# FUNDAMENTAL GROUPS OF PROPER VARIETIES ARE FINITELY PRESENTED

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ABSTRACT. It was proven in [ESS22], that the étale fundamental group of a connected smooth projective variety over an algebraically closed field $k$ is topologically finitely presented. In this note, we extend this result to all connected proper schemes over $k$.

## 1. INTRODUCTION

For a connected algebraic variety $X$ over an algebraically closed field $k$ of characteristic 0, the étale fundamental group $\pi_1^{\text{ét}}(X, \bar{x})$ of $X$ is a topologically finitely presented profinite group. This is proven by first reducing to the case of $k = \mathbf{C}$ and then applying the Riemann Existence Theorem [SGA1, Exp. XII, Thm. 5.1] together with the fact that the topological fundamental group $\pi_1^{\text{top}}(X(\mathbf{C}), x)$ is of finite presentation as a discrete group (see e.g. [Łoj64] or [Hir75]).

In characteristic $p > 0$, the picture is much more subtle due to the existence of Artin-Schreier covers of affine schemes, which makes $\pi_1^{\text{ét}}(\text{Spec}(A), \bar{x})$ typically not even finitely generated. Remark 5.7 of [SGA1, Exp. IX] raised doubts whether $\pi_1^{\text{ét}}(X, \bar{x})$ is finitely presented for proper varieties, even for proper smooth curves. In recent work, however, Shusterman (in the case of curves [Shu22]), and Esnault, Shusterman and the second named author [ESS22] have shown that for smooth projective varieties $\pi_1^{\text{ét}}(X, \bar{x})$ is still finitely presented. Both results are based on a criterion for finite presentation of profinite groups due to Lubotzky [Lub01].

**Theorem 1.1** (part of Thm. 1.1 of [ESS22]). *Let $X$ be a connected smooth projective variety over an algebraically closed field $k$. Then the étale fundamental group $\pi_1^{\text{ét}}(X, \bar{x})$ is topologically finitely presented.*

Our goal is to generalize Thm. 1.1 to all connected schemes that are proper over $\text{Spec}(k)$ (which is new only if $k$ has characteristic $p > 0$). Such a generalization responds affirmatively to a question raised by Esnault.

**Theorem 1.2.** *Let $X$ be a connected scheme that is proper over* $\text{Spec}(k)$ *for an algebraically closed field $k$. Then $\pi_1^{\text{ét}}(X, \bar{x})$ is topologically finitely presented.*

To prove the theorem, we use descent along an alteration map to $X$ and the van Kampen presentation of $\pi_1^{\text{ét}}(X, \bar{x})$ arising in this way. More precisely, we use this trick twice.

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## 2. The proof

For a scheme $T$, let $\mathsf{F\acute{E}t}_T$ denote the category of finite étale covers of $T$. This gives rise to a category fibred over schemes. Recall that a morphism $g : T' \to T$ of schemes is said to be of effective descent for $\mathsf{F\acute{E}t}$, if $g^*$ induces an equivalence of categories between $\mathsf{F\acute{E}t}_T$ and the category of descent data in $\mathsf{F\acute{E}t}$ along $g$.

**Proposition 2.1** (Exp. IX, Thm. 4.12 of [SGA1])**.** *Let $f : X' \to X$ be a proper surjective morphism of finite presentation. Then $f$ is of effective descent for $\mathsf{F\acute{E}t}$.*

Morphisms of effective descent $f : X' \to X$ for $\mathsf{F\acute{E}t}$ give rise to a van Kampen-like presentation of $\pi_1^{\text{ét}}(X, \bar{x})$ as the profinite completion of a quotient of the free topological product of the étale fundamental groups of the connected components of $X'$ and the usual topological fundamental group of a suitably defined "dual graph". This goes back to [SGA1, Exp. IX, § 5] and has been worked out in detail in [Sti06, Cor. 5.3].

The existence of such a presentation allows one to "descend" finite generation/presentation of the fundamental groups involved, as made precise in the following proposition. Every statement below is about *topological* finite generation/presentation. We will omit the word "topological" and the base points from now on.

**Proposition 2.2** (Exp. IX, Cor. 5.2 + Cor. 5.3 of [SGA1])**.** *Let $f : X' \to X$ be a morphism of effective descent for $\mathsf{F\acute{E}t}$. We denote $X' \times_X X'$ by $X''$ and $X' \times_X X' \times_X X'$ by $X'''$.*

- *(a) Assume that $X', X''$ have finite $\pi_0$'s and that $\pi_1^{\text{ét}}$'s of the connected components of $X'$ are finitely generated. Then $\pi_1^{\text{ét}}(X)$ is finitely generated.*
- *(b) Assume that $X', X'', X'''$ have finite $\pi_0$'s, that $\pi_1^{\text{ét}}$'s of the connected components of $X'$ are of finite presentation and that $\pi_1^{\text{ét}}$'s of the connected components of $X''$ are finitely generated. Then $\pi_1^{\text{ét}}(X)$ is of finite presentation.*

Let us now recall a result of de Jong specialized to our setting.

**Proposition 2.3** (see Thm. 4.1 of [dJ96])**.** *Let $X$ be a scheme that is proper over* $\operatorname{Spec}(k)$ *for an algebraically closed field $k$. Then there exists a proper surjective morphism (of finite presentation) $f : X' \to X$ from a smooth projective variety $X'$ over* $\operatorname{Spec}(k)$.

*Proof.* Let $\nu : X^\nu \to X$ be the normalization of $X$. It is finite, and thus the scheme $X^\nu$ is still proper over $\operatorname{Spec}(k)$. Let then $f_1 : X' \to X^\nu$ be the alteration map of [dJ96, Thm. 4.1] applied to each connected component of $X^\nu$. The map $f_1$ is proper, dominant and thus surjective. Moreover, *loc. cit.* guarantees that $X'$ is regular and projective (and not merely proper!) over $\operatorname{Spec}(k)$. Now, as $k$ is algebraically closed, $X'$ is smooth over $\operatorname{Spec}(k)$. The composition $f = \nu \circ f_1 : X' \to X$ has all the requested properties. □

We are now ready to finish the proof of the main result.

*Proof of Thm. 1.2.* Take $f : X' \to X$ as in Prop. 2.3. By Thm. 1.1 and Prop. 2.1 the map $f$ satisfies the assumptions for Prop. 2.2(a). This shows that $\pi_1^{\text{ét}}(X)$ is finitely generated, for any scheme $X$ that is proper over $\operatorname{Spec}(k)$. In fact, finite generation was already proven in [SGA1, Exp. X, Thm. 2.9], and we included the argument here for the convenience of the reader.

We are going to apply finite generation to the connected components of $X'' = X' \times_X X'$, which are connected proper schemes over $\operatorname{Spec}(k)$. Indeed, using Thm. 1.1 (this time crucially!) and Prop. 2.1 again, the map $f$ now satisfies the assumptions of Prop. 2.2(b). This shows that $\pi_1^{\text{ét}}(X)$ is finitely presented. □

## 3. More general base fields

Similarly to [ESS22, §5], our main result extends to more arithmetic settings. We thank Peter Haine for essentially suggesting the following corollary.

**Corollary 3.1.** *Let $X$ be a connected scheme that is proper over* $\mathrm{Spec}(k)$ *for a field $k$. Then* $\pi_1^{\text{ét}}(X)$ *is finitely presented if and only if the absolute Galois group* $\mathrm{Gal}_k$ *is finitely presented.*

*Proof.* We may assume $X$ is reduced and thus $k' = \mathrm{H}^0(X, \mathcal{O}_X)$ is a finite field extension of $k$. Let $\bar{k}$ be an algebraic closure of $k$ containing $k'$. Then $X \to \mathrm{Spec}(k')$ being the Stein factorization of $X \to \mathrm{Spec}(k)$ implies that $\bar{X} = X \times_{k'} \bar{k}$ is connected. By Thm. 1.2, the group $\pi_1(\bar{X})$ is finitely presented. The fundamental exact sequence [SGA1, Exp. IX, Thm. 6.1]

$$1 \to \pi_1(\bar{X}) \to \pi_1(X) \to \mathrm{Gal}_{k'} \to 1$$

shows that $\pi_1(X)$ is finitely presented if and only if $\mathrm{Gal}_{k'}$ is finitely presented. The latter is equivalent to $\mathrm{Gal}_k$ being finitely presented, see for example [ESS23, Prop. 2.3]. □

**Remark 3.2.** Examples of fields $k$ with finitely presented $\mathrm{Gal}_k$ include: fields algebraic over a finite field, local $p$-adic fields, $\mathbb{R}$ and more generally real closed fields, $K((t))$ for a field of characteristic 0 with $\mathrm{Gal}_K$ of finite presentation, and by [Jar74, Thm. 5.1] for a hilbertian field $k$, probabilistically almost always (for the Haar measure on $\mathrm{Gal}_k$) the fixed field $k^\Sigma$ in the separable closure $\bar{k}$ of a finite subset $\Sigma \subset \mathrm{Gal}_k$.

## References


[dJ96] Aise Johan de Jong. Smoothness, semi-stability and alterations. *IHES Publ. Math.*, 83:51–93, 1996.

[ESS22] Hélène Esnault, Mark Shusterman, and Vasudevan Srinivas. Finite presentation of the tame fundamental group. *Selecta Math. (N.S.)*, 28(2):Paper No. 37, 19, 2022.

[ESS23] Hélene Esnault, Vasudevan Srinivas, and Jakob Stix. An obstruction to lifting to characteristic 0. *Algebraic Geometry*, 10(3):327–347, 2023.

[Hir75] Heisuke Hironaka. Triangulations of algebraic sets. In *Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Arcata, Calif., 1974)*, pages 165–185. Amer. Math. Soc., Providence, R.I., 1975.

[Jar74] Moshe Jarden. Algebraic extensions of finite corank of Hilbertian fields. *Israel J. Math.*, 18:279–307, 1974.

[Łoj64] Stanisław Łojasiewicz. Triangulation of semi-analytic sets. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 18:449–474, 1964.

[Lub01] Alexander Lubotzky. Pro-finite presentations. *J. Algebra*, 242(2):672–690, 2001.

[SGA1] *Revêtements étales et groupe fondamental.* Lecture Notes in Mathematics, Vol. 224. Springer-Verlag, Berlin-New York, 1971. Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1), Dirigé par Alexandre Grothendieck. Augmenté de deux exposés de Michèle Raynaud.

[Shu22] Mark Shusterman. Balanced presentations for fundamental groups of curves over finite fields. *Mathematical Research Letters*, 29(4):1251–1259, 2022.

[Sti06] Jakob Stix. A general Seifert-Van Kampen theorem for algebraic fundamental groups. *Publ. Res. Inst. Math. Sci.*, 42(3):763–786, 2006.



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