagonal section Δ of $(\mathrm{pr}^*C)^0/C^0$ as above and the conjugation action $\tilde{\rho}$ maps K to the group of inner automorphisms $\mathrm{Inn}(\pi^{\mathbb{L}}_{s,c})$ which is a pro-$\mathbb{L}$ group. In particular $\tilde{\rho}\circ\Delta$ factors through a map $\pi_1^{(\mathbb{L})}(C^0,c) \to \mathrm{Aut}(\pi^{\mathbb{L}}_{s,c})$. The restriction of $\tilde{\rho}\circ\Delta$ to $\pi^{\mathbb{L}}_{s,c}$ coincides with the canonical map $\pi^{\mathbb{L}}_{s,c} \cong \mathrm{Inn}(\pi^{\mathbb{L}}_{s,c})$ being induced by the diagonal section of

$$1 \to \pi^{\mathbb{L}}_{s,c} \to \pi^{\mathbb{L}}_{s,c} \times \pi^{\mathbb{L}}_{s,c} \to \pi^{\mathbb{L}}_{s,c} \to 1 \ .$$

Here we used a pro-$\mathbb{L}$ product formula for π_1 which is valid as all primes from $\mathbb{L}$ are invertible on S. The vanishing of the center of $\pi^{\mathbb{L}}$, see Lemma 2.2 (3), shows that $\tilde{\rho}\circ\Delta$ is injective on $\pi^{\mathbb{L}}_{s,c}$ and thus also the injectivity of the map $\pi^{\mathbb{L}}_{s,c} \to \pi_1^{(\mathbb{L})}(C^0,c)$. $\square$

(2.8) Remarks. (a) Let $\mathbb{L}$ be a set of prime numbers. We set $\mathbb{Z}[\frac{1}{\mathbb{L}}]$ for the ring of rational numbers whose denominator only contains primes from $\mathbb{L}$. The construction of the $\mathrm{Out}(\pi^{\mathbb{L}})$-torsor in Proposition 2.6 may be applied to the universal curve above $\mathscr{M}_{g,n} \otimes \mathbb{Z}[\frac{1}{\mathbb{L}}]$. We obtain the universal outer pro-$\mathbb{L}$ representation

$$\rho^{\mathrm{univ}} : \ \pi_1\big(\mathscr{M}_{g,n} \otimes \mathbb{Z}[\tfrac{1}{\mathbb{L}}]\big) \longrightarrow \mathrm{Out}(\pi^{\mathbb{L}}) \ .$$

(b) Let G be a finite group. We may fuse the $\mathrm{Out}(\pi^{\mathbb{L}})$-torsor with the $\mathrm{Out}(\pi^{\mathbb{L}})$-set $\mathrm{Hom}^{\mathrm{out}}_{\mathrm{surj}}(\pi^{\mathbb{L}},G)$ of surjective, outer homomorphisms $\pi^{\mathbb{L}} \twoheadrightarrow G$. We obtain a finite étale cover ${}_G\mathscr{M}_{g,n} \to \mathscr{M}_{g,n} \otimes \mathbb{Z}[\frac{1}{\#G}]$ of curves that are equipped with a nonabelian Teichmüller level structure of level G, see [DM69], [PdJ94] §2.3, [BP00]. The stack ${}_G\mathscr{M}_{g,n}$ is an open substack of the normalisation $\overline{{}_G\mathscr{M}_{g,n}}$ of the stack $\overline{\mathscr{M}}_{g,n}$ of stable pointed curves in the function field of ${}_G\mathscr{M}_{g,n}$.

For d a product of powers of primes from $\mathbb{L}$ and $G = \pi^{\mathbb{L}}[d] := \pi^{\mathbb{L}}/[\pi^{\mathbb{L}},\pi^{\mathbb{L}}](\pi^{\mathbb{L}})^d$ we denote the space ${}_G\mathscr{M}_{g,n}$ of curves with abelian level structures of level d by $\mathscr{M}_{g,n}[d]$.

If G surjects to $\pi^{\mathbb{L}}[\ell^r]$ for some $\ell \in \mathbb{L}$ and $\ell^r \geq 3$, then $\overline{{}_G\mathscr{M}_{g,n}}$ is actually a scheme which is projective over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{\#G}])$, see [PdJ94] Prop 2.3.4 and moreover [De85] §3. Strictly speaking [PdJ94] and [De85] only treat the proper case ($n = 0$), but the general case follows along the same lines.

(c) Note that the assumption $2-2g-n < 0$ in Proposition 2.7 is essential. A counterexample for the case $\mathscr{M}_{0,2}$ is as follows. The complement of the sections of the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathscr{O}(-1) \oplus \mathscr{O})$ over $\mathbb{P}^1_{\mathbb{C}}$ equipped with the two sections 0 and ∞ is isomorphic to $\mathbb{A}^2_{\mathbb{C}} - \{0\}$. Hence it is simply connected, whereas a fibre has fundamental group isomorphic to $\widehat{\mathbb{Z}}$.

(2.9) We can formulate a second monodromy criterion for extending curves in terms of outer representations as follows.

Theorem 2.10 (monodromy criterion II). *Let $2-2g-n$ be negative. Let U be a dense open subscheme of a normal, connected, excellent scheme S.*

Let C/U be a U-curve in $\mathscr{M}_{g,n}(U)$ and let $\pi^{\mathbb{L}}$ be as above the pro-$\mathbb{L}$ completion of the complement of the sections of C/U in a geometric fibre. Then C/U extends to an S-curve in $\mathscr{M}_{g,n}(S)$ if and only if the following commutative diagram, where the solid arrows are *induced by the natural maps $U \subset S$ and the outer pro-$\mathbb{L}$ representation ρ associated to $C \otimes \mathbb{Z}[\frac{1}{\mathbb{L}}]/U \otimes \mathbb{Z}[\frac{1}{\mathbb{L}}]$,*

$$\begin{tikzcd}
\pi_1\big(U \otimes \mathbb{Z}[\frac{1}{\mathbb{L}}]\big) \arrow[rr, "\rho"] \arrow[dr] & & \mathrm{Out}\big(\pi^{\mathbb{L}}\big) \\
& \pi_1\big(S \otimes \mathbb{Z}[\frac{1}{\mathbb{L}}]\big) \arrow[ur, dotted] &
\end{tikzcd}$$

exists for one of the following collections of sets of prime numbers $\mathbb{L}$ and additional conditions on S:

(A) $\mathbb{L} = \{\ell\}$ for some prime number ℓ and S is of characteristic 0, or

(B) $\mathbb{L} = \{\ell_1, \ell_2\}$ for all pairs of sufficiently large prime numbers ℓ_1, ℓ_2 and no additional conditions on S.

(2.11) Remarks. (a) It follows from the universal outer pro-$\mathbb{L}$ representation ρ^{univ} and functoriality that Thereom 2.10 is stronger than Theorem 1.2.

(b) It is plausible though, that Theorem 2.10 also holds under the following condition on the collection of $\mathbb{L}$'s:

(C) $\mathbb{L} = \{\ell\}$ for some prime number ℓ that is invertible on S.

Indeed, assumption (B) which involves two prime numbers only comes into play to apply results from [Ta04] in the positive characteristic situation of Theorem 4.5, where alternative proofs are not out of sight, see the remarks there.

(c) In case S is the spectrum of a henselian discrete valuation ring, Theorem 2.10, under condition (C) above, is nothing but the criterion for good reduction of Oda–Tamagawa [Ta97] Thm 5.3. In particular, curves satisfying either monodromy criterion above extend into codimension 1 points of the base. It remains to deal with $U \subset S$ with boundary of codimension ≥ 2.

(d) In [MB85a] Moret-Bailly proves a purity theorem for relative curves over regular bases S. Namely, for U open dense in a regular S with $S - U$ of codimension at least 2, the restriction functor $\mathscr{M}_{g,n}(S) \to \mathscr{M}_{g,n}(U)$ is an equivalence of categories. Moret-Bailly's theorem together with Oda-Tamagawa's criterion as mentioned in (c) imply our Theorem 2.10, if the base S is regular along the boundary $S \setminus U$. In fact, by Zariski–Nagata purity for the branch locus, Moret-Bailly's theorem is equivalent to our monodromy criteria in the case of a regular base and codimension of the boundary at least 2.

The new contribution of the present note thus only consists in the case where S is not regular along the boundary.

In the beginning, the proof in [MB85a] and our treatment of the monodromy criterion Theorem 2.10 follow the same strategy while the methods to enforce isotriviality along subvarieties, that one needs to contract, are essentially different (even in the regular case). More precisely, [MB85a] uses first a deformation argument to reduce to bases of dimension 2 to the effect that by induction and the theory of regular surfaces it suffices to contract rational lines. Then [MB85a] uses results on pencils of stable curves over $\mathbb{P}^1$.

The deformation argument is not applicable in our case where the base is only assumed to be normal. Instead, the present note exploits a recent result of Tamagawa [Ta04] on the specialisation homomorphism for the fundamental group of curves in positive characteristic, see also [Saï03].

(e) The result of [MB85a] has been extended in various ways to stable or log smooth curves by de Jong–Oort [dJO97], Mochizuki [Mz99b] and T. Saito [Sai04]. The author believes that the monodromy criteria of this note extend to the logarithmic situation, cf. [St05] Thm 1.2.

(f) Theorem 5.2 at the end of this note contains an extended monodromy criterion where the open immersion $U \to S$ is replaced by a morphism of finite type.

3 Extension after modification of the base

(3.1) The rest of this note is devoted to the proof of Theorem 2.10 and thus of part (2) of Theorem 1.2. We only need to show that a curve C/U extends if the condition on the fundamental groups holds.

(3.2) We can use part (1) of Theorem 1.2 and étale descent for curves to work locally on S in the étale topology. The descent is effective by the presence of a canonical, relatively ample line bundle: the bundle of relative differentials twisted by the locus of the sections.

For example, we may replace S by a finite étale cover S' obtained by pullback of a finite étale cover $\mathscr{M}'$ of $\mathscr{M}_{g,n} \otimes \mathbb{Z}[\frac{1}{\ell}]$ that is induced via the universal outer pro-ℓ representation by a finite quotient of $\mathrm{Out}(\pi^\ell)$. Indeed, under the map $U \to \mathscr{M}_{g,n}$ the cover $\mathscr{M}'$ pulls back to a finite étale cover U' of U that extends to S' by the condition on the π_1's.

We apply the preceeding construction to the cover $\mathscr{M}_{g,n}[\ell^r]$ of curves with abelian level structure of level $\ell^r \geq 3$ for some prime ℓ.

(3.3) The first step will be to prove that if a curve in $\mathscr{M}_{g,n}[\ell^r](U)$ satisfies the monodromy criterion then it extends as a curve with abelian level structure to a proper modification $S' \to S$ which is an isomorphism over U. This part of the theorem holds unconditionally on $\ell \in \mathscr{O}_S^*$.

Theorem 3.4 (extension after modification). *Let U be an open dense subscheme in a normal, connected, excellent scheme S, and let ℓ be a prime number invertible on S. Let C/U be a U-curve in $\mathscr{M}_{g,n}[\ell^r](U)$ endowed with an abelian level structure of level ℓ^r for some $\ell^r \geq 3$.*

If the associated outer pro-ℓ representation factors through $\pi_1 U \to \pi_1 S$, then there exists a proper birational map $\sigma : S' \to S$ which is an isomorphism above U, such that

(i) S' is normal,

(ii) $\sigma_* \mathscr{O}_{S'} = \mathscr{O}_S$,

(iii) the U-curve C/U extends to an S'-curve with level structure in $\mathscr{M}_{g,n}[\ell^r](S')$ such that the corresponding outer pro-ℓ representation factors through $\pi_1\sigma : \pi_1 S' \to \pi_1 S$.

Moreover, if also the outer pro-$\mathbb{L}$ representation of C/U factors through $\pi_1 U \to \pi_1 S$ for some set of primes $\mathbb{L}$ invertible on S, then the correponding outer pro-$\mathbb{L}$ representation of the extension over S' factors through $\pi_1\sigma : \pi_1 S' \to \pi_1 S$.

Proof: The curve C/U is represented by a map $f_U : U \to \mathscr{M}_{g,n}[\ell^r]$ and we need to find a modification $\sigma : S' \to S$ such that f_U extends to a map $f' : S' \to \mathscr{M}_{g,n}[\ell^r]$. Let S' be the normalisation of the closure in $S \times \overline{\mathscr{M}}_{g,n}[\ell^r]$ of the graph of f_U. The first projection

$\sigma : S' \to S$ satisfies the requirements. To see this, we argue that the second projection $f' : S' \to \overline{\mathscr{M}}_{g,n}[\ell^r]$ actually has image contained in $\mathscr{M}_{g,n}[\ell^r]$. The other requirements for σ are immediate.

Assume on the contrary that $s' \in S'$ is mapped to $f'(s')$ in the boundary. We can find a strict henselian, discrete valuation ring R and $\mathrm{Spec}(R) \to S'$ with the closed point mapping to s' and the generic point η mapping into $U \subset S'$. The composition with f' yields a map $\mathrm{Spec}(R) \to \overline{\mathscr{M}}_{g,n}[\ell^r]$ such that the corresponding curve over $\mathrm{Spec}(R)$ has bad reduction. Since, on the other hand, the assumption on the outer pro-ℓ representation implies that the corresponding map $\pi_1\eta \to \pi_1 U \to \mathrm{Out}(\pi^\ell)$ is trivial, this contradicts the Oda–Tamagawa criterion for good reduction, [Ta97] Thm 5.3, that was already mentioned above in remark (2.11 (c)). □

(3.5) Remark. For $g \geq 2$ and $n = 0$ it follows from [PdJ94] Thm 3.1.3 that $\mathscr{M}_{g,n} \otimes \mathbb{Z}[\frac{1}{\ell}]$ is universally ramified in $\overline{\mathscr{M}}_{g,n} \otimes \mathbb{Z}[\frac{1}{\ell}]$ with respect to ℓ, see also [Br79] and [Lo94]. The latter includes a definition of universal ramification. Conjecturally $\mathscr{M}_{g,n} \otimes \mathbb{Z}[\frac{1}{\ell}]$ has universal ramification in $\overline{\mathscr{M}}_{g,n} \otimes \mathbb{Z}[\frac{1}{\ell}]$ with respect to ℓ for all (g,n) with $2-2g-n$ negative. There is a proof of Theorem 3.4 based on universal ramification, see [St04] §2. This would lead to a geometric proof based on the moduli space of the Oda–Tamagawa criterion for good reduction as a special case of Theorem 2.10.

4 Constant maps

(4.1) To complete the proof of Theorem 2.10 it remains to prove that the map f' from the proof of Theorem 3.4 factors through $\sigma : S' \to S$ under the monodromy conditions of Theorem 2.10. More precisely, the latter imply as indicated in Theorem 3.4, that the respective outer pro-$\mathbb{L}$ representations $\pi_1 S' \to \mathrm{Out}(\pi^{\mathbb{L}})$ factor through $\pi_1\sigma : \pi_1 S' \to \pi_1 S$.

(4.2) It suffices that f' factorises set-theoretically as $f' = f \circ \sigma$. Indeed, S carries the quotient topology under σ, hence f is automatically continuous. Then the property $\sigma_*\mathscr{O}_{S'} = \mathscr{O}_S$ is responsible for the enrichment of f to a map of (locally) ringed spaces.

(4.3) For $g \leq 2$ there is nothing to prove. Indeed, $\mathscr{M}_{g,n}[\ell^r]$ is consecutively fibred in affine curves over $\mathscr{M}_{2,0}[\ell^r]$, $\mathscr{M}_{1,1}[\ell^r]$ or $\mathscr{M}_{0,3}[\ell^r]$ respectively. These latter moduli spaces are affine, whereas the fibres of σ are proper. In particular Theorem 2.10 holds even under hypothesis (C) on ℓ and S in these cases.

(4.4) For $g > 2$ and a geometric fibre F of σ, the outer pro-$\mathbb{L}$ representations $\rho : \pi_1 F \to \mathrm{Out}(\pi^{\mathbb{L}})$ of the restriction of the S'-curve to F factors through $\pi_1 S$ for the various sets $\mathbb{L}$ in question. Hence ρ is trivial.

The following Theorem 4.5 accomplishes the last step in the proof of Theorem 2.10 and thus Theorem 1.2.

Theorem 4.5. *Let $2-2g-n$ be negative. Let F be a reduced, connected variety over an algebraically closed field k. Let $\varphi : F \to \mathscr{M}_{g,n}$ be a map such that the outer pro-$\mathbb{L}$ representation $\rho : \pi_1 F \to \mathrm{Out}(\pi^{\mathbb{L}})$ is the trivial homomorphism for one of the following collections of sets of prime numbers $\mathbb{L}$ and additional conditions on k:*

(A) $\mathbb{L} = \{\ell\}$ for some prime number ℓ and k is of characteristic 0, or

(B) $\mathbb{L} = \{\ell_1, \ell_2\}$ for all pairs of sufficiently large prime numbers ℓ_1, ℓ_2 invertible in k and k is of positive characteristic.

Then φ is constant in the sense that the correspondig F-curve $C/F \in \mathscr{M}_{g,n}(F)$ comes by base extension from an object in $\mathscr{M}_{g,n}\big(\mathrm{Spec}(k)\big)$.

(4.6) Remark. The characteristic 0 part of the above theorem follows also from the work of Mochizuki on pro-p anabelian geometry, see [Mz99a].

Proof: Let ℓ be a prime in $\mathbb{L}$. By assumption, the map φ lifts to a map $\tilde{\varphi} : F \to \mathscr{M}_{g,n}[\ell^r]$. As for $\ell^r \geq 3$ the target is now a scheme, it suffices to prove that $\tilde{\varphi}$ is a constant map of sets in this case.

By covering F with images of curves we may assume that F is a smooth curve. Now we apply Theorem 3.4 to $\tilde{\varphi} : F \to \mathscr{M}_{g,n}[\ell^r]$ and deduce that we may extend the curve to the smooth compactification of F. Still, the outer pro-$\mathbb{L}$ representation is trivial for the respective $\mathbb{L}$. Hence we may assume that F is even a proper smooth curve over k.

Next we observe that $\mathscr{M}_{g,n+1}[\ell^r] \to \mathscr{M}_{g,n}[\ell^r]$ is fibred in affine curves for $n \geq 1$. By induction on n we are reduced to the cases $n \leq 1$. The last step $\mathscr{M}_{g,1}[\ell^r] \to \mathscr{M}_{g,0}[\ell^r]$ is fibred in smooth proper curves of fixed genus g. Let us assume that we have successfully dealt with the case $n = 0$. Then any non constant lift corresponds to a dominant map from F to a fibre, i.e., a smooth proper curve X/k of genus g. This curve X is the base for the family $X \times X \setminus \Delta$ under second projection, where Δ is the diagonal section. By [As96] Prop 6 and (4.4.4), the corresponding outer pro-ℓ representation factors through the pro-ℓ completion and yields an injective map $\pi_1^\ell X \to \mathrm{Out}(\pi^\ell)$. Hence the induced map $\pi_1 F \to \pi_1^\ell X$ is trivial by the assumption on ρ (for $(g,1)$). Therefore the map $F \to X$ lifts to any finite ℓ-primary Galois cover of X. As the genus of such covers tends to infinity (here $g \geq 2$) this is impossible.

It remains to deal with the cases $g \geq 3$ and $n = 0$, because the moduli spaces are affine in the other minimal cases $(2,0)$, $(1,1)$ and $(0,3)$.

We fix geometric points $x \in F$ and $c \in C_x$, the fibre of C/F above x. The vanishing of the outer pro-$\mathbb{L}$ representation and Proposition 2.7 yield a canonical isomorphism

$$\pi_1^{(\mathbb{L})}(C,c) = \pi_{x,c}^{\mathbb{L}} \times \pi_1(F,x).$$

The finite étale cover $C_{x,\Phi}$ of C_x associated to a finite set Φ with continuous $\pi_{x,c}^{\mathbb{L}}$-action is the fibre above x of a finite étale cover $C_\Phi \to C$ associated to the action of $\pi_1^{(\mathbb{L})}(C,c)$ on Φ via projection to $\pi_{x,c}^{\mathbb{L}}$. If $C_{x,\Phi}$ is connected, then C_Φ is a family of geometrically connected curves over F. By construction we then have canonically $\pi_1^{(\mathbb{L})} C_\Phi = \pi_\Phi^{\mathbb{L}} \times \pi_1 F$ with the group $\pi_\Phi^{\mathbb{L}} = \pi_1^{\mathbb{L}}(C_{x,\Phi}) \subset \pi_{x,c}^{\mathbb{L}}$. It follows that any C_Φ/F as above with geometrically connected fibres has constant abelian ℓ^n level structure for all $n \in \mathbb{N}$ and $\ell \in \mathbb{L}$ and therefore obeys the following theorem.

Theorem 4.7. *Let B be a normal variety over an algebraically closed field k. Let ℓ be invertible in k. Let C/B be a curve with constant abelian ℓ^n level structure for all $n \in \mathbb{N}$. Then the following holds.*

(i) The relative jacobian $\mathrm{Jac}_{C/B}$ is radicially isogenous to a constant abelian variety.

(ii) If k has characteristic $p > 0$, then the p-rank of the fibres C_b of C/B is a constant function of $b \in B$.

Proof: Part (i) follows from [Gr66] Prop. 4.4 (or [O74] Thm 2.1), the essential ingredients are the theory of the k-trace of abelian varieties and the relative Mordell–Weil

Theorem of Lang–Néron. For (ii) just notice that isogenous abelian varieties have the same p-rank. □

(4.8) We come back to the proof of Theorem 4.5. If k is of characteristic 0 then by (i) of Theorem 4.7 the relative jacobian is constant. Thus also its canonical principal polarization and its abelian ℓ^n level structures are constant. By Torelli's theorem, see [OS79], the curve C/F must be constant and we are done. It remains to deal with positive characteristic.

(4.9) Let k have positive characteristic p. We deduce from Theorem 4.7 that all families C_Φ/F as above have constant p-rank.

(4.10) Following ideas of Raynaud (for new ordinarity), in [Ta04] Theorem 0.7, Proposition 0.8, Corollary 5.3, and Theorem 0.5 Tamagawa proved the existence of prime to p covers $C''_x \to C'_x \to C_x$ of $C_x = C \times_F x$, such that C''_x/C'_x is

(i) prime cyclic Galois,

(ii) new ordinary: the growth in p-rank equals the growth in genus, and

(iii) 'new Torelli': the infinitesimal deformation functor of C'_x is closedly immersed in the infinitesimal deformation functor of the new part of the relative jacobian of C''_x/C'_x, a generalised Prym variety.

A close inspection of [Ta04] shows that we may moreover assume first, that $C'_x \to C_x$ is ℓ_1-primary for some arbitrary but sufficiently large prime ℓ_1, and then secondly the degree of C''_x/C'_x can be chosen to be a prime ℓ_2, which is again arbitrary but sufficiently large. It is exactly here that our approach uses the outer pro-$\mathbb{L}$ representations for all pairs $\mathbb{L}$ of two sufficiently large prime numbers ℓ_1, ℓ_2.

By what was said above, the étale covers $C''_x \to C'_x \to C_x$ are isomorphic to the fibres above x of étale covers $C'' \to C' \to C$. The cover C''/C' remains prime cyclic Galois. It follows that the new part of the relative jacobian of the cyclic cover C''/C' is a projective family of ordinary abelian varieties, and hence constant by a theorem of Raynaud, Szpiro and Moret-Bailly, see [MB85b] XI Thm 5.1. Consequently, by property (iii) 'new Torelli' above, the deformation C' of C'_x is a constant family. A priori it is constant in the sense that all fibres are mutually isomorphic, but as C' carries a level structure this implies that C'/F is constant. The original curve C/F is a quotient of the Galois closure of C'/C and thus also constant, which proves Theorem 4.7. □

An application of Theorem 4.5 is the following immediate corollary.

Corollary 4.11. *Let X be a reduced, connected and simply connected variety, i.e. $\pi_1 X = 1$, over an algebraically closed field k. Then any smooth proper curve C over X of genus g with n disjoint sections, where $2-2g-n$ is negative, is a constant family of such curves, i.e, comes by base change from an object in $\mathscr{M}_{g,n}\big(\mathrm{Spec}(k)\big)$.*

(4.12) Remark. In particular, Corollary 4.11 claims that there are no $\mathbb{P}^1$'s on $\mathscr{M}_{g,n}$ for $2-2g-n<0$. Note that the corresponding result for polarized abelian varieties fails. Oort has constructed a family of abelian varieties over $\mathbb{P}^2_k$ with image of dimension 2 in the moduli space of polarized abelian varieties, [O74] Rmk 2.6. Moreover, the coarse moduli space M_g of proper smooth curves behaves also different from the fine moduli space $\mathscr{M}_g$ that we use in this note. Again Oort proves the existence of a complete rational curve in $M_g \otimes k$ for k of positive characteristic and suitable g, [O74] Thm 3.1.

5 Concluding remarks

(5.1) Remarks. (a) Let S be a connected local complete intersection scheme and U a dense open in S such that $S - U$ has codimension at least 3. In [Gr68] Exp. X Thm 3.4 Grothendieck proves that $\pi_1 U \to \pi_1 S$ is an isomorphism. Hence, if S is normal, by Theorem 1.2 the natural functor $\mathscr{M}_{g,n}(S) \to \mathscr{M}_{g,n}(U)$ is an equivalence of categories.

(b) In order to yield a more natural anabelian proof of Theorem 4.5, it would be nice if the natural map $\mathrm{H}^2(\pi_1 \mathscr{M}_{g,n}, \mathbb{Q}_\ell) \to \mathrm{H}^2(\mathscr{M}_{g,n}, \mathbb{Q}_\ell)$ is an isomorphism. Note that $\mathrm{H}^2(\mathscr{M}_{g,n}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell \cdot \mathrm{c}_1(\lambda)$. Then Theorem 4.5 allows a quick proof exploiting the ℓ-adic first Chern class of the Hodge bundle λ, see [St04]. In particular, this would give a proof of Theorem 2.10 unconditionally on $\ell \in \mathscr{O}_S^*$ for all characteristics.

The isomorphism $\mathrm{H}^2(\pi_1 \mathscr{M}_{g,n}, \mathbb{Q}_\ell) \cong \mathrm{H}^2(\mathscr{M}_{g,n}, \mathbb{Q}_\ell)$ has been announced by Boggi.

Theorem 5.2 (extended monodromy criterion). *Let $2 - 2g - n$ be negative. Let $f : U \to S$ be a map of finte type that factorizes as $f = \overline{f} \circ j$ where $j : U \to X$ is a dense open immersion into a normal scheme X, and $\overline{f} : X \to S$ is a proper map that satisfies $\overline{f}_* \mathscr{O}_X = \mathscr{O}_S$. Furthermore we assume that S is excellent. Then the following holds.*

(1) The pull back functor $f^* : \mathscr{M}_{g,n}(S) \to \mathscr{M}_{g,n}(U)$ is fully faithful.

(2) A U-curve $C \in \mathscr{M}_{g,n}(U)$ comes by base change from an S-curve in $\mathscr{M}_{g,n}(S)$ if and only if for all $N \in \mathbb{N}$ and geometric points $\overline{u} \in U \otimes \mathbb{Z}[\frac{1}{N}]$ the following commutative diagram exists, where the solid arrows are induced by the natural maps $U \to \mathscr{M}_{g,n}$ and $f : U \to S$:

$$\begin{tikzcd}
\pi_1(U \otimes \mathbb{Z}[\frac{1}{N}], \overline{u}) \arrow[rr] \arrow[dr] & & \pi_1(\mathscr{M}_{g,n} \otimes \mathbb{Z}[\frac{1}{N}], C_{\overline{u}}) \\
& \pi_1(S \otimes \mathbb{Z}[\frac{1}{N}], \overline{u}) \arrow[ur, dotted] &
\end{tikzcd}$$

Proof: By applying Theorem 1.2 to $U \subset X$ we may assume that f is proper. Using again the finiteness of the isom-scheme of hyperbolic curves (resp. abelian level structures plus descent), (1) and (2) come down to the property of $\overline{f} : X \to S$ that a map of schemes $h : X \to T$ factorizes through $\overline{f}$ if and only if it is constant along the fibres of $\overline{f}$ as a map of sets. In particular, we use again Theorem 4.5 for claim (2). $\square$

The obvious formulation of the analogous theorem in terms of outer pro-$\mathbb{L}$ representations is left to the reader.

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