Birational p-adic Galois sections in higher dimensions

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Abstract — This note explores the consequences of Koenigsmann's model theoretic argument from the proof of the birational p-adic section conjecture for curves in the context of higher dimensional varieties over p-adic local fields.

1. Introduction

A **birational Galois section** for a geometrically irreducible and reduced variety X/k is a continuous section of the restriction homomorphism

$$\mathrm{res}_{K/k} : \mathrm{Gal}_K \to \mathrm{Gal}_k,$$

where Gal_K and Gal_k are the absolute Galois groups of the function field $K = k(X)$ and of k, with respect to an algebraic closure $\overline{K}$ of K and the algebraic closure $\bar{k}$ of k contained in $\overline{K}$.

A proof of Grothendieck's anabelian section conjecture would imply, for a smooth projective curve X/k over a finitely generated extension $k/\mathbb{Q}$, that birational Galois sections come in packets indexed by the k-rational points $X(k)$. In particular, then $X(k) \neq \emptyset$ if and only if X/k admits birational Galois sections. These consequences of the section conjecture are known in some cases for number fields k by Stoll [St07] Remark 8.9, see also [HS10] Theorem 17 and [Sx12a]. For the section conjecture we refer to [Sx12b].

Even more strikingly, in [Ko05] Proposition 2.4(b), Koenigsmann proves, with a touch of model theory of p-adically closed fields, that birational Galois sections for smooth curves X/k over finite extensions $k/\mathbb{Q}_p$ come in packets indexed by $X(k)$. The packet of $a \in X(k)$ consists of those sections s with $s(\mathrm{Gal}_k) \subset D_{\bar{v}}$, where $D_{\bar{v}}$ is any decomposition group of a prolongation $\bar{v}$ to $\overline{K}$ of the k-valuation v corresponding to a with valuation ring $\mathcal{O}_{X,a} \subset k(X)$.

Our main result is the following theorem, see Theorem 8.

Theorem. *Let X/k be a geometrically irreducible, normal variety over a finite extension $k/\mathbb{Q}_p$ with function field K. Then every birational Galois section has image in the decomposition subgroup $D_{\bar{v}} \subset \mathrm{Gal}_K$ for a unique k-valuation $\bar{v}$ of $\overline{K}$ with residue field of $v = \bar{v}|_K$ equal to k.*

In particular, conjugacy classes of sections of $\mathrm{Gal}_K \to \mathrm{Gal}_k$ come in disjoint non-empty packets associated to k-valuations v of K with residue field k.

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2. Review of Koenigsmann's use of model theory

We keep the notation from the introduction and refer to [PR84] for the notion of a p-adically closed field, i.e., a field that is elementary equivalent to a finite extension of $\mathbb{Q}_p$. Koenigsmann has the following lemma in the case of smooth curves, [Ko05] Proposition 2.4(a). The general case, see [Ko05] Remark 2.5, admits a mathematically identical proof. We decide to nevertheless give a proof in order to hopefully make the argument more transparent for non-model theorists.

Lemma 1 (Koenigsmann's Lemma). *Let k be a p-adically closed field, and let X/k be a geometrically irreducible and reduced variety. If X/k admits a birational Galois section, then $X(k) \neq \emptyset$.*

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Proof: Let $s : \mathrm{Gal}_k \to \mathrm{Gal}_K$ be a section. We set $L = \overline{K}^{s(\mathrm{Gal}_k)}$ for the fixed field of the image, so that by construction, the restriction map $\mathrm{Gal}_L \to \mathrm{Gal}_k$ is an isomorphism. Thus $k \subset L$ is relatively algebraically closed and the subfields $k^{\mathrm{abs}} = L^{\mathrm{abs}}$ of absolutely algebraic elements, i.e., algebraic over $\mathbb{Q}$, agree. Since the restriction map $\mathrm{Gal}_k \to \mathrm{Gal}_{k^{\mathrm{abs}}}$ is an isomorphism [Po88] (E4), we deduce that $\mathrm{Gal}_L \to \mathrm{Gal}_{L^{\mathrm{abs}}}$ is an isomorphism as well. Now [Po88] Theorem (E12) implies that L is p-adically closed with respect to a valuation with same the p-rank as the p-adic valuation on k.

With Prestel and Roquette [PR84] Theorem 5.1 we conclude that $k \subset L$ is an elementary extension with respect to the model theory of valued fields. This implies that the statement in the language of fields with constants in k saying

$$\text{'}\textit{the set of rational points of } X \textit{ is non-empty}\text{'} \tag{2.1}$$

is true over k if and only if it is true over L where the constants from k are interpreted via $k \subset L$. Since X admits the tautological L-rational point

$$\mathrm{Spec}(L) \to \mathrm{Spec}(K) \to X,$$

we are done. □

Remark 2. To understand that (2.1) is a statement in the language of fields with constants in k one has to get back (for each affine open of a finite Zariski open covering) to the classical language that describes the set of rational points of X as the set of solutions of systems of polynomial equations with coefficients in k.

3. The limit argument

The fundamental group $\pi_1(X)$ of X/k fits in an extension

$$1 \to \pi_1(X_{\bar{k}}) \to \pi_1(X) \xrightarrow{\mathrm{pr}_*} \mathrm{Gal}_k \to 1, \tag{3.1}$$

where the geometric generic point $\mathrm{Spec}(\overline{K}) \to X_{\bar{k}} \to X$ is the implicit base point. The notation $\pi_1(X/k)$ will serve as a shorthand for the extension (3.1), i.e., the datum of the group $\pi_1(X)$ together with the projection pr_*.

The space of sections $\mathrm{Gal}_k \to \pi_1(X)$ of $\pi_1(X/k)$ up to conjugation by elements from $\pi_1(X_{\bar{k}})$ will be denoted by $\mathscr{S}_{\pi_1(X/k)}$ and the set of $\mathrm{Gal}_{\bar{k}K}$-conjugacy classes of sections of $\mathrm{Gal}_K \to \mathrm{Gal}_k$, will be denoted by $\mathscr{S}_{\pi_1(K/k)}$. By functoriality, to $a \in X(k)$ we associate a class of sections $s_a : \mathrm{Gal}_k \to \pi_1(X)$. This gives rise to the non-abelian Kummer map $a \mapsto \kappa(a) = s_a$, see [Sx12b] §2.4,

$$\kappa \ : \ X(k) \to \mathscr{S}_{\pi_1(X/k)}.$$

Let $j : \mathrm{Spec}\, K \to X$ be the inclusion of the generic point which on $\pi_1(-)$ induces the map $j_* : \mathrm{Gal}_K \to \pi_1(X)$, a surjection if X is normal, and furthermore a map $s \mapsto j_*(s) = j_* \circ s$

$$j_* : \mathscr{S}_{\pi_1(K/k)} \to \mathscr{S}_{\pi_1(X/k)}.$$

A section in the image of j_* is called **birationally liftable**.

Proposition 3. *Let k be a finite extension of $\mathbb{Q}_p$, and let X/k be a proper, smooth and geometrically irreducible variety. Then the image of the non-abelian Kummer map $\kappa : X(k) \to \mathscr{S}_{\pi_1(X/k)}$ agrees with the set of birationally liftable sections.*

Proof: A **neighbourhood** of a section $s : \mathrm{Gal}_k \to \pi_1(X)$ is a finite étale cover $X' \to X$ together with a lift s' of the section, i.e., an open subgroup $\pi_1(X') = H \subseteq \pi_1(X)$ containing the image of the section $s = s'$. By allowing a lift s', the map $\pi_1(X') \to \mathrm{Gal}_k$ is surjective and hence X'/k is necessarily geometrically connected. Since here X and thus X' are smooth over k, we conclude that X' is also geometrically irreducible over k.

The limit over all neighbourhoods yields a pro-étale cover

$$X_s = \varprojlim X' \to X$$

corresponding to $\pi_1(X_s) = s(\mathrm{Gal}_k) \subseteq \pi_1(X)$. Then we have $s = s_a$ if and only if

$$a \in \mathrm{im}\big(X_s(k) \to X(k)\big).$$

We have

$$X_s(k) = \varprojlim X'(k)$$

where X' ranges over all neighbourhoods of the section s. Since X/k is assumed proper, all sets $X'(k)$ are compact in the p-adic topology. Thus $X_s(k)$ is nonempty if and only if $\varprojlim X'(k)$ is a projective limit of nonempty sets. We conclude that $s = s_a$ for some $a \in X(k)$ if and only if all neighbourhoods (X', s') of s have k-rational points. The latter follows from being in the image of j_* by Koenigsmann's Lemma above, because with s also s' is birationally lifting as a section of $\pi_1(X'/k)$.

To conclude the converse, it suffices to construct a k-valuation v on $K = k(X)$, i.e., a valuation on K with k in its valuation ring, such that

(i) a given point $a \in X(k)$ is the center of the valuation on X,
(ii) the residue field of v agrees with k,
(iii) and the natural surjection $D_v \twoheadrightarrow \mathrm{Gal}_k$ splits where $D_v \subset \mathrm{Gal}_K$ is the decomposition subgroup of the valuation.

Then $j_*(D_v) = s_a(\mathrm{Gal}_k)$, and thus any splitting of $D_v \to \mathrm{Gal}_k$ will allow to lift s_a to a section of $\mathrm{Gal}_K \to \mathrm{Gal}_k$.

We construct such a valuation v. Let $t_1, \ldots, t_d$ be a system of parameters in the regular local ring $\mathcal{O}_{X,a}$. The prime ideals $\mathfrak{p}_i = (t_1, \ldots, t_i)$ for $0 \leq i \leq d$ correspond to the generic points $a_i \in X$ of irreducible cycles $Z_i \subseteq X$ which are regular at $a = a_d$. We consider the valuation v associated to the corresponding chain

$$X = Z_0 \supset Z_1 \supset \ldots \supset Z_d = a$$

which is the composition of the discrete rank 1 valuation rings $\mathcal{O}_{Z_i, a_{i+1}}$. Its decomposition group D_v sits in an extension

$$1 \to \hat{\mathbb{Z}}(1)^d \to D_v \to \mathrm{Gal}_k \to 1$$

that splits by choosing compatible nth roots for all t_i and all n. The splitting also follows by general valuation theory, see for example [HJP07] Proposition 7.3(a). □

Remark 4. (1) For birationally liftable sections to be contained in $\kappa(X(k))$ requires X/k to be proper, or at least a smooth compactification with no k-rational point in the boundary. Any affine smooth hyperbolic curve with a k-rational point at infinity provides a counter-example.

(2) That sections s_a are birationally liftable requires a certain amount of local regularity at k-rational points. It is not enough to assume X normal as the following example shows.

Example 5. Let $k/\mathbb{Q}_p$ be finite and let C/k be a smooth projective geometrically irreducible curve which is

(i) not hyperelliptic, in particular of genus $g \geq 3$,
(ii) and has no k-rational point.

We consider the difference map $(a, b) \mapsto a - b$

$$d : C \times C \to A = \mathrm{Alb}_C$$

to the Albanese variety Alb_C of C. We set $f : C \times C \to X$ for the Stein-factorization of d. This X is a normal variety and, due to assumption (i) and Riemann-Roch, the map d contracts exactly the diagonally embedded $\Delta = C \hookrightarrow C \times C$. So X can be identified with the contraction of Δ to a k-rational point $\star$

$$X = C \times C/\Delta \sim \star.$$

Thus X is birational to $C \times C$. If X/k were eligible for the conclusion of Proposition 3, then the section $s_\star$ associated to $\star \in X(k)$ admits birational lifting and in particular lifts to a birationally lifting section s of $\pi_1(C \times C/k)$. That contradicts Proposition 3 by assumption (ii).

We now forget about assumption (ii) but keep (i). The induced maps on fundamental groups are surjective

$$\pi_1(C \times C) \twoheadrightarrow \pi_1(X) \twoheadrightarrow \pi_1(A).$$

Since the diagonally embedded copy of $\pi_1(C_{\bar{k}})$ goes to zero, the computation in $\ker(\pi_1(f))$

$$(1,\alpha)(\beta,\beta)(1,\alpha)^{-1}(\beta,\beta)^{-1} = (1, \alpha\beta\alpha^{-1}\beta^{-1})$$

shows that we more precisely have

$$\operatorname{coker}\left(\pi_1(\Delta) : \pi_1^{\mathrm{ab}}(C_{\bar{k}}) \to \pi_1^{\mathrm{ab}}(C_{\bar{k}}) \times \pi_1^{\mathrm{ab}}(C_{\bar{k}})\right) \twoheadrightarrow \pi_1(X_{\bar{k}}) \twoheadrightarrow \pi_1(A_{\bar{k}})$$

with the composite map being an isomorphism by geometric class field theory. In particular, we have $\pi_1(X/k) = \pi_1(A/k)$ as extensions. The image of the p-adic analytic map

$$d \, : \, C(k) \times C(k) \to X(k) \to A(k)$$

lies in a 2-dimensional p-adic analytic subspace of the p-adic manifold $A(k)$ of dimension $g \geq 3$. Therefore, the map d is never surjective on k-rational points. Sections s_a of $\pi_1(A/k)$ for k-rational points in $a \in A(k) \setminus X(k)$ are thus sections of $\pi_1(X/k)$ which are not birationally liftable and for which there is no k-rational point on X that is responsible for it.

4. On birational sections and valuations

We keep the notation from above, but now assume X is normal. A neighbourhood of a section $s : \mathrm{Gal}_k \to \mathrm{Gal}_K$ is given by a normal finite branched cover $X' \to X$, i.e., a finite extension K'/K inside $\overline{K}$, such that the image of s lies in $\mathrm{Gal}_{K'} \subseteq \mathrm{Gal}_K$. In particular, X'/k has to be geometrically irreducible. The limit over all these neighbourhoods yields a pro-branched cover again denoted

$$X_s = \varprojlim X' \to X$$

corresponding to $\pi_1(k(X_s)) = s(\mathrm{Gal}_k) \subset \mathrm{Gal}_K$.

Proposition 6. *Let k be a finite extension of $\mathbb{Q}_p$, and let X/k be a normal projective, geometrically irreducible variety. Suppose X/k admits a birational Galois section $s : \mathrm{Gal}_k \to \mathrm{Gal}_K$. Then the set $X_s(k)$ consists of exactly one element.*

Proof: If we can show that the image of $X_s(k) \to X(k)$ consists of only one element, then we can apply this to all neighbourhoods X' of s and achieve the proof of the proposition.

That the image is nonempty follows from the first part of the proof of Proposition 3 above which only requires X/k to be proper.

A point $a \in \operatorname{im}\left(X_s(k) \to X(k)\right)$ necessarily has $j_*(s) = s_a$ as a section of $\pi_1(X/k)$. So let us assume that we have $a \neq b$ in the image. By the projective version of Noether's Normalization Lemma, we can choose a suitable finite map

$$f : X \to \mathbb{P}^n_k$$

with $n = \dim(X)$ by first choosing an immersion $X \hookrightarrow \mathbb{P}^N_k$ followed by a sufficiently generic linear projection. We furthermore assume that $f(a) \neq f(b)$, which is an open condition. Next, we pick our favourite abelian variety A/k of dimension n and apply Noether's Normalization Lemma again to obtain a finite map $A \to \mathbb{P}^n_k$. Let X' be defined as the normalization of X in the compositum of function fields of X and of A over that of $\mathbb{P}^n_k$, i.e., we have a commutative (not necessarily cartesian) square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & A \arrow[d] \\ X \arrow[r, "f"] & \mathbb{P}^n_k. \end{tikzcd}$$

The branched cover $X' \to X$ however, as can be seen by looking at $\mathrm{Gal}_{K'} \subseteq \mathrm{Gal}_K$ where K' is the function field of X', is hardly ever a neighbourhood of s. Nevertheless, we can choose a finite extension k'/k such that $s(\mathrm{Gal}_{k'}) \subset \mathrm{Gal}_{K'}$ which means that a scalar extension to k' (from the field of constants) of X' is a neighbourhood of $s' = s|_{\mathrm{Gal}_{k'}}$ as a section of the birational extension associated to $X \times_k k'$. Now $X_{s'} = X_s \times_k k'$ and the map of rational points factors as

$$X_s(k') \to X'(k') \to X(k'),$$

with $a \neq b$ in the image. Let a' and b' be the intermediate images in $X'(k')$. Then we have $s_{a'} = s_{b'}$ as sections of $\pi_1(X'/k')$, so that $f'(a') = f'(b')$ by the injectivity of the map from rational points to sections for abelian varieties over a p-adic local field recalled in Lemma 7 below. This contradicts our choice of f, namely $f(a) \neq f(b)$, and the proof is complete. □

Lemma 7. *Let $k/\mathbb{Q}_p$ be a finite extension, and let A/k be an abelian variety. If two rational points $a, b \in A(k)$ yield the same section $s_a = s_b : \mathrm{Gal}_k \to \pi_1(A)$, then a equals b.*

Proof: A proof can be found for example in [Sx12b] Proposition 73. We present an alternative for the convenience of the reader. By the translation between Galois sections and Gal_k-equivariant path torsors, see [Sx12b] Proposition 8 and Definition 20, the assumption $s_a = s_b$ means that the space

$$\pi_1(A_{\bar{k}}; a, b)$$

of étale paths on $A_{\bar{k}}$ from a to b, see [SGA1] exposé V §5 or [Sx12b] §2, contains a Gal_k-invariant path. Indeed, the section s_a is more precisely a homomorphism

$$s_a : \mathrm{Gal}_k \to \pi_1(A, a)$$

and similarly for s_b. The choice of an étale path $\gamma \in \pi_1(A_{\bar{k}}; a, b)$ yields an isomorphism

$$\gamma()\gamma^{-1} : \pi_1(A, a) \xrightarrow{\sim} \pi_1(A, b),$$

and the precise meaning of $s_a = s_b$ is that $\gamma()\gamma^{-1} \circ s_a$ is conjugate to s_b under an element of $\pi_1(A_{\bar{k}}, b)$. By modifying γ appropriately, we may assume that

$$\gamma()\gamma^{-1} \circ s_a = s_b.$$

This means that for arbitrary $\sigma \in \mathrm{Gal}_k$ we have

$$\sigma(\gamma) = s_b(\sigma) \circ \gamma \circ s_a(\sigma)^{-1} = s_b(\sigma) \circ s_b(\sigma)^{-1} \circ \gamma = \gamma,$$

and indeed γ is Gal_k-invariant.

The path space, together with its Galois action, is described by natural transformations from the fibre above a to the fibre above b in finite étale covers of $A_{\bar{k}}$. Cofinally these are the multiplication by n maps $[n] : A \to A$. The element $\gamma \in \pi_1(A_{\bar{k}}; a, b)$ is described by $\gamma_n \in A(\bar{k})$ for all $n \in \mathbb{N}$ such that translation by γ_n

$$[n]^{-1}(a) \xrightarrow{\sim} [n]^{-1}(b)$$

$$\alpha \mapsto \alpha + \gamma_n$$

yield bijections that are compatible in n in the tower of all $[n]$, and

$$\gamma = (\gamma_n) \in \varprojlim_n [n]^{-1}(b - a) = \pi_1(A_{\bar{k}}; a, b).$$

Now, if γ is Gal_k-invariant, then $b - a = n(\gamma_n)$ has a Gal_k-invariant nth root $\gamma_n \in A(k)$ for all n. Because $A(k)$ is a topologically finitely generated profinite group by a theorem of Mattuck [Ma55] Theorem 7, therefore

$$b - a \in \bigcap_{n \geq 1} nA(k) = 0,$$

which shows $a = b$ as claimed by the lemma. □

We are now ready to prove the main theorem.

Theorem 8. *Let X/k be a geometrically irreducible, normal, proper variety over a finite extension $k/\mathbb{Q}_p$. Let K be the function field of X. Then the image of every section of the restriction map*

$$\mathrm{Gal}_K \to \mathrm{Gal}_k$$

is contained in the decomposition subgroup $D_{\bar{v}} \subset \mathrm{Gal}_K$ for a unique k-valuation $\bar{v}$ of $\overline{K}$ with residue field of $v = \bar{v}|_K$ equal to k.

In particular, conjugacy classes of sections of $\mathrm{Gal}_K \to \mathrm{Gal}_k$ come in disjoint non-empty packets associated to each k-valuation v of K with residue field k.

Proof: We first show that for every k-valuation v of K the set of birational Galois sections with image in D_v is nonempty. Put differently, we need to show that the natural projection $D_v \to \mathrm{Gal}_k$ splits. This follows from the well know fact of general valuation theory that the corresponding extension

$$1 \to I_v \to D_v \to \mathrm{Gal}_k \to 1$$

with the inertia group I_v of v splits, see for example [HJP07] Proposition 7.3(a).

We now show that every birational Galois section s belongs to a packet of a k-valuation. By Chow's Lemma and Hironaka's resolution of singularities, we may assume that X/k is smooth and projective. For every normal, birational $X' \to X$ the section s gives rise to a tower of branched neighbourhoods that are again linked by a natural map $X'_s \to X_s$. The map that assigns to a k-valuation of $\overline{K}$ its center on a birational model for a finite branched cover of X leads to a bijection

$$\left\{ \bar{v} \ ; \ \begin{array}{c} k\text{-valuation on } \overline{K} \text{ with} \\ \text{residue field } \bar{k} \end{array} \right\} \xrightarrow{\sim} \varprojlim_{X' \to X} X'_s(\bar{k})$$

since $X'_s(\bar{k}) = (X_s \times_k \bar{k})(\bar{k})$ and $X'_s \times_k \bar{k}$ is the normalization of X'_s in $\overline{K}$. Indeed, the assumption on the residue field of the valuation implies that the center is a closed point on every birational model.

By definition of the decomposition group, a valuation $\bar{v}$ has $s(\mathrm{Gal}_k) \subseteq D_{\bar{v}}$ if and only if the action of Gal_k via s on the set of all valuations fixes $\bar{v}$. But the set of such fixed points is precisely

$$\varprojlim_{X' \to X} X'_s(k),$$

which is a set of cardinality one by Proposition 6. $\square$

5. An alternative proof based on valuation theory

Theorem 8 also admits the following valuation theoretic proof[1] and generalization to arbitrary field extensions. This generalization could also be obtained by a limit argument from Theorem 8.

Theorem 9. *Let F/k be a nontrivial field extension of a finite extension $k/\mathbb{Q}_p$. Then every section of the restriction map*

$$\mathrm{res}_{F/k} : \mathrm{Gal}_F \to \mathrm{Gal}_k$$

is contained in the decomposition group $D_{\bar{v}}$ for a unique k-valuation $\bar{v}$ of the algebraic closure $\overline{F}$ of F with k as residue field of $\bar{v}|_F$.

Proof: We may assume that F/k is a regular extension because otherwise $\mathrm{res}_{F/k}$ is not surjective and there is no section, thus nothing to be proven. In particular, since $F \neq k$, the extension F/k is not algebraic.

Let $s : \mathrm{Gal}_k \to \mathrm{Gal}_F$ be a section of $\mathrm{res}_{F/k}$. Denote the fixed field of $s(\mathrm{Gal}_k)$ in $\overline{F}$ by L. Note that $F \subseteq L \subseteq \overline{F}$ is an algebraic closure of L, so $\mathrm{Gal}_L = s(\mathrm{Gal}_k)$ as subgroups of Gal_F. Then

[1] I am grateful to Jochen Koenigsmann for allowing me to include his observation of an alternative valuation theoretic proof and to Moshe Jarden for his assistance with the details of the proof.

restriction $\mathrm{Gal}_L \xrightarrow{\sim} \mathrm{Gal}_k$ is an isomorphism, and we conclude[2] as in the proof of Lemma 1 that the field L is p-adically closed with respect to a p-adic valuation w_L that extends the p-adic valuation w_k on k. Since k is relatively algebraically closed in L, the residue field of L with respect to w_L coincides with the residue field of k with respect to w_k, see [PR84] page 38. The value group of w_k is $\mathbb{Z}$ and the value group of w_L contains a unique convex subgroup isomorphic to $\mathbb{Z}$ that contains the value $w_L(p)$, and in this sense

$$w_L(p) = w_k(p) \in \mathbb{N},$$

see [PR84] page 38. Let $\mathcal{O}_{w_L}$ be the valuation ring of w_L and let $\mathfrak{o}_k$ be the valuation ring of w_k. It follows by induction on n that for every $n \in \mathbb{N}$ the inclusion $\mathfrak{o}_k \subset \mathcal{O}_{w_L}$ induces isomorphisms

$$\mathfrak{o}_k/p^n\mathfrak{o}_k \xrightarrow{\sim} \mathcal{O}_{w_L}/p^n\mathcal{O}_{w_L}.$$

In the limit we find an $\mathfrak{o}_k$-algebra homomorphims

$$\varphi : \mathcal{O}_{w_L} \twoheadrightarrow \varprojlim_n \mathcal{O}_{w_L}/p^n\mathcal{O}_{w_L} \cong \varprojlim_n \mathfrak{o}_k/p^n\mathfrak{o}_k = \mathfrak{o}_k$$

due to the p-adic completeness of the valuation w_k. The map φ is surjective by $\varphi(\mathfrak{o}_k) = \mathfrak{o}_k$.

Now let v_L be the *relative rank-1* coarsening of w_L, i.e., the unique k-valuation of L with valuation ring

$$\mathcal{O}_{v_L} = \mathcal{O}_{w_L}[1/p],$$

see [PR84] page 27. The homomorphism φ extends to a surjective k-algebra homomorphism

$$\tilde{\varphi} : \mathcal{O}_{v_L} = \mathcal{O}_{w_L}[1/p] \twoheadrightarrow \mathfrak{o}_k[1/p] = k,$$

that identifies k with the residue field of v_L. It follows that the restriction $v = v_L|_F$ has also residue field k. Moreover, the morphism $\tilde{\varphi}$ shows $\mathcal{O}_{v_L} \neq L$, since $L \neq k$ (otherwise F/k were algebraic), and thus the valuation v_L is nontrivial.

By [PR84] Theorem 3, the p-adically closed valuation w_L is henselian. As a nontrivial coarsening of a henselian valuation, v_L is henselian as well, see [Ja91] Proposition 13.1. With the unique extension $\bar{v}$ of v_L to $\overline{F}$ we have

$$s(\mathrm{Gal}_k) = \mathrm{Gal}_L \subseteq D_{\bar{v}},$$

showing the existence of a k-valuation $\bar{v}$ as in the theorem.

In order to prove uniqueness of $\bar{v}$ we assume that we have a further valuation $\bar{u}$ of $\overline{F}$ with

$$s(\mathrm{Gal}_k) \subseteq D_{\bar{u}} \tag{5.1}$$

such that $u = \bar{u}|_F$ has residue field k. Let $u_L = \bar{u}|_L$ be the restriction to L. Now L admits the henselian valuation v_L with non-separably closed residue field k and, by (5.1), the valuation u_L is also henselian. It follows from a theorem of F. K. Schmidt and Engler, [En78] Corollary 2.6, that v_L and u_L are comparable valuations, more precisely, one is a coarsening of the other.

If u_L is a coarsening of v_L, then, by the assumption on the residue fields of u and v, the quotient valuation $\nu = v_L/u_L$ on the residue field of u_L is a henselian valuation on an algebraic extension of k with residue field k. This forces ν to be trivial and therefore $v_L = u_L$. If v_L is a coarsening of u_L, then the quotient valuation $\nu = u_L/v_L$ is a henselian valuation on k with a residue field that is algebraic over k. Again ν must be trivial and $v_L = u_L$.

The uniqueness of the extension to $\overline{F}$ of a henselian valuation on L shows $\bar{v} = \bar{u}$ as claimed by the theorem. □

[2] An algebraic proof based on p-rigid elements of the existence of the p-adic valuation on L follows from [Ko95] Theorem 4.1.

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