TOPOLOGICAL RIGIDITY OF MAPS IN POSITIVE CHARACTERISTIC AND ANABELIAN GEOMETRY

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Abstract. We study pairs of non-constant maps between two integral schemes of finite type over two (possibly different) fields of positive characteristic. When the target is quasi-affine, Tamagawa showed that the two maps are equal up to a power of Frobenius if and only if they induce the same homomorphism on their étale fundamental groups. We extend Tamagawa's result by adding a purely topological criterion for maps to agree up to a power of Frobenius.

Contents

1. Introduction

Anabelian geometry aims to describe geometry of schemes in terms of their étale fundamental groups, or more generally in terms of their étale homotopy types.

Originally, Grothendieck's anabelian geometry considered varieties over arithmetic fields of characteristic 0. In the late 1990s, Tamagawa discovered that anabelian geometry exists also in characteristic $p > 0$: for smooth affine curves over finite fields, see [Tam97], and, more surprisingly at the time, also for smooth curves X over algebraically closed fields $k \supseteq \mathbf{F}_p$. For example, Tamagawa shows in [Tam02, Theorem 3.1] that $\pi_1^{\text{ét}}(X, \bar{x})$ determines the genus of X and the degree of the divisor of a smooth compactification. It is formulated as a conjecture [Tam02, Conjecture 2.2] that the profinite group $\pi_1^{\text{ét}}(X, \bar{x})$ determines the isomorphism class of such a curve X as a scheme.

To further support his conjecture, Tamagawa [Tam02, Proposition 1.24] showed a remarkable étale rigidity property for morphisms between integral varieties over a field. If the target is quasi-affine then the induced map on étale fundamental groups determines non-constant maps up to a power of Frobenius.

Unlike in characteristic 0, where the homotopy type of maps is locally constant in families, in characteristic $p > 0$ finite étale covers can have continuous moduli and thus potentially determine geometry that moves in families. The most famous example here is the family of finite étale Artin–Schreier covers $\wp_t \colon \mathbf{A}^1_k \to \mathbf{A}^1_k$ with $\wp_t(x) = x^p - tx$. The maps $\wp_{t,*} \colon \pi_1^{\text{ét}}(\mathbf{A}^1_k, \bar{0}) \to \pi_1^{\text{ét}}(\mathbf{A}^1_k, \bar{0})$ vary with $t \in k^\times$, even when k is algebraically closed.

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In this note, we show a general topological rigidity property for morphisms between integral varieties over a field. Non-constant maps are determined up to a power of Frobenius (the identity if the varieties are of characteristic 0) by the map they induce on the underlying topological spaces. The rigidity of maps (in the affine case) fits well with the result by the first author who showed in [Ach17] that affine schemes of finite type are étale $K(\pi, 1)$ spaces in characteristic $p > 0$.

We once and for all fix a prime p. Furthermore, we denote by $|X|$ the underlying set of a scheme X. The étale fundamental group of a connected scheme X is denoted by $\pi_1(X)$ with base points omitted. The maximal abelian quotient is $\pi_1^{\mathrm{ab}}(X)$.

The goal of this note is to prove the following result.

Theorem A. *Let X and Y be connected quasi-compact and quasi-separated $\mathbf{F}_p$-schemes. Let $f, g\colon Y \to X$ be two morphisms, and consider the following properties:*

(a) The morphisms $f_*, g_*\colon \pi_1(Y) \to \pi_1(X)$ are equal up to conjugation.

(a′) The morphisms $f_*, g_*\colon \pi_1^{\mathrm{ab}}(Y) \to \pi_1^{\mathrm{ab}}(X)$ are equal.

(a″) The morphisms $f^*, g^*\colon \mathrm{H}^1_{\text{ét}}(X, \mathbf{F}_p) \to \mathrm{H}^1_{\text{ét}}(Y, \mathbf{F}_p)$ are equal.

(b) The maps f and g induce the same map of sets $|Y| \to |X|$.

(c) There exist integers $a, b \geq 0$ such that $F_X^a \circ f = F_X^b \circ g$, where $F_X\colon X \to X$ is the absolute Frobenius.

Then the following implications always hold:

$$(c) \Longrightarrow (a) \Longrightarrow (a') \Longrightarrow (a'') \qquad \text{and} \qquad (c) \Longrightarrow (b).$$

Moreover:

(1) If Y is integral of finite type over a field and X is integral of finite type and separated over a second field, and $f(Y)$ and $g(Y)$ have positive dimension, then (b) is equivalent to (c).

(2) If, moreover in addition to the assumptions in (1), X is quasi-affine, then all five properties are equivalent.

Remark 1.1. Tamagawa in [Tam02, Proposition 1.24] shows that (a) $\Longleftrightarrow$ (c) if Y is irreducible of finite type over a field, and X is quasi-affine and connected. The proof of the nontrivial direction (a) $\Rightarrow$ (c) essentially uses the equality of the pullback maps

$$f^*, g^*\colon \mathrm{H}^1_{\text{ét}}(X, \mathbf{F}_q) \to \mathrm{H}^1_{\text{ét}}(Y, \mathbf{F}_q)$$

for all p-powers q. This being a consequence of (a″), Tamagawa's proof in fact yields the implication (a″) $\Rightarrow$ (c).

Although it turns out that our proof in §4 of (a″) $\Rightarrow$ (b) is similar to Tamagawa's proof of (a) $\Rightarrow$ (c), we nevertheless decided to include our argument for completeness sake.

We start with the immediate implications: (a) $\Rightarrow$ (a′) is trivial and (a′) $\Rightarrow$ (a″) follows from the functorial isomorphism

$$\mathrm{Hom}(\pi_1^{\mathrm{ab}}(X), \mathbf{F}_p) \simeq \mathrm{H}^1_{\text{ét}}(X, \mathbf{F}_p).$$

The fact that F_X induces the identity on $\pi_1(X)$ and $|F_X|$ is the identity on $|X|$ proves the implication (c) $\Rightarrow$ (a) and (c) $\Rightarrow$ (b).

The remaining implication (b) $\Rightarrow$ (c) that completes the proof of Theorem A is a special case of the following slightly more general proposition proved in §3. Here by a variety we mean a separated scheme of finite type over a field.

Proposition B (see Proposition 3.1)**.** *Let Y be an integral scheme of finite type over a field and let X be an integral variety over a field. Let $f, g\colon Y \to X$ be two maps inducing the same map $|Y| \to |X|$, with image of positive dimension. Then, there exist $a, b \geq 0$ such that*

$$F_X^a \circ f = F_X^b \circ g.$$

In the case of varieties over finite fields, Proposition B has been proven earlier by Stix [Sti02, §2]. The method of [Sti02] applies in the more general case only partially, and at one point we need to make Y to be of finite type over an uncountable field in order to reduce the problem to the case of curves.

We also describe a proof for the implication (a″) ⇒ (b) §4 as Proposition 4.3 for the convenience of the reader. We again reduce to the case of curves, and use Artin–Schreier theory to conclude. This step is similar to the proof of [Tam02, Proposition 1.24] by Tamagawa.

Before we embark on the proof, we discuss examples illustrating that the assumptions integral, affine and finite type in Theorem A are actually necessary.

Example 1.2. Let $Y = X = \operatorname{Spec} \mathbf{F}_p[u,v]/(uv)$. Let $f\colon X \to X$ be defined by $f(u,v) = (u, v^p)$, i.e., identity on one component $C = \{v = 0\}$ and Frobenius on the other $D = \{u = 0\}$, and let $g = \mathrm{id}_X$. Since restriction

$$\mathrm{H}^1(X, \mathbf{F}_p) \hookrightarrow \mathrm{H}^1(C, \mathbf{F}_p) \oplus \mathrm{H}^1(D, \mathbf{F}_p)$$

is injective, and Frobenius acts trivially on cohomology, we find that $f^* = g^*$ on $\mathrm{H}^1(X, \mathbf{F}_p)$. But assertion (c) of Theorem A does not hold.

Example 1.3. The case $X = Y = \mathbf{P}^1_k$ over a field k of characteristic $p > 0$ illustrates that the assumption that X is affine cannot be dropped easily. Indeed, $\mathbf{P}^1_k$ has étale fundamental group $\pi_1(\mathbf{P}^1_k) = \pi_1(k)$ and many nontrivial separable k-linear endomorphisms which all induce the identity on $\pi_1(\mathbf{P}^1_k)$.

Example 1.4. Let $X = \mathbf{A}^1_k$ with k algebraically closed and let $Y = \operatorname{Spec}(\mathcal{O}^{\mathrm{sh}}_{X,0})$ be the strict henselisation in $0 \in \mathbf{A}^1_k$. Then $\pi_1(Y) = 1$ and so the distinct maps $Y \to X$ by composing the standard map with a translation of $\mathbf{A}^1_k$ all induce the same map $\pi_1(Y) \to \pi_1(\mathbf{A}^1_k)$.

2. Preliminaries on base fields

2.1. **Reminder on inverse perfection.** Recall that for an $\mathbf{F}_p$-algebra R, the inverse perfection is the ring

$$R^{\mathrm{perf}} = \varprojlim_F R = \lim\left(\cdots \to R \xrightarrow{F} R \xrightarrow{F} R \xrightarrow{F} R\right)$$

where $F\colon R \to R$ is the Frobenius. The ring R^{perf} is perfect (Frobenius is an isomorphism) and inverse perfection R^{perf} is right adjoint to the inclusion of perfect $\mathbf{F}_p$-algebras.

Lemma 2.1. *Inverse perfection has the following effect on fields.*

(1) *If R is a reduced $\mathbf{F}_p$-algebra, then $R^{\mathrm{perf}} = \bigcap_{n \geq 0} R^{p^n}$.*
(2) *For a field K the inverse perfection K^{perf} is a field.*
(3) *The natural map $K^{\mathrm{perf}} \to K(T)^{\mathrm{perf}}$ is an isomorphism.*
(4) *If L/K is a finite purely inseparable field extension, then $K^{\mathrm{perf}} \to L^{\mathrm{perf}}$ is an isomorphism.*
(5) *Let L/K be a finitely generated field extension. Then $L^{\mathrm{perf}}/K^{\mathrm{perf}}$ is a finite separable extension.*

Proof. (1) is obvious, and (2) follows from (1).

(3) If $f \in K(T)$ is a p^n-th power for all n, then for all discrete valuations v of $K(T)$ we have $v(f) \in \mathbf{Z}$ is arbitary p-divisible, hence $v(f) = 0$. It follows that $f \in K$. If $f = g^{p^n}$, then also $g \in K$, hence f is also a p^n-th power for all n as an element of K. This proves the claim.

(4) For large enough n we have $L^{p^n} \subseteq K$. Assertion (4) follows immediately from (1).

(5) By (3), it suffices to treat the case where L/K is a finite extension. By (4) we may even assume that L/K is finite and separable. Let $x \in L^{\mathrm{perf}}$ and, for all $n \geq 0$, let $P_n(T)$ be the minimal polynomial over K of $y_n \in L$ where $(y_n)^{p^n} = x$.

Let $Q_n(T)$ be the polynomial $P_n(T)$ with coefficients raised to p^n-th powers. Then $Q_n(x) = (P_n(y_n))^{p^n} = 0$, and since L^{p^n}/K^{p^n} is isomorphic to L/K as field extension via the n-th power of Frobenius, the polynomial $Q_n(T)$ is the minimal polynomial of $x \in L^{p^n}$ over K^{p^n}. This means that for $m \geq n$ the polynomial $Q_m(T)$ divides $Q_n(T)$ in K^{p^n}, but since both polynomials are monic of the same degree, they are in fact equal. This means that $Q_n(T)$ is independent of n, and hence has coefficients in K^{perf}. As $Q_0(T) = P(T)$, this polynomial is separable and of degree bounded by $[L:K]$. It follows that L^{perf} is a separable extension of K^{perf} with all elements of degree bounded by $[L:K]$. This shows that $[L^{\mathrm{perf}} : K^{\mathrm{perf}}] \leq [L:K]$ is finite. ☐

Notation 2.2. For a scheme X over $\mathbf{F}_p$ we write

$$k_X := \mathrm{H}^0(X, \mathcal{O}_X)^{\mathrm{perf}}.$$

The ring k_X is functorial in X and the canonical map $X \to \mathrm{Spec}(k_X)$ is the universal map to affine schemes $\mathrm{Spec}(R)$ with R a perfect $\mathbf{F}_p$-algebra.

2.2. **Reminder on the field of constants.** A variety over a field k is a scheme X together with a separated morphism $X \to \mathrm{Spec}(k)$ of finite type. The field k is not unique, but there is usually a universal one.

Proposition 2.3. *Let X be an integral variety over a field k. Then there is a field $L_X \subseteq \mathrm{H}^0(X, \mathcal{O}_X)$ such that, for every field L, every map $X \to \mathrm{Spec}(L)$ uniquely factors over the map induced by an inclusion $L \hookrightarrow L_X$. In particular, L_X is the unique maximal subfield of $\mathrm{H}^0(X, \mathcal{O}_X)$ containing all other subfields and is functorial in X.*

Moreover, L_X is a finite extension of k, and if X is normal then X is geometrically integral as a variety over L_X.

Proof. The proof is essentially identical to the proof in the affine case [Tam97, Lemma 4.2] by Tamagawa. We omit the details. ☐

If X is not normal, then it might not be geometrically irreducible over L_X. A concrete example is $\mathrm{Spec}(A)$ where $A = \{f \in \mathbf{C}[x] : f(0) \in \mathbf{R}\}$.

2.3. **Inverse perfection as a field of constants.** If the base field is perfect, then we can use the inverse perfection to detect the maximal field of constants.

Definition 2.4. A **variety over a perfect field** is a scheme X such that there is a perfect field k and a separated map of finite type $X \to \mathrm{Spec}(k)$.

Lemma 2.5. *Let X be an integral scheme over $\mathbf{F}_p$. Then the following are equivalent.*

(a) X is a variety over a perfect field.

(b) The ring k_X is a perfect field and the map $X \to \mathrm{Spec}(k_X)$ is separated of finite type.

If both properties hold, then k_X is the unique maximal subfield L_X of $\mathrm{H}^0(X, \mathcal{O}_X)$.

Proof. Assertion (b) clearly implies (a). So we prove the converse and assume that we have a perfect field k and a separated map of finite type $X \to \mathrm{Spec}(k)$.

Denote by $k(X)$ the function field of X. The inclusions $k \hookrightarrow \mathrm{H}^0(X, \mathcal{O}_X) \hookrightarrow k(X)$ yield inclusions

$$k = k^{\mathrm{perf}} \hookrightarrow k_X \hookrightarrow k(X)^{\mathrm{perf}}.$$

By Lemma 2.1 (5) applied to $k(X)/k$ the extension k_X/k is contained in a finite (separable) extension. So k_X is perfect and (b) holds.

We now assume that both properties hold and keep the notation above. Proposition 2.3 shows that L_X is a finite extension of k, and hence is perfect. In particular, L_X is contained in $\mathrm{H}^0(X, \mathcal{O}_X)^{\mathrm{perf}} = k_X$. The converse inclusion is obvious. ☐

Remark 2.6. Let X be a variety over a perfect field k. Any open subscheme $U \subseteq X$ is a variety over a perfect field. Note that the field extension $k_X \to k_U$ can be nontrivial if X is not normal.

3. Topological coincidence of maps, revisited

The goal of this section is to prove the following rigidity property of maps between integral varieties. In case the base fields have characteristic 0, the Frobenius maps have to be interpreted as the identity, and some steps in the proof can be left out. We mainly deal with the case of base fields of characteristic p.

Proposition 3.1. *Let Y be an integral scheme of finite type over a field, and let X be an integral variety over a field. Let $f, g\colon Y \to X$ be two maps inducing the same map $|Y| \to |X|$, with image of positive dimension.*

Then, there exist $a, b \geq 0$ such that $F_X^a \circ f = F_X^b \circ g$.

Take note that we do not assume that X and Y are defined over the same field. If both X and Y are of finite type over $\mathbf{F}_p$, the proposition follows from [Sti02, Proposition 2.3]. Since we did not succeed to reduce to this case, we give a completely independent proof in the complementary case that instead exploits passing to uncountable fields. Surprisingly, this second proof strategy below needs the additional assumption that L_X **is infinite**, hence complementing [Sti02, Proposition 2.3]. Below we give a complete proof in both cases.

3.1. Reduction to functions.

Before embarking on the technical heart of the proof of Proposition 3.1, we will perform a few easy reductions.

Lemma 3.2. *In the situation described in Proposition 3.1.*

(1) Let $h\colon Y' \to Y$ be a dominant map with Y' an integral scheme of finite type over some field, and suppose that fh (and equivalently gh) still has image of positive dimension.
If the assertion of Proposition 3.1 holds for $fh, gh\colon Y' \to X$, then it holds for $f, g\colon Y \to X$ as well.

(2) Let $i\colon X \to X'$ be a locally closed immersion with X' an integral variety. If the assertion of Proposition 3.1 holds for $if, ig\colon Y \to X'$, then it holds for $f, g\colon Y \to X$ as well.

(3) Let $U \subseteq X$ be a dense open subvariety such that the intersection of U with the image of f (and equivalently for g) still has positive dimension. Denote the preimage $f^{-1}(U) = g^{-1}(U)$ by V and the restrictions of f and g by $f_U, g_U : V \to U$. If the assertion of Proposition 3.1 holds for $f_U, g_U\colon V \to U$, then it holds for $f, g\colon Y \to X$ as well.

Proof. Since X is separated, the equality of $f'h = g'h$ implies that f' agrees with g' for any $f', g'\colon Y \to X$ and part (1) follows. Part (2) similarly follows from the fact that i is a monomorphism. For (3), let $j\colon U \to X$ and $h\colon V \to Y$ be the inclusions. The assertion for $(f_U, g_U)\colon V \to U$ implies the assertion for $(jf_U, jg_U) = (fh, gh)\colon V \to X$, which implies the assertion for $(f, g)\colon Y \to X$ by (1). □

Proposition 3.3. *In Proposition 3.1, we may assume that Y is an integral normal variety over an uncountable algebraically closed field $K = k_Y$ and $X = \mathbf{A}_k^1$ with k a field.*

Proof. The proof proceeds in three steps. We first pick an uncountable algebraically closed field K that is an extension of L_Y and consider the normalization Y_0' of an irreducible component Y_0 of the base change $Y_K = Y \otimes_{L_Y} K$. Note that Y_0' is also integral and of finite type over the perfect field K. The composition with the projection

$$h\colon Y_0' \to Y_0 \hookrightarrow Y_K = Y \otimes_{L_Y} K \to Y$$

is schematically dominant. By choosing Y_0 appropriately, we may also assume that for the maps f and g for which we are trying to prove the claim of Proposition 3.1, the compositions fh and gh have still an image of positive dimension. Now apply Lemma 3.2(1).

In the second step we consider an affine open U in X that contains two points of $f(Y) = g(Y)$ that show this image has dimension > 0. By Lemma 3.2(3) we may replace X by U (and Y by $f^{-1}(U) = g^{-1}(U)$; this keeps the properties of Y we already adjusted) and thus assume X is an affine variety over a field k. We now choose a closed embedding $\iota : X \hookrightarrow \mathbf{A}^n_k$, and Lemma 3.2(2) lets us assume $X = \mathbf{A}^n_k$.

In the third step we choose good coordinates on $\mathbf{A}^n_k$ so that we may reduce to the coordinate projections.[1] We set $B = \mathrm{H}^0(Y, \mathcal{O}_Y)$ and have $k_Y = \bigcap_{n \geq 0} B^{p^n} = K$ by Lemma 2.5. Let $x_1, \ldots, x_n$ be the linear standard coordinates on $\mathbf{A}^n_k$, and we set $f_i = f^*(x_i)$ and $g_i = g^*(x_i)$. Then not all f_i are contained in K because otherwise the map $f : Y \to \mathbf{A}^n_k$ would factor over a point $\operatorname{Spec}(K)$ and not have image of positive dimension. After replacing f by a map $f' : Y \to \mathbf{A}^n_k$ with $f = F^a_{\mathbf{A}^n_k} \circ f'$ we may assume that there is an index j with $f_j \in B \setminus B^p$. Indeed, if all $f_i \in B^p$, then the map $f^* \colon k[x_1, \ldots, x_n] \to B$ has image contained in B^p (as k maps into $K = K^p$) and so f factors as $f = f' \circ F_Y$. But then also $f = F_{\mathbf{A}^n_k} \circ f'$.

As $B^p \subseteq B$ is a proper subgroup, we may replace (by a linear coordinate change $x_i \mapsto x_i + x_j$ of coordinates of $\mathbf{A}^n_k$) any $f_i \in B^p$ by $f_i + f_j$ to assume that for all $i = 1, \ldots, n$ we have $f_i \in B \setminus B^p$.

Now we turn our attention to g. By factoring over a suitable power of Frobenius, again for one index j we have $g_j \in B \setminus B^p$. Since we prepared the coordinates of $\mathbf{A}^n_k$ carefully with respect to f, we are guaranteed that with the same index j we have $f_j, g_j \in B \setminus B^p$. Note that $f_j(Y) = g_j(Y)$ in $\mathbf{A}^1_k$ does not factor over a closed point because otherwise f_j and g_j would be p-th powers.

We are now modifying the coordinates on $\mathbf{A}^n_k$ yet again by a transformation of the form $x_i \mapsto x_i + h_i(x_j)$ for all $i \neq j$. We pick polynomials $h_i \in k[T]$ such that $f_i + h_i(f_j)$ and $g_i + h_i(g_j)$ are not contained in B^p. The polynomials h_i that are unsuitable here are contained in two proper affine linear subspaces. More precisely, we consider the maps of k-vector spaces

$$\begin{aligned} \varphi \colon k[T] &\to B/B^p, \qquad h(T) \mapsto h(f_j) + B^p, \\ \psi \colon k[T] &\to B/B^p, \qquad h(T) \mapsto h(g_j) + B^p, \end{aligned}$$

and need to show that there is an h_i such that $\varphi(h_i) \notin -f_i + B^p$ and $\psi(h_i) \notin -g_i + B^p$. The subgroup

$$\ker(\varphi) = \{h \in k[T] \ ; \ h(f_j) \in B^p\}$$

is in fact a normal subring of $k[T]$, because B is normal. It contains $k[T^p]$ but not T, because $f_j \notin B^p$. It follows that $\ker(\varphi) = k[T^p]$, and similarly $\ker(\psi) = k[T^p]$. Since we established that $\ker(\varphi) = \ker(\psi)$ has codimension ∞ in $k[T]$, we always find a polynomial h_i avoiding the two forbidden affine subspaces.[2]

We have now achieved that the maps $f, g : Y \to \mathbf{A}^n_k$ are coordinatewise not p-th powers. In particular, the coordinate functions f_i, g_i are not constant since otherwise $k(f_i)$ or $k(g_i)$ are contained in K, the maximal subfield of $B = \mathrm{H}^0(Y, \mathcal{O}_Y)$, and thus p-th powers contrary to our preparations. Since $|f| = |g|$ also implies $|f_i| = |g_i|$, we may apply now Proposition 3.1 to these coordinate functions and deduce that there are $a_i, b_i \geq 0$ with

$$F^{a_i}_{\mathbf{A}^1_k} \circ f_i = F^{b_i}_{\mathbf{A}^1_k} \circ g_i.$$

After canceling powers of Frobenius we may assume that $\min\{a_i, b_i\} = 0$ for each i. But since neither f_i nor g_i is a p-th power, we obtain $a_i = b_i = 0$ for all i. It follows that $f = g$. $\square$

3.2. **Reduction to smooth curves.** We are going to reduce further to "generically étale" maps from a connected smooth affine curve to the affine line.

Lemma 3.4. *In Proposition 3.1, in addition to the reduction of Proposition 3.3, we may assume that Y is a connected smooth affine curve.*

[1] The reduction to coordinate projections is obvious in characteristic 0.

[2] Unless $k = \mathbf{F}_2$, the precise form of $\ker(\varphi)$ and $\ker(\psi)$ is not important, only that these are proper subspaces.

Proof. We need to argue that Proposition 3.1 holds in the scenario of Proposition 3.3. So let Y be an integral variety over an uncountable algebraically closed field K, and let $f, g : Y \to \mathbf{A}^1_k$ be maps with $|f| = |g|$ having image of dimension > 0. For $a, b \geq 0$, let

$$Z_{a,b} \subseteq Y$$

be the equalizer of $F^a_X \circ f$ and $F^b_X \circ g$, a closed subscheme of Y. We need to show that there are suitable a, b with $Z_{a,b} = Y$.

We first note that, f, g having image of dimension > 0, implies that the generic point and at least one closed point of $\mathbf{A}^1_k$ is in the image. In particular, the induced map of f and g on closed points $Y(K) \subseteq |Y| \to |\mathbf{A}^1_k|$ is non-constant.

For every $y \in Y(K)$, pick an integral curve $C_y \subseteq Y$ passing through y and another point z, depending on y, with $f(y) \neq f(z)$, see [Mum08, Lemma, p. 56]. Let D_y be an affine open subset of the normalization of C_y such that the image of the induced map $\gamma\colon D_y \to Y$ contains y and z. Thus γf and γg are not constant, and by the assumed case of the result applied to $f\gamma, g\gamma\colon D_y \to \mathbf{A}^1_k$, we see that $C_y(K) \subseteq Z_{a,b}(K)$ for some $a, b \geq 0$. In particular, $y \in Z_{a,b}$. Since this holds for all $y \in Y(K)$, by the Lemma 3.5 below there is a nonempty open $U \subseteq Y$ contained in some $Z_{a,b}$. Since $Z_{a,b}$ is closed and Y is reduced, we have $Y = Z_{a,b}$ and Proposition 3.1 holds for $f, g : Y \to \mathbf{A}^1_k$. □

Lemma 3.5. *Let Y be an integral scheme of finite type over an uncountable algebraically closed field K and let $\{Z_i\}_{i\in I}$ be a countable family of constructible subsets of Y such that*

$$Y(K) = \bigcup_{i\in I} Z_i(K).$$

Then there exists an $i \in I$ such that Z_i contains a dense open subset of Y.

Proof. This fact is well-known, though we do not know of a reference; here is a sketch of the proof. We may first replace Y by a dense affine open. Then Noether normalization allows us to reduce to the case $Y \simeq \mathbf{A}^n_K$. We argue by induction on n, the case $n = 0$ being obvious. Suppose that $n > 0$ and that none of the Z_i is dense. Then each of their closures contains at most a finite number of hyperplanes. By uncountability of K, there exists a hyperplane $\mathbf{A}^{n-1}_K \simeq H \subseteq \mathbf{A}^n_K$ not contained in the closure of any Z_i, contradicting the induction assumption for the intersections $H \cap Z_i$ whose K-points cover $H(K)$. □

In the curve case, we can factor out powers of Frobenius to assume that f and g are both "generically étale." To make this precise, we need to introduce some notation. Consider the situation of Lemma 3.4: $Y = \mathrm{Spec}(B)$ is a connected smooth affine curve over the big field K and we have two nonconstant maps $f, g\colon Y \to X = \mathbf{A}^1_k$. Let $f_0, g_0\colon k \to K$ be the induced field extensions, see the functoriality of the field of constants in Proposition 2.3. Focusing on f, it corresponds to a map $k[x] \to B$ inducing f_0 on k and mapping $x \mapsto f'$. Geometrically, the map $f'\colon Y \to \mathbf{A}^1_K$ appears in the factorization

$$\begin{tikzcd} Y \arrow[r, "f'"] \arrow[dr, "f"'] & X_K = \mathbf{A}^1_K \arrow[d, "f_0"] \\ & X = \mathbf{A}^1_k \end{tikzcd} \tag{3.1}$$

where by abuse of notation we also denote the projection $X_K \to X$ of the base change along $f_0 : k \to K$ by f_0. Thus f' is not constant, and since $B^{\mathrm{perf}} = K$ there exists an integer $a \geq 1$ and an element $f'_{\mathrm{new}} \in B \setminus B^p$ such that $f' = (f'_{\mathrm{new}})^{p^a}$. Since Y is a connected smooth curve, the fact that f'_{new} is not a p-th power means that $df'_{\mathrm{new}} \neq 0$, so that the induced map $f'_{\mathrm{new}}\colon Y \to \mathbf{A}^1_K$ is generically étale. We write $f_{\mathrm{new}}\colon Y \to \mathbf{A}^1_k$ for the composition with $f_0\colon \mathbf{A}^1_K \to \mathbf{A}^1_k$, and find

$f = F^a_{\mathbf{A}^1_k} \circ f_{\rm new}$. We may therefore for our purposes replace f with $f_{\rm new}$ and hence assume that in addition $f' \colon Y \to \mathbf{A}^1_K$ is generically étale. Doing the same with g, we conclude that Proposition 3.1 follows from the assertion below.

Proposition 3.6. *Let Y be a connected affine smooth curve over $k_Y = K$ and let $f, g \colon Y \to \mathbf{A}^1_k$ be two maps inducing the same map $|Y| \to |\mathbf{A}^1_k|$ such that the induced maps $f', g' \colon Y \to \mathbf{A}^1_K$ are generically étale.*

Then, there exist $a, b \geq 0$ such that $F^a_{\mathbf{A}^1_k} \circ f = F^b_{\mathbf{A}^1_k} \circ g$.

We are going to prove Proposition 3.6 in the following two sections depending on the cardinality $\#k$.

Remark 3.7. In fact, in view of our reduction to the generically étale case, the conclusion of Proposition 3.6 should be f equals g. This is indeed what we prove below when $\#k$ is infinite. However, when $\#k$ is finite, we again trade in *generically étale* for another desirable property, namely that f and g agree on the field of constants. We decided to weaken the assertion of Proposition 3.6 in favour of a more transparent structure of our reduction steps in the proof.

3.3. **Infinite base field.** We first assume that $\#k$ is infinite. By Lemma 3.2(1), we may shrink Y further to an open subscheme so that both maps $f', g' : Y \to \mathbf{A}^1_K$ are étale.

Let $U \subseteq \mathbf{A}^1_k$ be the image of f and equally g. We verify that U is an open subset of $\mathbf{A}^1_k$: factoring f into $f' \colon Y \to \mathbf{A}^1_K$ and $f_0 \colon \mathbf{A}^1_K \to \mathbf{A}^1_k$, the set $U = f_0(f'(Y))$ is the image of $f'(Y) \subseteq \mathbf{A}^1_K$, which is the complement of a finite subset. Then $f_0^{-1}(U)$ and $g_0^{-1}(U)$ are open subschemes of $\mathbf{A}^1_K$ which receive the maps f' and g'.

We claim that for $y_1, y_2 \in Y(K)$ we have $f'(y_1) = f'(y_2)$ if and only if $g'(y_1) = g'(y_2)$. In other words, the étale equivalence relation

$$R(f') = Y \times_{f', \mathbf{A}^1_K, f'} Y \subseteq Y \times_K Y$$

agrees with the analogous equivalence relation $R(g')$. Note that $Y/R(f') \simeq f'(Y)$ canonically induced by f', and similarly for g'.

Indeed, it suffices to check this for y_1 and y_2 belonging to an infinite subset $Z \subseteq Y(K)$. Consider Z to be the set of all $y \in Y(K)$ whose image in $\mathbf{A}^1_k$ (via f or g) is a k-rational point. The set Z is infinite by our assumption that $\mathbf{A}^1(k) = k$ is infinite(!). Now, the preimage of any k-rational point under the projections $f_0, g_0 : \mathbf{A}^1_K \to \mathbf{A}^1_k$ is a singleton. This shows that for $y_1, y_2 \in Z$ we have

$$f'(y_1) = f'(y_2) \iff f(y_1) = f(y_2) \iff g(y_1) = g(y_2) \iff g'(y_1) = g'(y_2),$$

where the equivalence in the middle makes use of $|f| = |g|$.

Therefore f' and g' define the same étale equivalence relation on Y, which implies that there exists a K-linear isomorphism

$$\varphi \colon f'(Y) \xleftarrow{f'} Y/R(f') = Y/R(g') \xrightarrow{g'} g'(Y).$$

Then, we have the diagram

$$\begin{tikzcd}
& f'(Y) \arrow[dd, "\simeq"', "\varphi"] \arrow[dr, "f_0"] & \\
Y \arrow[ur, two heads, "f'"] \arrow[dr, two heads, "g'"'] & & U \\
& g'(Y) \arrow[ur, "g_0"'] &
\end{tikzcd}$$

where the left triangle commutes by construction and the right triangle commutes on the level of topological spaces as a consequence of $|f| = |g|$.

Let $\overline{\varphi}\colon \mathbf{P}^1_K \to \mathbf{P}^1_K$ be the unique extension of φ to smooth compactifications. For $\alpha \in U(k)$ the preimage under f_0 (resp. g_0) is a singleton, more precisely only the point $[f_0(\alpha):1]$ (resp. the point $[g_0(\alpha):1]$), hence we have

$$\overline{\varphi}([f_0(\alpha):1]) = [g_0(\alpha):1], \qquad \text{for } \alpha \in U(k). \tag{3.2}$$

By a change of coordinates defined over k we may assume that 0 and 1 belong to U. Then (3.2) yields in particular that $\overline{\varphi}$ fixes $0 = [0:1]$ and $1 = [1:1]$. After a further change of coordinates by $t \mapsto t/(t-1)$, now 0 and $\infty = [1:0]$ are preserved. In this coordinate the map $\overline{\varphi}$ must be of the form, for some $\lambda \in K$,

$$\overline{\varphi}(t) = \lambda \cdot t, \qquad t \in K \cup \{\infty\}.$$

Now (3.2) implies

$$\lambda = g_0(\alpha)/f_0(\alpha) \qquad \text{for all } \alpha \in U(k), \alpha \neq 0.$$

As $U(k)$ contains all but finitely many elements of $k^\times$ and k is an infinite field, we find elements $\alpha_1, \alpha_2 \in U(k)$ with $\alpha_1\alpha_2 \in U(k)$. Then we conclude $\lambda = 1$ from

$$\lambda^2 = \frac{g_0(\alpha_1)}{f_0(\alpha_1)} \cdot \frac{g_0(\alpha_2)}{f_0(\alpha_2)} = \frac{g_0(\alpha_1\alpha_2)}{f_0(\alpha_1\alpha_2)} = \lambda.$$

Thus $\overline{\varphi}$ is the identity $g' = \varphi \circ f'$ equals f'. Moreover, also $f_0 = g_0$ holds for the cofinite set $U(k) \cap k^\times$, and thus for all of k. This completes the proof of Proposition 3.6 if $\#k$ is infinite.

3.4. **Finite base field: intersection theory.** We will now deal with the case $X = \mathbf{A}^1_k$ with $k = k_X$ a finite field of cardinality q. We use f_0 to identify k with a subfield of $K = k_Y$. Then, as K only contains a unique subfield of cardinality q, we have $g_0(k) = k$, and so there is some $b \geq 0$ such that $f_0 = g_0 \circ F^b$. We may replace g by the composition $F^b \circ g$, and thus assume that $f_0 = g_0$ at the expense of giving up that g' is actually generically étale (f' still is). We thus have achieved that in the factorization of f and g as in (3.1) both K-varieties denoted X_K and the projections $X_K \to X$ agree. Since we will be mostly working with the K-linear maps now, we rename $\mathrm{pr} := f_0 = g_0$ and denote f' by f (resp. g' by g). So we are left to prove the following.

Proposition 3.8. *Let k be a finite field contained in an algebraically closed field K. Let Y be a connected affine smooth curve of finite type over K and let $f, g\colon Y \to \mathbf{A}^1_K$ be two maps of K-schemes inducing the same map $|Y| \to |\mathbf{A}^1_k|$ after composition with the projection* $\mathrm{pr}\colon \mathbf{A}^1_K \to \mathbf{A}^1_k = X$, *and with image in $|\mathbf{A}^1_k|$ of positive dimension. We further assume that f is generically étale.*

Then, there exist $a, b \geq 0$ such that $F^a_X \circ f = F^b_X \circ g$.

Proof. Let η_Y (resp. η_{X_K} and η_X) denote the generic point of Y (resp. X_K and X). Let $\bar{k}$ denote the algebraic closure of k inside K. Then the maps of topological spaces to be considered factor as

$$|Y| = \{\eta_Y\} \cup Y(K) \xrightarrow{|f|,|g|} |X_K| = \{\eta_{X_K}\} \cup K \xrightarrow{|\mathrm{pr}|} |X| = \{\eta_X\} \cup \bar{k}/\operatorname{Gal}(\bar{k}/k)$$

where by $\bar{k}/\operatorname{Gal}(\bar{k}/k)$ we denote the set of Galois orbits of $\operatorname{Gal}(\bar{k}/k)$ acting on $\bar{k}$. The fibre of pr in a Galois orbit agrees with precisely this orbit.

For every $d \geq 1$ there is a unique subfield $k_d \subseteq K$ of order q^d. The set $G_d := \mathrm{pr}(k_d)$ in $|X|$ consists of precisely those Galois orbits of length dividing d. Since $k_d = \mathrm{pr}^{-1}(G_d)$ and since $|\mathrm{pr} \circ f|$ agrees with $|\mathrm{pr} \circ g|$, the preimages with respect to f, g

$$|f|^{-1}(k_d) = |\mathrm{pr} \circ f|^{-1}(G_d) = |\mathrm{pr} \circ g|^{-1}(G_d) = |g|^{-1}(k_d)$$

agree and will be denoted by S_d. Let

$$M_d = \left\{ m \in \mathbf{Z} \,;\, -\frac{d}{2} < m \leq \frac{d}{2} \right\}$$

be a minimal set of representatives of $\mathbf{Z}/d\mathbf{Z}$ in terms of absolute value. It follows that for every $y \in S_d$ there is an $m \in M_d$ with

$$\begin{cases} f(y) = g(y)^{q^m} & \text{if } m \geq 0, \\ f(y)^{q^{-m}} = g(y) & \text{if } m \leq 0, \end{cases} \tag{3.3}$$

because $\mathrm{Gal}(k_d/k)$ is generated by the q-power Frobenius.

For $m \geq 0$, let $\Gamma_m \subseteq \mathbf{P}^1_K \times \mathbf{P}^1_K$ denote the graph of Frobenius $F^m : \mathbf{P}^1_K \to \mathbf{P}^1_K$, given by $[u : v] \mapsto [u^{q^m} : v^{q^m}]$. For $m < 0$, we denote by Γ_m the image of Γ_{-m} under the transposition of factors of $\mathbf{P}^1_K \times \mathbf{P}^1_K$.

Let $\overline{Y}$ denote the smooth projective completion of Y, and let $\bar{f}$ and $\bar{g}$ denote the extensions of f and g to maps $\bar{f}, \bar{g} \colon \overline{Y} \to \mathbf{P}^1_K$. Define $\overline{Z}_m$ as the fibre product

$$\begin{tikzcd} \overline{Z}_m \arrow[r, hook] \arrow[d] & \overline{Y} \arrow[d, "h=(\bar{f},\bar{g})"] \\ \Gamma_m \arrow[r, hook] & \mathbf{P}^1_K \times \mathbf{P}^1_K. \end{tikzcd}$$

If $\overline{Z}_m = \overline{Y}$, then $g = F^m \circ f$ for $m \geq 0$ or $F^{-m} \circ g = f$ for $m < 0$, so Proposition 3.8 is proven. We argue by contradiction and assume that for all $m \in \mathbf{Z}$ the subscheme $\overline{Z}_m$ has dimension 0. Intersection theory on $\mathbf{P}^1_K \times \mathbf{P}^1_K$ computes the degree $\deg(\overline{Z}_m) = \dim_K \mathrm{H}^0(\overline{Z}_m, \mathcal{O}_{\overline{Z}_m})$ as

$$\deg(\overline{Z}_m) = \deg_{\mathbf{P}^1_K \times \mathbf{P}^1_K}\left([\Gamma_m] \cdot [\overline{Y}]\right) = \begin{cases} q^m \deg(\overline{f}) + \deg(\overline{g}) & \text{if } m \geq 0, \\ \deg(\overline{f}) + q^{-m} \deg(\overline{g}) & \text{if } m \leq 0. \end{cases}$$

Here the Chow ring $\mathrm{CH}^*(\mathbf{P}^1_K \times \mathbf{P}^1_K) = \mathbf{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$ is generated by the classes $\alpha = [\mathbf{P}^1_K \times \{0\}]$ and $\beta = [\{0\} \times \mathbf{P}^1_K]$ with $\deg(\alpha\beta) = 1$, and the cycle classes of Γ_m and $\overline{Y}$ are

$$[\overline{Y}] = \deg(\overline{f})\alpha + \deg(\overline{g})\beta \quad \text{ and } \quad [\Gamma_m] = \begin{cases} \alpha + q^m\beta & \text{if } m \geq 0, \\ q^{-m}\alpha + \beta & \text{if } m \leq 0. \end{cases}$$

By (3.3) we have

$$S_d \subseteq \bigcup_{m \in M_d} \bar{Z}_m(K).$$

Now, since f is generically étale, there is a fixed B, uniform in d, taking into account the ramification and boundary of $f \colon Y \to \mathbf{A}^1_K$ in comparison with $\bar{f} \colon \overline{Y} \to \mathbf{P}^1_K$, such that

$$\deg(\overline{f}) \cdot q^d - B \leq \#S_d.$$

Combining the above we obtain the inequality

$$\deg(\overline{f}) \cdot q^d - B \leq \#S_d \leq \sum_{m \in M_d} \deg \overline{Z}_m \leq d \cdot q^{d/2}\big(\deg(\overline{f}) + \deg(\overline{g})\big).$$

Letting d tend to infinity leads to a contradiction, and that concludes the proof of Proposition 3.8 and thus also, finally, the proof of Proposition 3.1. $\square$

4. Artin–Schreier theory

The goal of this section is to prove the implication (a″)⇒(b) of Theorem A, thus completing its proof. We first deal with the case of a "formal punctured disc".

4.1. **Local curve case.** Temporarily, let K be a perfect field of characteristic $p > 0$. The field $K(\!(t)\!)$ of formal Laurent series is endowed with the discrete valuation $v\colon K(\!(t)\!) \to \mathbf{Z} \cup \{\infty\}$ normalized so that $v(t) = 1$. We write $d\colon K(\!(t)\!) \to \Omega$ for the universal continuous derivation, so that $\Omega = K(\!(t)\!)dt$. We extend the valuation v to differentials by setting $v(dt) = 1$. We then have $v(z) \leq v(dz)$ for all z.

Proposition 4.1. *Let $f, g \in K(\!(t)\!)$ with $v(f) < 0$ and $g \notin K$, and suppose that for every $n \geq 1$ there exists an $h_n \in K(\!(t)\!)$ such that*

$$f^n - g^n = h_n - h_n^p.$$

Then there exist integers $a, b \geq 0$ such that $f^{p^a} = g^{p^b}$.

Proof. Since K is perfect, an element z of $K(\!(t)\!)$ is a p-th power if and only if $dz = 0$. Repeatedly taking p-th roots of f or g, we may therefore assume that *df* and *dg* are nonzero. Note that we keep the assumption of the lemma by taking p-th roots because if for example $f = (f_1)^p$, then

$$f_1^n - g^n = f_1^n - (f_1^n)^p + f^n - g^n = (f_1^n + h_n) - (f_1^n + h_n)^p.$$

After these preparations, our goal is now to show that $f = g$. Possibly exchanging f and g, we may assume that $v(f) \leq v(g)$. Set $\varepsilon = g/f$, so that $v(\varepsilon) \geq 0$. We need to show that $\varepsilon = 1$, so suppose otherwise.

The basic idea is to differentiate both sides of

$$f^n(1 - \varepsilon^n) = f^n - g^n = h_n - h_n^p$$

and look at valuations, making use of the fact that $d(h_n^p)$ disappears. This forces the valuation of $d(f^n - g^n)$ to be much less negative than the valuation of $f^n - g^n$ forcing a number (that grows linearly with n) of certain initial coefficients of f^n and g^n to agree.

The actual argument below does not refer to coefficients. Rather, we need to consider n coprime to p and such that $\varepsilon^n \neq 1$. All values of n from now on will be assumed to satisfy these assumptions. Since we assumed $\varepsilon \neq 1$, the set of such n is infinite, and it makes sense to talk about asymptotics as $n \gg 0$. We use the notation $O(1)$ to denote a bounded function in n.

Lemma 4.2. *Consider values of $n \geq 0$ which are coprime to p and such that $\varepsilon^n \neq 1$. Then:*

(a) $v(1 - \varepsilon^n) = O(1)$,
(b) $v(f^n - g^n) = nv(f) + O(1)$,
(c) $v(f^n - g^n) < 0$ *for* $n \gg 0$,
(d) $v(f^n - g^n) = pv(h_n)$ *for* $n \gg 0$,
(e) $v(d(f^n - g^n)) \geq v(h_n)$.

Proof. Assertion (a) follows from

$$1 - \varepsilon^n = \prod_{\xi^n = 1} (1 - \xi\varepsilon).$$

and $(1 - \xi\varepsilon)(0) = 1 - \xi\varepsilon(0) \neq 0$ unless $\xi = \varepsilon(0)^{-1}$. So in fact $v(1 - \varepsilon^n)$ takes at most two values. Part (b) follows since $f^n - g^n = f^n(1 - \varepsilon^n)$, and part (c) holds since $v(f) < 0$, so that $nv(f) + O(1) < 0$ for $n \gg 0$.

For (d), note that (c) implies that $v(h_n - h_n^p) < 0$ for $n \gg 0$. But then $v(h_n)$ is negative and $v(h_n - h_n^p) = pv(h_n)$ by the triangle inequality. For (e), we write

$$d(f^n - g^n) = d(h_n - h_n^p) = dh_n,$$

and $v(dh) \geq v(h)$ holds for any h. □

Now, we combine parts (d) and (e) of the lemma to obtain the inequality:

$$v\left(d(f^n - g^n)\right) \geq v(h_n) = \frac{1}{p} v(f^n - g^n) = \frac{n}{p} v(f) + O(1). \tag{4.1}$$

On the other hand, we have

$$d(f^n - g^n) = nf^{n-1}x_n, \qquad x_n := (1-\varepsilon^n)df - f\varepsilon^{n-1}d\varepsilon$$

and

$$v\left(d(f^n-g^n)\right) = (n-1)v(f) + v(x_n) = nv(f) + v(x_n) + O(1). \tag{4.2}$$

Combining (4.1) and (4.2) we obtain

$$v(x_n) \geq cn + O(1), \qquad c = -\left(1 - \frac{1}{p}\right)v(f) > 0. \tag{4.3}$$

In the rest of the proof, we shall estimate $v(x_n)$ from above for certain values of n, obtaining a contradiction with (4.3), showing that $\varepsilon = 1$. We first note that

$$x_n = \frac{d(f^n - g^n)}{nf^{n-1}} = df - \varepsilon^{n-1}dg.$$

Case 1. Suppose that $v(\varepsilon) > 0$. Then for $n \gg 0$ we have

$$v(x_n) = v(df - \varepsilon^{n-1}dg) = v(df) = O(1),$$

contradicting (4.3).

Case 2. Suppose that ε is a root of unity. Then x_n takes only finitely many values and so $v(x_n) = O(1)$ is bounded, again contradicting (4.3).

Case 3. Suppose that $v(\varepsilon) = 0$, but ε is not a root of unity. For n prime to p

$$v(x_{n+p} - x_n) = v\left((\varepsilon^{n-1} - \varepsilon^{n+p-1})dg\right) = (n-1)v(\varepsilon) + v(1-\varepsilon^p) + v(dg) = O(1).$$

On the other hand, (4.3) implies the following contradiction

$$v(x_{n+p} - x_n) \geq \min\{v(x_{n+p}), v(x_n)\} \geq cn + O(1). \qquad \square$$

4.2. **Concluding the proof.** The goal of this section is to prove the implication (a″) ⇒ (b) of Theorem A, concluding its proof.

Proposition 4.3. *Let Y be an integral $\mathbf{F}_p$-scheme of finite type over a field, and let X be a quasi-affine $\mathbf{F}_p$-scheme of finite type over a field. Let $f, g\colon Y \to X$ be a pair of maps whose image has positive dimension. Suppose that the two maps*

$$f^*, g^*\colon \mathrm{H}^1(X, \mathbf{F}_p) \to \mathrm{H}^1(Y, \mathbf{F}_p)$$

are equal. Then f and g induce the same map $|f| = |g| : |Y| \to |X|$ on the underlying sets.

Proof. If $h\colon Y' \to Y$ is a surjective map of integral $\mathbf{F}_p$-schemes respectively of finite type over a field, then we may replace f and g by fh and gh. We apply this reduction to the normalization of a suitably chosen irreducible component of the base change of Y to an algebraically closed field K.

Next, we choose an embedding $i : X \hookrightarrow X'$ into an affine variety X'. Because the map $|i|$ is a monomorphism $|X| \hookrightarrow |X'|$, we may replace X by X', compose f and g by i, and thus assume that X is affine from the start.

The map $|Y| \to |X|$ associated to f (resp. g) is determined by the restriction to closed points $Y(K) = |Y|_0$ of $|Y|$. Indeed, for any $y \in |Y|$ let $Z_y \subseteq Y$ denote the Zariski closure of y in Y. Then $f(y)$ is the generic point of the Zariski closure of $f(|Z_y| \cap |Y|_0)$ in $|X|$.

Because $f(Y)$ and $g(Y)$ have positive dimension, both f and g are not constant when restricted to closed points $|Y|_0$.

For every closed point $y \in Y$ pick an affine irreducible curve $C_y \subseteq Y$ passing through y and points z_1, z_2 depending on y with $f(y) \neq f(z_1)$ and $g(y) \neq g(z_2)$, see [Mum08, Lemma, p. 56][3].

[3]Note that the proof of [Mum08, Lemma, p. 56] only constructs a curve passing through two points. But the proof immediately generalizes to any finite set of closed points.

Thus f and g restricted to C_y are nonconstant maps $f_y := f|_{C_y}$ and $g_y := g|_{C_y}$. If we assume Proposition 4.3 in the case of $\dim(Y) = 1$, then the restrictions $|f_y| = |g_y|$ agree. As the union of these curves C_y with y ranging over all closed points of Y covers $|Y|_0$, the two maps $|f|$ and $|g|$ agree on all of $|Y|$ and the proof is achieved. This reduces the proof to the case $\dim(Y) = 1$ with Y affine. By replacing Y by its normalization, we may assume that Y is smooth because normal curves over K are smooth, K being algebraically closed.

Let $Y = \mathrm{Spec}(B)$ and $X = \mathrm{Spec}(A)$, and the maps f and g are given by homomorphisms $f, g : A \to B$. By assumption, neither f nor g is constant, hence their image is not contained in $K = \bigcap_{n \geq 0} B^{p^n}$. Let n be maximal with $f(A) \subseteq B^{p^n}$, in other words we can factor $f = F_X^n \circ f'$ with $f'(A) \nsubseteq B^p$. Since F_X acts as identity on $|X|$, we may replace f by f' and similarly for g in order to reduce to the case where $f(A)$ and $g(A)$ are not contained in B^p. The result then follows from the more precise proposition below. $\square$

Proposition 4.4. *Let $Y = \mathrm{Spec}(B)$ be an affine smooth connected curve over the algebraically closed field K of characteristic $p > 0$. Let $X = \mathrm{Spec}(A)$ be an affine $\mathbf{F}_p$-scheme of finite type over a field. Let $f, g \colon A \to B$ be a pair of maps such that their respective images as maps $f, g \colon Y \to X$ have positive dimension, and such that $f(A)$ and $g(A)$ are not contained in B^p. Suppose that the two maps*

$$f^*, g^* \colon \mathrm{H}^1(X, \mathbf{F}_p) \to \mathrm{H}^1(Y, \mathbf{F}_p)$$

are equal. Then f equals g.

Proof. For each $t \in A$ we may compose f and g with the map $t \colon X \to \mathbf{A}^1_{\mathbf{F}_p}$. The assumptions of the proposition are preserved by tf and tg if

$$t \in S := A \setminus \Big(f^{-1}(B^p) \cup g^{-1}(B^p)\Big).$$

Note that if for example tf is constant, then $\mathbf{F}_p[t] \to B$, $t \mapsto f(t)$ factors over a subfield of B, hence over the maximal such $K \subseteq B$. In that case t would be a p-th power in B. So asking $t \in S$ also guarantees that tf and tg are both not constant.

If the proposition holds for all such compositions $tf, tg : Y \to \mathbf{A}^1_{\mathbf{F}_p}$, then $f(t) = g(t)$ for all $t \in S$. Since S is the complement in A by two proper $\mathbf{F}_p$-subvector spaces, the set S generates A as an $\mathbf{F}_p$-algebra. This reduces the proof of the proposition to the case $X = \mathbf{A}^1_{\mathbf{F}_p} = \mathrm{Spec}\big(\mathbf{F}_p[t]\big)$.

By assumption all $\mathbf{F}_p$-torsors $C \to \mathbf{A}^1_{\mathbf{F}_p}$ pull back with f^* and g^* to isomorphic torsors. We apply this to the Artin-Schreier torsor and also its pullback via $t \mapsto t^n$, i.e, the torsor $t^n = x - x^p$. That isomorphism of torsors is expressed by the existence of $h_n \in B$ for each $n \geq 1$ such that

$$f^n - g^n = h_n - (h_n)^p. \tag{4.4}$$

We factor $f, g \colon Y \to \mathbf{A}^1_{\mathbf{F}_p}$ as K-linear maps $\varphi, \gamma \colon Y \to \mathbf{A}^1_K$ followed by the projection $\mathbf{A}^1_K \to \mathbf{A}^1_{\mathbf{F}_p}$. Still $d\varphi = \varphi^* dt$ and $d\gamma = \gamma^* dt$ are non-zero and (4.4) holds for $f = \varphi$ and $g = \gamma$ in B. We now compactify Y to a smooth projective curve $\overline{Y}$ and extend the maps to $\overline{\varphi}, \overline{\gamma} : \overline{Y} \to \mathbf{P}^1_K$. Since $\overline{\varphi}$ is not constant, we may look at the local field $K(\!(T)\!)$ at a closed point $y \in Y$ where $\overline{\varphi}$ has a pole. Here Proposition 4.1 applies and yields $a, b \geq 0$ with $f^{p^a} = g^{p^b}$. Since df and dg are both non-zero, we may cancel powers of p on both sides until none remain and so $f = g$. $\square$

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