Étale contractible varieties in positive characteristic

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Abstract — Unlike in characteristic 0, there are no non-trivial smooth varieties over an algebraically closed field k of characteristic $p > 0$ that are contractible in the sense of étale homotopy theory.

Introduction

Homotopy theory is founded on the idea of contracting the interval, either as a space, or as an actual homotopy, i.e., a path in a space of maps. In algebraic geometry, the affine line $\mathbb{A}^1_k$ serves as an algebraic equivalent of the interval, at least in characteristic 0.

Matters differ in characteristic $p > 0$ where $\pi_1(\mathbb{A}^1_k)$ is an infinite group: a group G occurs as a finite quotient of $\pi_1(\mathbb{A}^1_k)$ precisely if G is a quasi-p group[1] due to Abhyankar's conjecture for the affine line as proven by Raynaud. This raises the question whether there is an étale contractible variety in positive characteristic.

Theorem 1. *Let k be an algebraically closed field of characteristic $p > 0$ and let U/k be a smooth variety. Then U is étale contractible, if and only if $U = \mathrm{Spec}(k)$ is the point.*

We recall the relevant terminology and results of étale homotopy from [AM69].

Definition 2. Let k be an algebraically closed field.

(1) A variety U over k is **(étale) contractible** if the map $U_{\text{ét}} \to \mathrm{Spec}(k)_{\text{ét}}$ is a weak equivalence of its étale homotopy type $U_{\text{ét}}$ with $\mathrm{Spec}(k)_{\text{ét}}$.

(2) A variety U over k is **(étale) n-connected** for an $n \in \mathbb{N}$, if its étale homotopy groups $\pi_i^{\text{ét}}(U)$ vanish for all $i \leq n$.

The étale Hurewicz and Whitehead theorems, see [AM69, §4], lead to the following equivalent characterisations. The variety U/k is contractible if and only if it is n-connected for all $n \in \mathbb{N}$. Moreover, for $n \geq 1$, a normal variety U/k is n-connected if and only if

(i) U is connected, and

(ii) U is simply connected: $\pi_1^{\text{ét}}(U) = 1$, and

(iii) U is cohomologically acyclic in degrees $\leq n$: for all $0 < i \leq n$ the groups $\mathrm{H}^i_{\text{ét}}(U, A)$ vanish for all finite abelian groups A.

It turns out that our discussion in positive characteristic uses only H^1 and H^2, and moreover covers more than just smooth varieties. Here is the more precise result which proves Theorem 1 because smooth varieties are locally factorial and hence locally $\mathbb{Q}$-factorial everywhere.

Theorem 3. *Let k be an algebraically closed field of characteristic $p > 0$ and let U/k be a 2-connected normal variety such that one of the following holds:*

(a) U *is quasi-projective, or*

(b) U *is locally $\mathbb{Q}$-factorial everywhere, or*

(c) $\dim(U) \leq 2$.

Then U has dimension 0.

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[1] A finite group is a **quasi-p group** if it is generated by its p-Sylow subgroups.

More precisely, we can prove the following Theorem 4 that has Theorem 3 with the assumptions (a) or (b) as a special case. Proposition 6 shows the existence of big Cartier divisors in this case. We recall big Cartier divisors on general varieties in Section §1.

Theorem 4. *Let k be an algebraically closed field of characteristic $p > 0$ and let U/k be a connected normal variety with a big Cartier divisor and such that*

(i) the group $\mathrm{H}^1_{\text{ét}}(U, \mathbb{F}_p)$ vanishes, and
(ii) there is a prime number $\ell \neq p$ such that $\mathrm{H}^2_{\text{ét}}(U, \mu_\ell) = 0$.

Then U has dimension 0.

The proof of Theorem 3 with the assumption (c) will be completed in Section 3 for normal surfaces, the case of normal curves being covered by assumption (a).

In the proof of Theorem 4 one would like to work with a completion $U \subseteq X$ and the geometry of line bundles on U versus X. For that strategy to work, we need a completion that is locally factorial along $Y = X \setminus U$. Since in characteristic $p > 0$ resolution of singularities is presently absent in dimension ≥ 4 we resort to desingularisation by alterations due to de Jong. Unfortunately, the alteration typically destroys the étale contractibility assumption. The strategy consists in first deducing more *coherent* properties from étale 2-connectedness that can be transferred to the alteration.

The key difference with characteristic 0 comes from Artin–Schreier theory relating $\mathrm{H}^1_{\text{ét}}(U, \mathbb{F}_p)$ to global rational functions.

Remark 5. Some further comments on the situation in characteristic 0 in contrast to Theorem 1:

(1) In characteristic 0 smoothness seems to be a crucial property to get an interesting classification problem. Indeed, let $k = \mathbb{C}$, then the affine cone U over any projective variety has a $\mathbb{C}^*$-action with the cone point $0 \in U$ as its only attracting fixed point. It follows that $U(\mathbb{C})$ is homotopy equivalent to 0, and there are far too many (singular) contractible varieties for a manageable classification.
(2) There are contractible complex smooth surfaces other than $\mathbb{A}^2_{\mathbb{C}}$, the first such example is due to Ramanujam [Ra71, §3], see also tom Dieck and Petrie [tDP90] for explicit equations. All of them are affine and have rational smooth projective completions.
(3) Smooth varieties $U/\mathbb{C}$ different from affine space $\mathbb{A}^n_{\mathbb{C}}$ but with $U(\mathbb{C})$ diffeomorphic to $\mathbb{C}^n$ are known as exotic algebraic structures on $\mathbb{C}^n$. These varieties are contractible and we recommend the Bourbaki talk on $\mathbb{A}^n$ by Kraft [Kr94], or the survey by Zaĭdenberg [Za99]. A remarkable non-affine (but quasi-affine) example U was obtained by Winkelmann [Wi90] as a quotient $U = \mathbb{A}^5/\mathbb{G}_a$ and more concretely as the complement in a smooth projective quadratic hypersurface in $\mathbb{P}^5_{\mathbb{C}}$ of the union of a hyperplane and a smooth surface.

Notation. We keep the following notation throughout the note: k will be an algebraically closed field. By definition, a variety over k is a separated scheme of finite type over k. We will denote the étale fundamental group by π_1 and its maximal abelian quotient by π_1^{ab}. The sheaf μ_ℓ for ℓ different from the characteristic denotes the (locally) constant sheaf of ℓ-th roots of unity.

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1. Big divisors on varieties

Recall that for a Cartier divisor D on a normal variety U/k, its **Iitaka dimension** is

$$\kappa(D) = \begin{cases} \max_{m \in \mathbb{N}(D)} \{\dim(\varphi_{|mD|}(U))\} & \text{if } \mathbb{N}(D) = \{m \in \mathbb{N} \,;\, |mD| \neq \varnothing\} \neq \varnothing, \\ -\infty & \text{otherwise.} \end{cases}$$

Here $\varphi_{|mD|}$ denotes the rational map associated to the linear system $|mD|$, and $\varphi_{|mD|}(U)$ denotes the closure of its image. A Cartier divisor is **big** if

$$\kappa(D) = \dim(U),$$

i.e., if $\varphi_{|mD|}$ is generically finite for $m \gg 0$. The pullback of a big divisor under a generically finite morphism is obviously big itself.

Proposition 6. *Let k be an algebraically closed field and let U/k be a variety such that one of the following holds:*

(a) *U is quasi-projective.*
(b) *U is normal and locally $\mathbb{Q}$-factorial everywhere.*

Then U has a big Cartier divisor.

Proof. Let us first assume (a). Since any ample divisor is big, the conclusion holds.

If (b) holds, then we first choose a dense affine open $V \subseteq U$ and an effective big Cartier divisor D on V by the above, since V is quasi-projective. Let $B = U \setminus V$ be the boundary, in fact a Weil-divisor since V is affine, and let D' be the Zariski closure of D as a Weil divisor on U. By assumption (b) there is an $m \geq 1$ such that mD' and mB are both effective Cartier divisors and there are sections $s_0, \ldots, s_d \in \mathrm{H}^0(V, mD)$ such that the induced map $V \to \mathbb{P}^d_k$ is generically finite. For $r \gg 0$ the sections s_i extend to sections of

$$\mathrm{H}^0(U, mD + mrB)$$

so that $mD + mrB$ is the desired big Cartier divisor on U. □

2. Geometry of varieties with vanishing H^1 and H^2

2.1. **Line bundles.** Let U be a variety over k with $\mathrm{H}^2_{\text{ét}}(U, \mu_\ell) = 0$ for some prime number ℓ different from the characteristic of k. The Kummer sequence $0 \to \mu_\ell \to \mathbb{G}_m \to \mathbb{G}_m \to 0$ on U yields in étale cohomology the exact sequence

$$\mathrm{Pic}(U) \xrightarrow{\ell} \mathrm{Pic}(U) \to \mathrm{H}^2_{\text{ét}}(U, \mu_\ell) = 0.$$

So $\mathrm{Pic}(U)$ is an ℓ-divisible abelian group.

2.2. **Regular functions.** The following argument crucially depends on k being a field of positive characteristic.

Proposition 7. *Let k be of characteristic $p > 0$ and let U/k be a connected variety such that $\pi_1^{\mathrm{ab}}(U) \otimes \mathbb{F}_p$ is finite. Then there is no non-constant map $f : U \to \mathbb{A}^1_k$. In other words*

$$\mathrm{H}^0(U, \mathcal{O}_U) = k.$$

Proof. We argue by contradiction and assume that there is a dominant map $f : U \to \mathbb{A}^1_k$. Then the induced map

$$f_* : \pi_1^{\mathrm{ab}}(U) \otimes \mathbb{F}_p \to \pi_1^{\mathrm{ab}}(\mathbb{A}^1_k) \otimes \mathbb{F}_p$$

has image of finite index in the infinite group $\pi_1^{\mathrm{ab}}(\mathbb{A}^1_k) \otimes \mathbb{F}_p$, a contradiction. □

By the tautological duality $\mathrm{H}^1_{\text{ét}}(U, \mathbb{F}_p) = \mathrm{Hom}(\pi_1^{\mathrm{ab}}(U), \mathbb{F}_p)$ the vanishing of $\mathrm{H}^1_{\text{ét}}(U, \mathbb{F}_p)$ implies the assumption of Proposition 7.

2.3. **Using alterations.** Clearly, the content of Sections §2.1 and §2.2 reduce the proof of Theorem 4 to the following proposition.

Proposition 8. *Let k be an algebraically closed field and let U/k be a connected normal variety with a big Cartier divisor and such that*

(i) $\mathrm{H}^0(U, \mathcal{O}_U) = k$, *and*
(ii) *there is a prime number ℓ such that $\mathrm{Pic}(U)$ is ℓ-divisible.*

Then U has dimension 0.

Proof. By [dJ96, Theorem 7.3], there exists an alteration, i.e, a generically finite projective map $h : \tilde{U} \to U$ such that $\tilde{U}$ can be embedded into a connected smooth projective variety $\tilde{X}$.

Step 1: Since U is normal, the maximal open $V \subset U$ such that the restriction

$$h|_{\tilde{V}} : \tilde{V} = \pi^{-1}(V) \to V$$

is a finite map has boundary $U \setminus V$ of codimension at least 2.

The k-algebra $\mathrm{H}^0(\tilde{V}, \mathcal{O}_{\tilde{V}})$ is an integral domain inside the function field of $\tilde{V}$. The minimal polynomial for a section $s \in \mathrm{H}^0(\tilde{V}, \mathcal{O}_{\tilde{V}})$ with respect to the function field of V has coefficients that are regular functions on V by normality and uniqueness of the minimal polynomial. Hence these coefficients are elements of $\mathrm{H}^0(V, \mathcal{O}_V) = \mathrm{H}^0(U, \mathcal{O}_U) = k$, and so

$$\mathrm{H}^0(\tilde{V}, \mathcal{O}_{\tilde{V}}) = k.$$

Step 2: The Picard scheme of $\tilde{X}$ exists by [Kl05, Theorem 4.18.2] and satisfies the theorem of the base by [Kl71, Theorem 5.1], see also [Kl05, Theorem 6.16 and Remark 6.19], namely the Néron–Severi group

$$\mathrm{NS}(\tilde{X}) = \mathrm{Pic}(\tilde{X}) / \mathrm{Pic}^0(\tilde{X})$$

is a finitely generated abelian group. Since the restriction map $\mathrm{Pic}(\tilde{X}) \twoheadrightarrow \mathrm{Pic}(\tilde{U})$ is surjective the induced composite map

$$h^* : \mathrm{Pic}(U) \to \mathrm{coker}\left(\mathrm{Pic}^0(\tilde{X}) \to \mathrm{Pic}(\tilde{U})\right) \tag{2.1}$$

maps an ℓ-divisible group to a finitely generated abelian group, hence has finite image of order prime to ℓ.

Step 3: Let D be a big Cartier divisor on U. Since $h : \tilde{U} \to U$ is generically finite, also the divisor h^*D is a big Cartier divisor on $\tilde{U}$. Moreover, as in the proof of Proposition 6, there is a big divisor $\tilde{D}$ on $\tilde{X}$ that restricts to h^*D on $\tilde{U}$. Upon replacing D and $\tilde{D}$ by a positive multiple we may assume by the finiteness of the image of the map (2.1) that $\tilde{D}$ is algebraically and thus numerically equivalent to a divisor B on $\tilde{X}$ that is supported in $\tilde{X} \setminus \tilde{U}$.

Since bigness on projective varieties only depends on the numerical equivalence class, see [Laz04, Corollary 2.2.8], the divisor B is also big. But by restriction to $\tilde{V}$

$$\bigcup_{n \geq 0} \mathrm{H}^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(nB)) \subseteq \mathrm{H}^0(\tilde{V}, \mathcal{O}_{\tilde{V}}) = k$$

by Step 1 above, and we conclude that $\dim(U) = \dim(\tilde{X}) = \kappa(B) \leq 0$. □

2.4. **Complementing examples.** We present a few examples that illustrate the assumptions in Theorem 4 or Proposition 8 or properties of the variety U in question that can easily be derived from the cohomology vanishing assumption.

2.4.1. *No non-constant functions and no proper curves.* If we assume that U is normal and quasi-projective and the assumptions (i) and (ii) of Proposition 8 hold, then U cannot contain a curve $C \hookrightarrow U$ that is proper over k (by arguments that are more elementary than the proof of Proposition 8). Indeed, let $\mathcal{L}$ be an ample line bundle on U, so that $\deg \mathcal{L}|_C > 0$. On the other hand, we have $\deg \mathcal{L}|_C = 0$, because $\mathcal{L} \in \mathrm{Pic}(U)$ is ℓ-divisible and zero is the only ℓ-divisible value in $\mathbb{Z}$, contradiction.

It is therefore tempting to raise the following question: is necessarily $U = \mathrm{Spec}(k)$ for a connected normal variety U/k over an algebraically closed field k such that

(i) $\mathrm{H}^0(U, \mathcal{O}_U) = k$, and
(ii) there is no non-constant map $C \to U$ from a proper smooth curve C/k?

The answer is no as the following example based on work of Totaro shows.

Example 9. Let k be the algebraic closure of $\mathbb{F}_p$ and consider a smooth curve $D \hookrightarrow \mathbb{P}^1_k \times \mathbb{P}^1_k$ of bi-degree $(2,3)$. Let Y be the strict transform of D in the blow-up $\sigma : X \to \mathbb{P}^1_k \times \mathbb{P}^1_k$ of 12 points on D. Let U be the complement $X \setminus Y$. For a choice of parameters (the curve D and the 12 points) in a non-empty Zariski open of the space of all parameters, Totaro shows in [To09] that

$$\mathrm{H}^0(U, \mathcal{O}_U) = \bigcup_{n \geq 0} \mathrm{H}^0(X, \mathcal{O}_X(nY)) = k$$

contains only the constants, and Y is nef. In fact, Y meets all curves except those $C \hookrightarrow X$ such that $\sigma(C)$ meets D only in a subset of the 12 points and only at most transversally. Let $\sigma(C)$ be such a curve of bi-degree (a,b) moving in a family of dimension $(a+1)(b+1)-1$. Then $2b + 3a = (\sigma(C) \cdot Y) \leq 12$ and for generic parameters the subspace with the described intersection locus has dimension

$$d = (a+1)(b+1) - 1 - (2b+3a) = (a-1)(b-2) - 2.$$

In the range $0 < 2b + 3a \leq 12$ and $a, b \geq 0$ we have $d < 0$ and so there are simply no such curves C for generic parameter values.

Nevertheless, this U is smooth and thus fails to be étale contractible by Theorem 1.

2.4.2. *Absence of big divisors.* The condition in Theorem 4 that U contains a big divisor cannot be omitted.

We first recall two facts about complete toric varieties that are standard analytically over $\mathbb{C}$ and which have étale counterparts for toric varieties over arbitrary algebraically closed base fields, in particular of characteristic $p > 0$.

Lemma 10. *Let k be an algebraically closed field. Any complete toric variety X/k is étale simply connected: $\pi_1(X) = 1$.*

Proof. By toric resolution, see [CLS11, §11.1], there is a resolution of singularities $\tilde{X} \to X$ with a smooth projective toric variety $\tilde{X}$. Since $\tilde{X}$ is rational, birational invariance of the étale fundamental group shows $\pi_1(\tilde{X}) = \pi_1(\mathbb{P}^n_k) = 1$, and the surjection $\pi_1(\tilde{X}) \twoheadrightarrow \pi_1(X)$ shows that X is étale 1-connected. □

Lemma 11. *Let k be an algebraically closed field of characteristic p, and let X/k be a complete toric variety. Then for all $\ell \neq p$ we have*

$$\mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_\ell(1)) \simeq \mathrm{Pic}(X) \otimes \mathbb{Z}_\ell.$$

Proof. In the context of toric varieties over $\mathbb{C}$ and with respect to singular cohomology this is [CLS11, Theorem 12.3.3]. The ℓ-adic case for toric varieties over an algebraically closed field k of characteristic $\neq \ell$ follows with a parallel proof. □

The examples showing that a big divisor is essential in Theorem 4 come from toric geometry.

Example 12. Let $U = X$ be a complete normal non-projective toric variety X of dimension 3 with trivial Picard group. Such toric varieties have been constructed in [Eik92, Example 3.5], or [Ful93, pp. 25–26, 65]. These sources construct X over $\mathbb{C}$ but the constructions work mutatis mutandis over any algebraically closed base field k. Then

(i) $\mathrm{H}^1_{\text{ét}}(X, \mathbb{F}_p) = 0$ by Lemma 10, and
(ii) $\mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_\ell(1)) = 0$ for all $\ell \neq p$ by Lemma 11, and since there is non-trivial torsion in ℓ-adic cohomology only for finitely many primes [Ga83], we conclude that $\mathrm{H}^2(X, \mu_\ell) = 0$ for almost all $\ell \neq p$.

Therefore the assumptions of Theorem 4 hold with the exception of the presence of a big Cartier divisor. Nevertheless, these toric varieties are not étale contractible since $\mathrm{H}^6_{\text{ét}}(X, \mathbb{Z}_\ell(3)) = \mathbb{Z}_\ell$.

Of course, also the geometric assumptions of Proposition 8 hold for the above toric variety examples:

(i) $\mathrm{H}^0(X, \mathcal{O}_X) = k$, and
(ii) $\mathrm{Pic}(X)$ is ℓ-divisible for some prime number $\ell \neq p$,

since these were deduced from the étale cohomological vanishing without the help of a big Cartier divisor. Yet again, these examples are showing that one cannot conclude $X = \mathrm{Spec}(k)$ in Proposition 8 without a big Cartier divisor.

3. Normal surfaces

In this section we give a proof of Theorem 3 in the case of assumption (c) for surfaces. Not every normal surface admits a big Cartier divisor, so something needs to be done. Examples of proper normal surfaces with trivial Picard group, in particular without big divisors, can be found in [Na58] and [Sch99]. Nevertheless, a specialisation argument allows us to conclude the existence of an ample Cartier divisor on a hypothetical normal 2-contractible surface in general.

Theorem 13. *There is no normal connected surface U/k over an algebraically closed field k of characteristic $p > 0$ such that*

(i) $\mathrm{H}^1_{\text{ét}}(U, \mathbb{F}_p) = 0$, *and*
(ii) $\mathrm{H}^2_{\text{ét}}(U, \mu_\ell) = 0$ *for some prime number $\ell \neq p$.*

Proof. By Nagata's embedding theorem and resolution of singularities for surfaces, the variety U is a dense open in a normal proper surface X/k with boundary $Y = X \setminus U$ being a normal crossing divisor. In particular, the surface X is smooth in a neighbourhood of Y. By the usual limit arguments, we may choose an integral scheme S of finite type over $\mathbb{F}_p$ together with a proper flat $f : \mathscr{X} \to S$, a relative Cartier divisor $\mathscr{Y}$ in $\mathscr{X}/S$ with normal crossing relative to S and complement $\mathscr{U} = \mathscr{X} \setminus \mathscr{Y}$ such that all fibres are normal proper surfaces and such that there is a point $\eta : \mathrm{Spec}(k) \to S$ over the generic point of S such that the fibre over η agrees with the original $\mathscr{X}_\eta = X$ together with $\mathscr{U}_\eta = U$ and $\mathscr{Y}_\eta = Y$. We may further assume that the set of irreducible components of the fibres of $\mathscr{Y}$ forms a constant system, and each component of $\mathscr{Y}$ is a Cartier divisor.

It follows from [SGA4$\frac{1}{2}$, Finitude, Theorem 1.9] that we may further assume that $\mathrm{R}^2 f|_{\mathscr{U}*}\mu_\ell$ is locally constant and commutes with arbitrary base change. Since its fibre in η

$$\left(\mathrm{R}^2 f|_{\mathscr{U}*}\mu_\ell\right)_\eta = \mathrm{H}^2_{\text{ét}}(U, \mu_\ell) = 0$$

vanishes we conclude that for all geometric points $\bar{s} \in S$ we have $\mathrm{H}^2_{\text{ét}}(\mathscr{U}_{\bar{s}}, \mu_\ell) = 0$, where $\mathscr{U}_{\bar{s}}$ is the fibre of $\mathscr{U} \to S$ in $\bar{s}$. As in the proof of Theorem 4 this implies that for every Cartier divisor D on $\mathscr{X}_{\bar{s}}$ there is an $m \geq 1$ and a Cartier divisor E on $\mathscr{X}_{\bar{s}}$ supported in $\mathscr{Y}_{\bar{s}}$ such that

$$mD \equiv E$$

are numerically equivalent.

We apply this insight to geometric fibres $\mathscr{X}_{\bar{t}}$ above closed points $t \in S$. Since by a theorem of Artin [Ar62, Corollary 2.11], all proper normal surfaces over the algebraic closure of a finite field are projective, we conclude that there is a very ample Cartier divisor $H_{\bar{t}}$ on $\mathscr{X}_{\bar{t}}$ with support contained in $\mathscr{Y}_{\bar{t}}$.

Let $\mathscr{H} \hookrightarrow \mathscr{X}$ be the relative Cartier divisor with support in $\mathscr{Y}$ that specialises to $H_{\bar{t}}$. By [EGA3, Theorem 4.7.1], the divisor $\mathscr{H}$ is ample relative to S in an open neighbourhood of $t \in S$. Consequently, the normal proper surface X is projective, and in particular U admits a big divisor. The proof now follows from Theorem 4. □

Remark 14. It follows from the proof of Theorem 13 that any proper non-projective normal surface X with trivial Picard group, in particular the examples of [Na58] and [Sch99], must have $\mathrm{H}^2_{\text{ét}}(X, \mu_\ell) \neq 0$ and a forteriori must contain non-trivial ℓ-torsion classes in the cohomological Brauer group $\mathrm{Br}(X)$ for all ℓ different from the characteristic. The existence of nontrivial torsion classes in $\mathrm{Br}(X)$ under the above assumptions was proven by different methods in [Sch01, proof of Theorem 4.1 on page 453].

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