

# On a functional contraction method

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## Abstract

Methods for proving functional limit laws are developed for sequences of stochastic processes which allow a recursive distributional decomposition either in time or space. Our approach is an extension of the so-called contraction method to the space  $\mathcal{C}[0, 1]$  of continuous functions endowed with uniform topology and the space  $\mathcal{D}[0, 1]$  of càdlàg functions with the Skorokhod topology. The contraction method originated from the probabilistic analysis of algorithms and random trees where characteristics satisfy natural distributional recurrences. It is based on stochastic fixed-point equations, where probability metrics can be used to obtain contraction properties and allow the application of Banach's fixed-point theorem. We develop the use of the Zolotarev metrics on the spaces  $\mathcal{C}[0, 1]$  and  $\mathcal{D}[0, 1]$  in this context. Applications are given, in particular a short proof of Donsker's functional limit theorem and to recurrences arising in the probabilistic analysis of algorithms.

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## 1 Introduction

The contraction method is an approach for proving convergence in distribution for sequences of random variables which satisfy recurrence relations in distribution. Such recurrence relations for a sequence  $(Y_n)_{n \geq 0}$  are often of the form

$$Y_n \stackrel{d}{=} \sum_{r=1}^K A_r(n) Y_{I_r^{(n)}}^{(r)} + b(n), \quad n \geq n_0, \quad (1)$$

where  $\stackrel{d}{=}$  denotes that the left and right hand side are identically distributed, and  $(Y_j^{(r)})_{j \geq 0}$  have the same distribution as  $(Y_n)_{n \geq 0}$  for all  $r = 1, \dots, K$ , where  $K \geq 1$  and  $n_0 \geq 0$  are fixed integers. Moreover  $I^{(n)} = (I_1^{(n)}, \dots, I_K^{(n)})$  is a vector of random integers in  $\{0, \dots, n\}$ . The basic independence assumption that fixes the distribution of the right hand side is that  $(Y_j^{(1)})_{j \geq 0}, \dots, (Y_j^{(K)})_{j \geq 0}$  and  $(A_1(n), \dots, A_K(n), b(n), I^{(n)})$  are independent. Note however, that dependencies between the coefficients  $A_r(n)$ ,  $b(n)$  and the integers  $I_r^{(n)}$  are allowed.

Recurrences of the form (1) come up in diverse fields, e.g., in the study of random trees, the probabilistic analysis of recursive algorithms, in branching processes, in the context of random fractals and in models from stochastic geometry where a recursive decomposition can be found,

as well as in information and coding theory. For surveys of such occurrences see [25, 27, 33]. In some applications one may need  $K$  to depend on  $n$  or the case  $K = \infty$ , where generalizations of the results for our case of fixed  $K$  can be stated, cf. [25, Section 4.3] for such extensions in the finite dimensional case.

The sequence  $(Y_n)_{n \geq 0}$  satisfying (1) often is a sequence of real random variables with real coefficients  $A_r(n)$ ,  $b(n)$ . However, the same recurrence appears also for sequences of random vectors  $(Y_n)_{n \geq 0}$  in  $\mathbb{R}^d$ . Then the  $A_r(n)$  are random linear maps from  $\mathbb{R}^d$  to  $\mathbb{R}^d$  and  $b(n)$  is a random vector in  $\mathbb{R}^d$ . We will also review below work that considered random sequences  $(Y_n)_{n \geq 0}$  into a separable Hilbert space satisfying (1) where  $A_r(n)$  become random linear operators on the space and  $b(n)$  a random vector in the Hilbert space. In the present work we develop a limit theory for such sequences in separable Banach spaces, where our main applications are first to the space  $\mathcal{C}[0, 1]$  endowed with the uniform topology. Secondly, although not a Banach space, we will also be able to cover the space  $\mathcal{D}[0, 1]$  equipped with the Skorokhod topology. Hence, we consider sequences  $(Y_n)_{n \geq 0}$  of stochastic processes with state space  $\mathbb{R}$  and time parameter  $t \in [0, 1]$  with continuous respectively càdlàg paths and are interested in conditions that together with (1) allow to deduce functional limit theorems for rescaled versions of  $(Y_n)_{n \geq 0}$ .

For functions  $f \in \mathcal{C}[0, 1]$  or  $f \in \mathcal{D}[0, 1]$  we denote the uniform norm by

$$\|f\|_\infty := \sup_{x \in [0, 1]} |f(x)|.$$

For functions  $f, g \in \mathcal{D}[0, 1]$  the Skorokhod distance  $d_{sk}(f, g)$  is used, see section 2.2.

The rescaling of the process  $(Y_n)_{n \geq 0}$  can be done by centering and normalization by the order of the standard deviation in case moments of sufficient order are available. Subsequently, we assume that the scaling has already been done and we denote the scaled process by  $(X_n)_{n \geq 0}$ . Note that affine scalings of the  $Y_n$  implies that the sequence  $(X_n)_{n \geq 0}$  also does satisfy a recurrence of type (1), where only the coefficients are changed:

$$X_n \stackrel{d}{=} \sum_{r=1}^K A_r^{(n)} X_{I_r^{(n)}}^{(r)} + b^{(n)}, \quad n \geq n_0, \quad (2)$$

with conditions on identical distributions and independence similar to recurrence (1). The coefficients  $A_r^{(n)}$  and  $b^{(n)}$  in the modified recurrence (2) are typically directly computable from the original coefficients  $A_r(n)$ ,  $b(n)$  and the scaling used, see e.g., for the case of random vectors in  $\mathbb{R}^d$ , [25, equations (4)]. Subsequently we consider equations of type (2) together with assumptions on the moments of  $X_n$  which in applications have to be obtained by an appropriate scaling.

For the asymptotic distributional analysis of sequences  $(X_n)_{n \geq 0}$  satisfying (2) the so-called contraction method has become a powerful tool. In the seminal paper [30] Rösler introduced this methodology for deriving a limit law for a special instant of this equation that arises in the analysis of the complexity of the Quicksort algorithm. In the framework of the contraction method first one derives limits of the coefficients  $A_r^{(n)}$ ,  $b^{(n)}$ ,

$$A_r^{(n)} \rightarrow A_r, \quad b^{(n)} \rightarrow b, \quad (n \rightarrow \infty) \quad (3)$$

in an appropriate sense. If with  $n \rightarrow \infty$  also the  $I_r^{(n)}$  become large and it is plausible that the quantities  $X_n$  converge, say to a random variable  $X$ , then, by letting formally  $n \rightarrow \infty$ , equation (2) turns into

$$X \stackrel{d}{=} \sum_{r=1}^K A_r X^{(r)} + b, \quad (4)$$

with  $X^{(1)}, \dots, X^{(K)}$  distributed as  $X$  and  $X^{(1)}, \dots, X^{(K)}, (A_1, \dots, A_k, b)$  independent. Hence, one can use the distributional fixed-point equation (4) to characterize the limit distribution  $\mathcal{L}(X)$ . The idea from Rösler [30] to formalize such an approach and to derive at least weak convergence  $X_n \rightarrow X$  consists of first using the right hand side of (4) to define a map as follows: If  $X_n$  are  $B$ -valued random variables, denote by  $\mathcal{M}(B)$  the space of all probability measures on  $B$  and

$$T : \mathcal{M}(B) \rightarrow \mathcal{M}(B) \quad (5)$$

$$T(\mu) = \mathcal{L} \left( \sum_{r=1}^K A_r Z^{(r)} + b \right), \quad (6)$$

where  $(A_1, \dots, A_K, b), Z^{(1)}, \dots, Z^{(K)}$  are independent and  $Z^{(1)}, \dots, Z^{(K)}$  have distribution  $\mu$ . Then, a random variable  $X$  solves (4) if and only if its distribution  $\mathcal{L}(X)$  is a fixed-point of the map  $T$ . To obtain fixed-points of  $T$  appropriate subspaces of  $\mathcal{M}(B)$  are endowed with a complete metric, such that the restriction of  $T$  becomes a contraction. Then, Banach's fixed-point theorem yields a (in the subspace) unique fixed-point of  $T$  and one can as well use the metric to also derive convergence of  $\mathcal{L}(X_n)$  to  $\mathcal{L}(X)$  in this metric. If the metric is also strong enough to imply weak convergence one has obtained the desired limit law  $X_n \rightarrow X$ .

This approach has been established and applied to a couple of examples in Rösler [30, 31] and Rachev and Rüschendorf [29]. In the latter paper also the flexibility of the approach by using various probability metrics has been demonstrated. Later on general convergence theorems have been derived stating conditions under which convergence of the coefficients of the form (3) together with a contraction property of the map (5) implies convergence in distribution  $X_n \rightarrow X$ . For random variables in  $\mathbb{R}$  with the minimal  $\ell_2$  metric see Rösler [32], and Neininger [24] for  $\mathbb{R}^d$  with the same metric. For a more widely applicable framework for random variables in  $\mathbb{R}^d$  see Neininger and Rüschendorf [25], where in particular various problems with normal limit laws could be solved which seem to be beyond the scope of the minimal  $\ell_p$  metric, see also [26]. An extension of these theorems to continuous time, i.e., to processes  $(X_t)_{t \geq 0}$  satisfying recurrences similar to (2) was given in Janson and Neininger [18].

For the case of random variables in a separable Hilbert space leading to functional limit laws general limit theorems for recurrences (1) have been developed in Drmota, Janson and Neininger [12]. The main application there was a functional limit law for the profile of random trees which, via a certain encoding of the profile, led to random variables in the Bergman space of square integrable analytic functions on a domain in the complex plane. In Eickmeyer and Rüschendorf [13] general limit theorems for recurrences in  $\mathcal{D}[0, 1]$  under the  $L_p$ -topology were developed. Note, that the uniform topology for  $\mathcal{C}[0, 1]$  and the Skorokhod topology for  $\mathcal{D}[0, 1]$  considered in the present paper are finer than the  $L_p$ -topology. In  $\mathcal{C}[0, 1]$ , the uniform topology provides more continuous functionals such as the supremum  $f \mapsto \sup_{t \in [0, 1]} f(t)$  or projections  $f \mapsto f(s_1, \dots, s_k)$ , for fixed  $s_1, \dots, s_k \in [0, 1]$ , to which the continuous mapping theorem can be applied. In  $\mathcal{D}[0, 1]$  these functionals are also appropriate for the continuous mapping theorem if the limit random variable has continuous sample paths.

Besides the minimal  $\ell_p$  metrics the probability metrics that have proved useful in most of the papers mentioned above is the family of Zolotarev metrics  $\zeta_s$  being reviewed and further developed here in section 2. All generalizations from  $\mathbb{R}$  via  $\mathbb{R}^d$  to separable Hilbert spaces are based on the fact that convergence in  $\zeta_s$  implies weak convergence, see section 2. However, for Banach spaces this is not true in general. Counterexamples have been reported in Bentkus and Rachkauskas [4], sketched here in section 2.1. Also completeness of the  $\zeta_s$  metrics on appropriate subspaces of  $\mathcal{M}(B)$  is only known for the case of separable Hilbert spaces, see [12, Theorem 5.1].

Our study of the spaces  $(\mathcal{C}[0, 1], \|\cdot\|_\infty)$  and  $(\mathcal{D}[0, 1], d_{sk})$  is also based on the Zolotarev metrics  $\zeta_s$ . Hence, we mainly have to deal with implications that can be drawn from convergence

in the  $\zeta_s$  metrics as well as with the lack of knowledge about completeness of  $\zeta_s$ . In section 2.3 implications of convergence in the Zolotarev metric are discussed together with additional conditions that enable to deduce in general weak convergence from convergence in  $\zeta_s$ . A key ingredient here is a technique developed in Barbour [2] in the context of Stein's method, see also Barbour and Janson [3]. We also obtain criteria for the uniform integrability of  $\{\|X_n\|_\infty^s \mid n \geq 0\}$  for  $0 \leq s \leq 3$  in the presence of convergence in the Zolotarev metric. This enables in applications as well to obtain moments convergence of the sup-functional.

In section 3 we give general convergence theorems in the framework of the contraction method first for a general separable Banach space and then apply and refine this to the space  $(\mathcal{C}[0, 1], \|\cdot\|_\infty)$  and develop a technique to also apply this to the metric space  $(\mathcal{D}[0, 1], d_{sk})$ . In particular, based on Janson and Kaijser [19], we give a criterion for the finiteness of the Zolotarev metric on appropriate subspaces that can easily be checked in applications.

To compensate for the lack of knowledge about completeness of the  $\zeta_s$  metrics we need to assume that the map  $T$  in (5) has a fixed-point in an appropriate subspace of  $\mathcal{M}(\mathcal{C}[0, 1])$  and  $\mathcal{M}(\mathcal{D}[0, 1])$  respectively. In applications one may verify this existence of a fixed-point either by guessing one successfully: In the application of our framework to Donsker's functional limit theorem in section 4.1 the Wiener measure can easily be guessed and be seen to be the fixed-point of the map  $T$  coming up there. Alternatively, in general the existence of a fixed-point may arise from infinite iteration of the map  $T$ : Applied to some probability measure such an iteration has a series representation for which one may be able to show that it is the desired fixed-point. This path is being taken in an application of our framework outlined in section 4.2.

In section 4.1 we apply our functional contraction method to derive a short proof of Donsker's functional limit theorem. This does not require the full generality of our setting but illustrates how self-similarities can easily be exploited with this approach. The application in section 4.2 is on the asymptotic study of fundamental complexities in Computer Science. Here, the full generality of our approach is needed to obtain a functional limit law. We highlight and discuss the use of our conditions C1–C5 formulated in section 3 on the recurrence (2) at this example. Details on the verification of the conditions are contained in Broutin, Neininger and Sulzbach [6] where, based on the functional limit law, also various long open standing problems on the complexities in Computer Science are solved.

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## 2 The Zolotarev metric

Let  $(B, \|\cdot\|)$  be a real Banach space and  $\mathcal{B}$  its Borel  $\sigma$ -algebra. In the section 2.1, we assume that the norm on  $B$  induces a separable topology. We denote by  $\mathcal{M}(B)$  the set of all probability measures on  $(B, \mathcal{B})$ . First, we introduce the Zolotarev metric  $\zeta_s$  and collect some of its basic properties, mainly covered in [39, 41]. In the second subsection we define our use of the Zolotarev metrics on the metric space  $(\mathcal{D}[0, 1], d_{sk})$ . Although not a Banach space we will be able to declare the Zolotarev metrics  $\zeta_s$  on  $(\mathcal{D}[0, 1], d_{sk})$  using the notion of differentiability of functions  $\mathcal{D}[0, 1] \rightarrow \mathbb{R}$  induced by the supremum norm on  $\mathcal{D}[0, 1]$ . We also comment in Remarks 6 and 7 on delicate measurability issues for the non-separable Banach space  $(B, \|\cdot\|) = (\mathcal{D}[0, 1], \|\cdot\|_\infty)$

and the realm of our methodology when working with the coarser (separable) topology on  $\mathcal{D}[0, 1]$  induced by the Skorokhod metric. In the third subsection conditions that allow to conclude from convergence in  $\zeta_s$  to weak convergence are studied for the case  $(B, \|\cdot\|) = (\mathcal{C}[0, 1], \|\cdot\|_\infty)$  as well as for the case  $(\mathcal{D}[0, 1], d_{sk})$ . We also discuss further implications from  $\zeta_s$ -convergence in these two spaces as well as criteria for finiteness of  $\zeta_s$ . Additional material to the content of this section can be found in the second author's dissertation [38, chapter 2].

## 2.1 Definition and basic properties

For functions  $f : B \rightarrow \mathbb{R}$  which are Fréchet differentiable the derivative of  $f$  at a point  $x$  is denoted by  $Df(x)$ . Note that  $Df(x)$  is an element of the space  $L(B, \mathbb{R})$  of continuous linear forms on  $B$ . We also consider higher order derivatives, where  $D^m f(x)$  denotes the  $m$ -th derivative of  $f$  at a point  $x$ . Thus,  $D^m f(x)$  is a continuous  $m$ -linear (or multilinear) form on  $B$ . The space of continuous multilinear forms  $g : B^m \rightarrow \mathbb{R}$  is equipped with the norm

$$\|g\| = \sup_{\|h_1\| \leq 1, \dots, \|h_m\| \leq 1} |g(h_1, \dots, h_m)|.$$

For a comprehensive account on differentiability in Banach spaces we refer to Cartan [7]. Subsequently  $s > 0$  is fixed and for  $m := \lceil s \rceil - 1$  and  $\alpha := s - m$  we define

$$\mathcal{F}_s = \{f : B \rightarrow \mathbb{R} : \|D^m f(x) - D^m f(y)\| \leq \|x - y\|^\alpha \ \forall x, y \in B\} \quad (7)$$

For  $\mu, \nu \in \mathcal{M}(B)$  the Zolotarev distance between  $\mu$  and  $\nu$  is defined by

$$\zeta_s(\mu, \nu) = \sup_{f \in \mathcal{F}_s} |\mathbf{E}[f(X) - f(Y)]|, \quad (8)$$

where  $X$  and  $Y$  are  $B$ -valued random variables with  $\mathcal{L}(X) = \mu$  and  $\mathcal{L}(Y) = \nu$ . Here  $\mathcal{L}(X)$  denotes the distribution of the random variable  $X$ . The expression in (8) does not need to be finite or even well-defined. However, we have  $\zeta_s(\mu, \nu) < \infty$  if

$$\int \|x\|^s d\mu(x), \int \|x\|^s d\nu(x) < \infty \quad (9)$$

and

$$\int f(x, \dots, x) d\mu(x) = \int f(x, \dots, x) d\nu(x), \quad (10)$$

for any bounded  $k$ -linear form  $f$  on  $B$  and any  $1 \leq k \leq m$ . For random variables  $X, Y$  in  $B$  we use the abbreviation  $\zeta_s(X, Y) := \zeta_s(\mathcal{L}(X), \mathcal{L}(Y))$ . Finiteness of  $\zeta_s(X, Y)$  in  $\mathbb{R}^d$  fails to hold if  $X$  and  $Y$  do not have the same mixed moments up to order  $m$ . The assumption on the finite absolute moment of order  $s$  can be relaxed slightly, see Theorem 4 in [40].

We denote

$$\mathcal{M}_s(B) := \left\{ \mu \in \mathcal{M}(B) \mid \int \|x\|^s d\mu(x) < \infty \right\}$$

and for all  $\nu \in \mathcal{M}_s(B)$  denote

$$\mathcal{M}_s(\nu) := \left\{ \mu \in \mathcal{M}_s(B) \mid \mu \text{ and } \nu \text{ satisfy (10)} \right\}.$$

Then,  $\zeta_s$  is a metric on the space  $\mathcal{M}_s(\nu)$  for any  $\nu \in \mathcal{M}_s(B)$ , see [42, Remark 1, page 198]. A crucial property of  $\zeta_s$  in the context of recursive decompositions of stochastic processes is the following lemma, see Theorem 3 in [40]. A short proof is given for the reader's convenience.

**Lemma 1.** Let  $B'$  be a Banach space and  $g : B \rightarrow B'$  a linear and continuous operator. Then we have

$$\zeta_s(g(X), g(Y)) \leq \|g\|^s \zeta_s(X, Y), \quad \mathcal{L}(X), \mathcal{L}(Y) \in \mathcal{M}_s(\nu).$$

Here,  $\|g\|$  denotes the operator norm of  $g$ , i.e.,  $\|g\| = \sup_{x \in B, \|x\| \leq 1} \|g(x)\|$ .

*Proof.* Note, that  $g$  is also bounded. It suffices to show that

$$\{\|g\|^{-s} f \circ g : f \in \mathcal{F}'_s\} \subseteq \mathcal{F}_s,$$

where  $\mathcal{F}'_s$  is defined analogously to  $\mathcal{F}_s$  in  $B'$ . Let  $f \in \mathcal{F}_s$  and  $\eta := \|g\|^{-s} f \circ g$ . Then  $\eta$  is  $m$ -times continuously differentiable and we have  $D^m \eta(x) = \|g\|^{-s} (D^m(f(g(x))) \circ g^{\otimes m})$  for  $x \in B$ . Here,  $g^{\otimes m} : B^m \rightarrow (B')^m$  denotes the mapping  $g^{\otimes m}(h_1, \dots, h_m) = (g(h_1), \dots, g(h_m))$ . This implies

$$\begin{aligned} \|D^m \eta(x) - D^m \eta(y)\| &= \|\|g\|^{-s} (D^m f(g(x))) \circ g^{\otimes m} - (D^m f(g(y))) \circ g^{\otimes m}\| \\ &\leq \|g\|^{-\alpha} \|g(x) - g(y)\|^\alpha \\ &= \|g\|^{-\alpha} \|g(x - y)\|^\alpha \leq \|x - y\|^\alpha. \end{aligned}$$

The assertion follows.  $\square$

Another basic property is that  $\zeta_s$  is  $(s, +)$  ideal:

**Lemma 2.** The metric  $\zeta_s$  is ideal of order  $s$  on  $\mathcal{M}_s(\nu)$  for any  $\nu \in \mathcal{M}_s(B)$ , i.e., we have

$$\begin{aligned} \zeta_s(cX, cY) &= |c|^s \zeta_s(X, Y) \\ \zeta_s(X + Z, Y + Z) &\leq \zeta_s(X, Y) \end{aligned}$$

for any  $c \in \mathbb{R} \setminus \{0\}$ ,  $\mathcal{L}(X), \mathcal{L}(Y) \in \mathcal{M}_s(\nu)$  and random variables  $Z$  in  $B$ , such that  $(X, Y)$  and  $Z$  are independent.

The lemma directly implies

$$\zeta_s(X_1 + X_2, Y_1 + Y_2) \leq \zeta_s(X_1, Y_1) + \zeta_s(X_2, Y_2) \quad (11)$$

for  $\mathcal{L}(X_1), \mathcal{L}(Y_1) \in \mathcal{M}_s(\nu_1)$  and  $\mathcal{L}(X_2), \mathcal{L}(Y_2) \in \mathcal{M}_s(\nu_2)$  with arbitrary  $\nu_1, \nu_2 \in \mathcal{M}_s(B)$  such that  $(X_1, Y_1)$  and  $(X_2, Y_2)$  are independent.

We want to give a result similar to Lemma 1 where the linear operator may also be random itself. We focus on the case that  $B'$  either equals  $B$  or  $\mathbb{R}$  where an extension to  $\mathbb{R}^d$  for  $d > 1$  is straightforward. Let  $B^*$  be the topological dual of  $B$  and  $\widehat{B}$  be the space of all continuous linear maps from  $B$  to  $B$ . Endowed with the operator norms

$$\|f\|_{\text{op}} = \sup_{x \in B, \|x\| \leq 1} |f(x)|, \quad \|f\|_{\text{op}} = \sup_{x \in B, \|x\| \leq 1} \|f(x)\|,$$

both spaces,  $B^*$  and  $\widehat{B}$  respectively, are Banach spaces. However, these spaces are typically non-separable, hence not suitable for our purposes of measurability. Therefore, we will equip them with smaller  $\sigma$ -algebras. Similar to the use of weak-\* convergence, let  $\mathcal{B}^*$  be the  $\sigma$ -algebra on  $B^*$  that is generated by all continuous (with respect to  $\|\cdot\|_{\text{op}}$ ) linear forms  $\varphi$  on  $B^*$  (i.e., elements of the bidual  $B^{**}$ ) of the form  $\varphi(a) = a(x)$  for some  $x \in B$ . Note that the set of these continuous linear forms coincides with the bidual  $B^{**}$  if and only if  $B$  is reflexive, a property that is not satisfied in our applications. We move on to  $\widehat{B}$  and define  $\widehat{\mathcal{B}}$  to be the  $\sigma$ -algebra generated by all continuous (with respect to  $\|\cdot\|_{\text{op}}$ ) linear maps  $\psi$  from  $\widehat{B}$  to  $B$  of the form  $\psi(a) = a(x)$  for some  $x \in B$ . By Pettis' theorem, we have  $\mathcal{B} = \sigma(\ell \in B^*)$ . Hence, if  $S \subseteq B^*$  with  $\mathcal{B} = \sigma(\ell \in S)$ , then  $\widehat{\mathcal{B}}$  is also generated by the continuous linear forms  $\varrho$  on  $\widehat{B}$  that can be written as  $\varrho(a) = \ell(a(x))$  for  $\ell \in S$  and  $x \in B$ .

Using the separability of  $B$  it is now easy to see that the norm-functionals  $B^* \rightarrow \mathbb{R}, f \mapsto \|f\|_{\text{op}}$  and  $\widehat{B} \rightarrow \mathbb{R}, f \mapsto \|f\|_{\text{op}}$  are  $\mathcal{B}^*$ - $\mathcal{B}(\mathbb{R})$  measurable and  $\widehat{\mathcal{B}}$ - $\mathcal{B}(\mathbb{R})$  measurable respectively.



**Definition 3.** By a random continuous linear form on  $B$  we denote any random variable with values in  $(B^*, \mathcal{B}^*)$ . Analogously, random continuous linear operators on  $B$  are random variables with values in  $(\widehat{B}, \widehat{\mathcal{B}})$ .

Note that the definition of the  $\sigma$ -algebras  $\mathcal{B}^*$  and  $\widehat{\mathcal{B}}$  implies in particular that for any  $a \in B^*$  or  $a \in \widehat{B}$ ,  $x \in B$ , random continuous linear form or operator  $A$  and random variable  $X$  in  $B$ , we have that the compositions  $a(X)$ ,  $A(x)$  and  $A(X)$  are again random variables. The latter property follows from measurability of the map  $(a, x) \mapsto a(x)$  with respect to  $(\mathcal{B}^* \otimes \mathcal{B})$ - $\mathcal{B}(\mathbb{R})$  and  $(\widehat{\mathcal{B}} \otimes \mathcal{B})$ - $\mathcal{B}$  respectively. In the case of the dual space this follows as for any  $r \in \mathbb{R}$  we have

$$\begin{aligned} & \{(a, x) \in B^* \times B : a(x) < r\} \\ &= \bigcup_{k \geq 1} \bigcup_{m \geq 1} \bigcap_{n \geq m} \bigcup_{i \geq 1} \{a \in B^* : a(e_i) < r - 1/k\} \times \{x \in B : \|x - e_i\| < 1/n\}, \end{aligned}$$

where  $\{e_i | i \geq 1\}$  denotes a countable dense subset of  $B$ ; the case  $\widehat{B}$  being analogous.

The following lemma follows from Lemma 1 by conditioning.

**Lemma 4.** Let  $\mathcal{L}(X), \mathcal{L}(Y) \in \mathcal{M}_s(\nu)$  for some  $\nu \in \mathcal{M}_s(B)$ . Then, for any random linear continuous form or operator  $A$  with  $\mathbf{E} [\|A\|_{\text{op}}^s] < \infty$  independent of  $X$  and  $Y$ , we have

$$\zeta_s(A(X), A(Y)) \leq \mathbf{E} [\|A\|_{\text{op}}^s] \zeta_s(X, Y).$$

Zolotarev gave upper and lower bounds for  $\zeta_s$ , most of them being valid if more structure on  $B$  is assumed. Subsequently, only an upper bound in terms of the minimal  $\ell_p$  metric is needed. For  $p > 0$  and  $\mu, \nu \in \mathcal{M}_p(B)$  the minimal  $\ell_p$  distance between  $\mu$  and  $\nu$  is defined by

$$\ell_p(\mu, \nu) = \inf \mathbf{E} [\|X - Y\|^p]^{(1/p) \wedge 1},$$

where the infimum is taken over all common distributions  $\mathcal{L}(X, Y)$  with marginals  $\mathcal{L}(X) = \mu$  and  $\mathcal{L}(Y) = \nu$ . We abbreviate  $\ell_p(X, Y) := \ell_p(\mathcal{L}(X), \mathcal{L}(Y))$ .

The next lemma gives an upper bound of  $\zeta_s$  in terms of  $\ell_s$  where the first statement follows from the Kantorovich-Rubinstein theorem and the second essentially coincides with Lemma 5.7 in [12].

**Lemma 5.** Let  $\mathcal{L}(X), \mathcal{L}(Y) \in \mathcal{M}_s(\nu)$  for some  $\nu \in \mathcal{M}_s(B)$  with  $B$  separable. If  $s \leq 1$  then

$$\zeta_s(X, Y) = \ell_s(X, Y). \quad (12)$$

If  $s > 1$  then

$$\zeta_s(X, Y) \leq \left( \mathbf{E} [\|X\|^s]^{1-1/s} + \mathbf{E} [\|Y\|^s]^{1-1/s} \right) \ell_s(X, Y).$$

If  $X_n, X$  are real-valued random variables,  $n \geq 1$ , then  $\zeta_s(X_n, X) \rightarrow 0$  implies convergence of absolute moments of order up to  $s$  since there is a constant  $C_s > 0$  such that the function  $x \mapsto C_s |x|^s$  is an element of  $\mathcal{F}_s$ , hence  $|\mathbf{E} [|X_n|^s] - \mathbf{E} [|X|^s]| \leq C_s^{-1} \zeta_s(X_n, X)$ .

We proceed with the fundamental question of how convergence in the  $\zeta_s$  distance relates to weak convergence on  $B$ . By the first statement of the previous lemma, or more elementary, by the proof of the Portmanteau Lemma [5, Theorem 2.1, ii)  $\Rightarrow$  iii)] one obtains that for  $0 < s \leq 1$  convergence in the  $\zeta_s$  metric implies weak convergence, see also [12, page 300].

If  $B$  is a separable Hilbert space, then for any  $s > 0$  convergence in the  $\zeta_s$  metric implies weak convergence. This was first proved by Giné and León in [17], see also Theorem 5.1 in [12]. In infinite-dimensional Banach spaces convergence in the  $\zeta_s$  metric does not need to imply weak convergence: For any probability distribution  $\mu$  on  $B = \mathcal{C}[0, 1]$  with zero mean and

$\int \|x\|_\infty^s d\mu(s) < \infty$  for some  $s > 2$  that is pregaussian, i.e. there exists a gaussian measure  $\nu$  on  $\mathcal{C}[0, 1]$  with zero mean and the same covariance as  $\mu$ , one has  $\zeta_s$ -convergence of a rescaled sum of independent random variables with distribution  $\mu$  towards  $\nu$ , see inequality (48) in [39]. However, pregaussian probability distributions supported by a bounded subset of  $\mathcal{C}[0, 1]$  that do not satisfy the central limit theorem can be found in [37]. For the central limit theorem in Banach spaces see [21]. Note that convergence with respect to  $\zeta_s$  implies convergence of the characteristic functions, hence  $\zeta_s(X_n, X) \rightarrow 0$  implies that  $\mathcal{L}(X)$  is the only possible accumulation point of  $(\mathcal{L}(X_n))_{n \geq 0}$  in the weak topology.

## 2.2 The Zolotarev metric on $(\mathcal{D}[0, 1], d_{sk})$

In this section we discuss our use of the Zolotarev metric on the metric space  $(\mathcal{D}[0, 1], d_{sk})$  of càdlàg functions on  $[0, 1]$  endowed with the Skorokhod metric defined by

$$d_{sk}(f, g) = \inf\{\varepsilon > 0 \mid \max\{|f(t) - g(\tau(t))|, |\tau(t) - t|\} < \varepsilon \text{ for all } t \in [0, 1]\}$$

with monotonically increasing and bijective  $\tau : [0, 1] \rightarrow [0, 1]$ ,

The Borel  $\sigma$ -algebra of the induced topology is denoted by  $\mathcal{B}_{sk}$ . For a general introduction to this space see Billingsley [5, chapter 3]. In particular,  $(\mathcal{D}[0, 1], d_{sk})$  is a Polish space,  $\mathcal{B}_{sk}$  coincides with the  $\sigma$ -algebra generated by the finite dimensional projections, the  $\sigma$ -algebra generated by the open spheres (with respect to the uniform metric) and the  $\sigma$ -algebra generated by all norm-continuous linear forms on  $\mathcal{D}[0, 1]$ , see [28, Theorem 3]. Subsequently, norm on  $\mathcal{D}[0, 1]$  will always refer to the uniform norm  $\|\cdot\|_\infty$ . Moreover, the norm function  $\mathcal{D}[0, 1] \rightarrow \mathbb{R}, f \mapsto \|f\|_\infty$  is  $\mathcal{B}_{sk}$ - $\mathcal{B}(\mathbb{R})$  measurable. By Theorem 2, respectively Theorem 4, in [28], any norm-continuous linear form on  $\mathcal{D}[0, 1]$  is  $\mathcal{B}_{sk}$ - $\mathcal{B}(\mathbb{R})$  measurable and any norm-continuous linear map from  $\mathcal{D}[0, 1]$  to  $\mathcal{D}[0, 1]$  is  $\mathcal{B}_{sk}$ - $\mathcal{B}_{sk}$  measurable. Recently, Janson and Kaijser [19, Theorem 15.8] generalized the latter result and proved that any norm-continuous  $k$ -linear form on  $\mathcal{D}[0, 1]$  is  $(\mathcal{B}_{sk})^{\otimes k}$ - $\mathcal{B}(\mathbb{R})$  measurable. We do however not know whether  $\mathcal{F}_s$  defined in (7) based on the uniform norm on  $\mathcal{D}[0, 1]$  is a subset of the  $\mathcal{B}_{sk}$ - $\mathcal{B}(\mathbb{R})$  measurable functions. Hence, we denote the  $\mathcal{B}_{sk}$ - $\mathcal{B}(\mathbb{R})$  measurable functions by  $\mathcal{E}$  and define the Zolotarev metrics analogously to (8) by

$$\zeta_s(\mu, \nu) = \sup_{f \in \mathcal{F}_s \cap \mathcal{E}} |\mathbf{E}[f(X) - f(Y)]|,$$

where  $X$  and  $Y$  are  $(\mathcal{D}[0, 1], d_{sk})$ -valued random variables with  $\mathcal{L}(X) = \mu$  and  $\mathcal{L}(Y) = \nu$ .

We denote by  $\mathcal{M}_s(\mathcal{D}[0, 1])$  the set of probability distributions  $\mu$  on  $\mathcal{D}[0, 1]$  with  $\int \|x\|_\infty^s d\mu(x) < \infty$  and for  $\nu \in \mathcal{M}_s(\mathcal{D}[0, 1])$ , we define  $\mathcal{M}_s(\nu)$  to be the subset of measures  $\mu$  from  $\mathcal{M}_s(\mathcal{D}[0, 1])$  satisfying (10). Then,  $\zeta_s$  is a metric on  $\mathcal{M}_s(\nu)$  for all  $\nu \in \mathcal{M}_s(\mathcal{D}[0, 1])$ , Lemma 1, Lemma 2, inequality (11), Lemma 5 where (12) is to be replaced by  $\zeta_s(X, Y) \leq \ell_s(X, Y)$ , and the implication  $\zeta_s(X_n, X) \rightarrow 0 \Rightarrow X_n \rightarrow X$  in distribution if  $0 < s \leq 1$  remain valid.

The situation becomes more involved concerning random linear forms and operators as defined in Definition 3 in the separable Banach case. Let  $\mathcal{D}[0, 1]^*$  and  $\widehat{\mathcal{D}[0, 1]}$  be the dual space respectively the space of norm-continuous endomorphisms on  $\mathcal{D}[0, 1]$  as in the Banach case. For reasons of measurability we need to restrict to smaller subspaces. Let  $\mathcal{D}[0, 1]_c^* \subseteq \mathcal{D}[0, 1]^*$  be the subset of functions that are additionally continuous with respect to  $d_{sk}$ . Analogously,  $\widehat{\mathcal{D}[0, 1]}_c \subseteq \widehat{\mathcal{D}[0, 1]}$  are those endomorphism which are continuous regarded as maps from  $(\mathcal{D}[0, 1], d_{sk})$  to  $(\mathcal{D}[0, 1], d_{sk})$ . We endow  $\mathcal{D}[0, 1]_c^*$  with the  $\sigma$ -algebra generated by the function  $f \mapsto \|f\|_{\text{op}}$  and all elements  $\varphi$  of  $\mathcal{D}[0, 1]^{**}$  of the form  $\varphi(a) = a(x)$  for some  $x \in \mathcal{D}[0, 1]$ . Also the  $\sigma$ -algebra on  $\widehat{\mathcal{D}[0, 1]}_c$  is generated by the function  $f \mapsto \|f\|_{\text{op}}$  and the continuous linear maps  $\psi: \widehat{\mathcal{D}[0, 1]} \rightarrow \mathcal{D}[0, 1]$  of the form  $\varphi(a) = a(x)$  for some  $x \in \mathcal{D}[0, 1]$ . Under these conditions, we have the same measurability results as in the Banach case and Lemma 4 remains valid.



**Remark 6.** Note that we could as well develop the use of the Zolotarev metric together with the contraction method for the Banach space  $(\mathcal{D}[0, 1], \|\cdot\|_\infty)$ . This can be done analogously to the discussion of sections 2.3 and 3 and in fact would lead to a proof of Donsker's theorem similar to the one given in section 4.1.1 when replacing the linear interpolation  $S^n = (S_t^n)_{t \in [0, 1]}$  by a constant (càdlàg) interpolation of the random walk. However, the applicability of such a framework seems to be limited due to measurability problems in the non-separable space  $(\mathcal{D}[0, 1], \|\cdot\|_\infty)$ : For example, the random function  $X$  defined by

$$X_t = \mathbf{1}_{\{t \geq U\}}, \quad t \in [0, 1],$$

with  $U$  being uniformly distributed on the unit interval is known to be non-measurable with respect to the Borel- $\sigma$ -algebra on  $(\mathcal{D}[0, 1], \|\cdot\|_\infty)$ . However, we have applications of the functional contraction method developed here in mind on processes with jumps at random times. A typical example in the context of random trees is given in section 4.2, see also [6]. Hence, in order to even have measurability of the processes considered it requires to work with the coarser Skorokhod topology than the uniform topology and this is our reason for using the Zolotarev metric on  $(\mathcal{D}[0, 1], d_{sk})$  instead of  $(\mathcal{D}[0, 1], \|\cdot\|_\infty)$ .

**Remark 7.** Although the methodology developed below covers sequences  $(X_n)_{n \geq 0}$  of processes with jumps at random times these times will typically need to be the same for all  $n \geq n_0$ . In particular sequences of processes with jumps at random times that require a (uniformly small) deformation of the time scale to be aligned cannot be covered by this methodology. The technical reason is that in condition **C1** below, see section 3, the convergence of the random continuous endomorphisms  $\|A_r^{(n)} - A_r\|_s$  is with respect to the operator norm based on the uniform norm which in general does not allow a deformation of the time scale.

### 2.3 Weak convergence on $(\mathcal{C}[0, 1], \|\cdot\|_\infty)$ and $(\mathcal{D}[0, 1], d_{sk})$

In this subsection we only consider the spaces  $(\mathcal{C}[0, 1], \|\cdot\|_\infty)$  and  $(\mathcal{D}[0, 1], d_{sk})$ .

For random variables  $X = (X(t))_{t \in [0, 1]}$ ,  $Y = (Y(t))_{t \in [0, 1]}$  in  $(\mathcal{C}[0, 1], \|\cdot\|_\infty)$  with  $\zeta_s(X, Y) < \infty$  we have

$$\zeta_s((X(t_1), \dots, X(t_k)), (Y(t_1), \dots, Y(t_k))) \leq k^{s/2} \zeta_s(X, Y) \quad (13)$$

for all  $0 \leq t_1 \leq \dots \leq t_k \leq 1$ . This follows from Lemma 1 using the continuous and linear function  $g : \mathcal{C}[0, 1] \rightarrow \mathbb{R}^k$ ,  $g(f) = (f(t_1), \dots, f(t_k))$  and observing that  $\|g\| = \sqrt{k}$ . The bound  $\zeta_s((X(t_1), \dots, X(t_k)), (Y(t_1), \dots, Y(t_k))) \leq \zeta_s(X, Y)$  can be obtained if  $\mathbb{R}^k$  is endowed with the max-norm instead of the Euclidean norm. However, no use of this is made here. Hence, we obtain for random variables  $X_n, X$  in  $(\mathcal{C}[0, 1], \|\cdot\|_\infty)$ ,  $n \geq 1$ , the implication

$$\zeta_s(X_n, X) \rightarrow 0 \quad \Rightarrow \quad X_n \xrightarrow{\text{fdd}} X.$$

Here,  $\xrightarrow{\text{fdd}}$  denotes weak convergence of all finite dimensional marginals of the processes. Additionally, if  $Z$  is a random variable in  $[0, 1]$ , independent of  $(X_n)$  and  $X$ , then applying Lemma 4 with the random continuous linear form  $A$  defined by  $A(f) = f(Z)$  implies

$$\zeta_s(X_n(Z), X(Z)) \leq \mathbf{E}[Z^s] \zeta_s(X_n, X). \quad (14)$$

In the càdlàg case, i.e.  $X = (X(t))_{t \in [0, 1]}$ ,  $Y = (Y(t))_{t \in [0, 1]}$  being random variables in  $(\mathcal{D}[0, 1], d_{sk})$  inequality (13) remains true by Lemma 1. (The fact that  $g$  is not continuous with respect to the product Skorokhod topology does not cause problems since measurability is sufficient here.) Next, in general, the operator  $A$  is no element of  $\mathcal{D}[0, 1]_c^*$ . Hence, we cannot apply

Lemma 4 to deduce (14). Nevertheless, by Theorem 2 in [40], the convergence of the characteristic functions of  $X_n(t)$  is uniform in  $t$ , hence we also have convergence in distribution of  $X_n(Z)$  to  $X(Z)$ . The same argument works for the moments of  $X_n(Z)$ . We summarize these properties in the following proposition, where  $\xrightarrow{d}$  denotes convergence in distribution.

**Proposition 8.** *For random variables  $X_n, X$  in  $(\mathcal{C}[0, 1], \|\cdot\|_\infty)$  or  $(\mathcal{D}[0, 1], d_{sk})$ ,  $n \geq 1$ , with  $\zeta_s(X_n, X) \rightarrow 0$  for  $n \rightarrow \infty$  we have*

$$X_n \xrightarrow{\text{fdd}} X.$$

$\mathcal{L}(X)$  is the only possible accumulation point of  $(\mathcal{L}(X_n))_{n \geq 1}$  in the weak topology. For all  $t \in [0, 1]$  we have

$$X_n(t) \xrightarrow{d} X(t), \quad \mathbf{E}[|X_n(t)|^s] \rightarrow \mathbf{E}[|X(t)|^s].$$

For any random variable  $Z$  in  $[0, 1]$  being independent of  $(X_n)$  and  $X$  we have

$$\mathbf{E}[|X_n(Z)|^s] \rightarrow \mathbf{E}[|X(Z)|^s], \quad X_n(Z) \xrightarrow{d} X(Z).$$

To conclude from convergence in the  $\zeta_s$  metric to weak convergence on  $(\mathcal{C}[0, 1], \|\cdot\|_\infty)$  or  $(\mathcal{D}[0, 1], d_{sk})$  further assumptions are needed. Let, for  $r > 0$ ,

$$\begin{aligned} \mathcal{C}_r[0, 1] := \{f \in \mathcal{C}[0, 1] \mid \exists 0 = t_1 < t_2 < \dots < t_\ell = 1 \ \forall i = 1, \dots, \ell : \\ |t_i - t_{i-1}| \geq r, f|_{[t_{i-1}, t_i]} \text{ is linear} \} \end{aligned} \quad (15)$$

denote the set of all continuous functions for which there is a decomposition of  $[0, 1]$  into intervals of length at least  $r$  such that the function is piecewise linear on those intervals. Analogously, we define

$$\begin{aligned} \mathcal{D}_r[0, 1] := \{f \in \mathcal{D}[0, 1] \mid \exists 0 = t_1 < t_2 < \dots < t_\ell = 1 \ \forall i = 1, \dots, \ell : \\ |t_i - t_{i-1}| \geq r, f|_{[t_{i-1}, t_i]} \text{ is constant, continuous in } 1 \}. \end{aligned} \quad (16)$$

**Theorem 9.** *Let  $X_n$  be random variables in  $\mathcal{C}_{r_n}[0, 1]$ ,  $n \geq 0$ , and  $X$  a random variable in  $\mathcal{C}[0, 1]$ . Assume that for  $0 < s \leq 3$  with  $s = m + \alpha$  as in (7)*

$$\zeta_s(X_n, X) = o\left(\log^{-m}\left(\frac{1}{r_n}\right)\right). \quad (17)$$

*Then  $X_n \rightarrow X$  in distribution. The assertion remains valid if  $\mathcal{C}[0, 1], \mathcal{C}_{r_n}[0, 1]$  are replaced by  $\mathcal{D}[0, 1], \mathcal{D}_{r_n}[0, 1]$  endowed with the Skorokhod topology and  $X$  has continuous sample paths.*

As discussed above,  $\zeta_s$  convergence does not imply weak convergence in the spaces  $\mathcal{C}[0, 1]$  and  $\mathcal{D}[0, 1]$  without any further assumption such as (17). In the counterexample from [37], the sequence  $S_n/\sqrt{n}$  there converges to a gaussian limit with respect to  $\zeta_s$  for  $2 < s \leq 3$  where the rate of convergence is upper bounded by the order  $n^{1-s/2}$ , see [39] or [38]. Moreover, the sequence is piecewise linear but the sequence  $r_n$  can only be chosen of the order  $(cn)^{-2n}$  for some  $c > 0$ . Hence, (17) is not satisfied.

In applications such as our proof of Donsker's functional limit law in section 4.1.1 or the application of the present methodology to a problem from the probabilistic analysis of algorithms in [6] the rate of convergence will typically be of polynomial order which is fairly sufficient.

We postpone the proof of the theorem to the end of this section and state two variants, where the first one, Corollary 10, contains a slight relaxation of the assumptions that is useful in applications such as in the analysis of the complexity of partial match queries in quadrees, see section ?? or [6]. The second one will be needed in the case  $s > 2$ , see section 4.1.

**Corollary 10.** Let  $X_n, X$  be  $\mathcal{C}[0, 1]$  valued random variables,  $n \geq 0$ , and  $0 < s \leq 3$  with  $s = m + \alpha$  as in (7). Suppose  $X_n = Y_n + h_n$  with  $Y_n$  being  $\mathcal{C}[0, 1]$  valued random variables and  $h_n \in \mathcal{C}[0, 1]$ ,  $n \geq 0$ , such that  $\|h_n - h\|_\infty \rightarrow 0$  for a  $h \in \mathcal{C}[0, 1]$  and

$$\mathbf{P}(Y_n \notin \mathcal{C}_{r_n}[0, 1]) \rightarrow 0. \quad (18)$$

If

$$\zeta_s(X_n, X) = o\left(\log^{-m}\left(\frac{1}{r_n}\right)\right),$$

then

$$X_n \xrightarrow{d} X.$$

The statement remains true if  $\mathcal{C}[0, 1]$  and  $\mathcal{C}_{r_n}[0, 1]$  are replaced by  $\mathcal{D}[0, 1]$  and  $\mathcal{D}_{r_n}[0, 1]$  endowed with the Skorokhod topology respectively,  $X$  has continuous sample paths and  $h$  remains continuous.

**Corollary 11.** Let  $X_n, Y_n, X$  be  $\mathcal{C}[0, 1]$  valued random variables,  $n \geq 0$ , and  $0 < s \leq 3$  with  $s = m + \alpha$  as in (7). Suppose  $X_n \in \mathcal{C}_{r_n}[0, 1]$  for all  $n$  and  $Y_n \rightarrow X$  in distribution. If

$$\zeta_s(X_n, Y_n) = o\left(\log^{-m}\left(\frac{1}{r_n}\right)\right),$$

then

$$X_n \xrightarrow{d} X.$$

The statement remains true if  $\mathcal{C}[0, 1]$  and  $\mathcal{C}_{r_n}[0, 1]$  are replaced by  $\mathcal{D}[0, 1]$  and  $\mathcal{D}_{r_n}[0, 1]$  endowed with the Skorokhod topology respectively and  $X$  has continuous sample paths.

In  $\mathcal{C}[0, 1]$  (or  $\mathcal{D}[0, 1]$  if the limit  $X$  has continuous paths), convergence in distribution implies distributional convergence of the supremum norm  $\|X_n\|_\infty$  by the continuous mapping theorem. In applications, one is also interested in convergence of moments of the supremum. For random variables  $X$  in  $\mathcal{C}[0, 1]$  or  $\mathcal{D}[0, 1]$ , we denote by

$$\|X\|_s := (\mathbf{E}[\|X\|_\infty^s])^{(1/s) \wedge 1}$$

the  $L_s$ -norm of the supremum norm.

**Theorem 12.** Let  $X_n, X$  be  $\mathcal{C}[0, 1]$  valued random variables and  $0 < s \leq 3$  with  $\|X_n\|_s, \|X\|_s < \infty$  for all  $n \geq 0$ . Suppose one of the following conditions is satisfied:

1.  $X_n \in \mathcal{C}_{r_n}[0, 1]$  for all  $n$  and

$$\zeta_s(X_n, X) = o\left(\log^{-m}\left(\frac{1}{r_n}\right)\right). \quad (19)$$

2.  $X_n = Y_n + h_n$  with  $Y_n$  being  $\mathcal{C}[0, 1]$  valued random variables and  $h_n \in \mathcal{C}[0, 1]$ ,  $n \geq 0$ , such that  $\|h_n - h\|_\infty \rightarrow 0$  for a  $h \in \mathcal{C}[0, 1]$ ,

$$\mathbf{E}[\|X_n\|_\infty^s \mathbf{1}_{\{Y_n \notin \mathcal{C}_{r_n}[0, 1]\}}] \rightarrow 0 \quad (20)$$

and

$$\zeta_s(X_n, X) = o\left(\log^{-m}\left(\frac{1}{r_n}\right)\right).$$

3.  $(Y_n)_{n \geq 0}$  is a sequence of  $\mathcal{C}[0, 1]$  valued random variables with  $Y_n \leq Z$  almost surely for a  $\mathcal{C}[0, 1]$  valued random variable  $Z$  with  $\|Z\|_s < \infty$ ,  $X_n \in \mathcal{C}_{r_n}[0, 1]$  for all  $n$  and

$$\zeta_s(X_n, Y_n) = o\left(\log^{-m}\left(\frac{1}{r_n}\right)\right).$$

Then  $\{\|X_n\|_\infty^s \mid n \geq 0\}$  is uniformly integrable. All statements remain true if  $\mathcal{C}[0, 1], \mathcal{C}_{r_n}[0, 1]$  are replaced by  $\mathcal{D}[0, 1], \mathcal{D}_{r_n}[0, 1]$  and  $h$  in item 2 remains continuous.

It is of interest whether the metric space  $(\mathcal{M}_s(\nu), \zeta_s)$  is complete. This is true for  $0 < s \leq 1$ . Also, in the case that  $B$  is a separable Hilbert space, this holds true, see Theorem 5.1 in [12]. Nevertheless, the problem remains open in the general case, in particular in the cases  $\mathcal{C}[0, 1]$  and  $\mathcal{D}[0, 1]$  with  $s > 1$ . We can only state the following proposition.

**Proposition 13.** *Let  $B = (\mathcal{C}[0, 1], \|\cdot\|_\infty)$  or  $B = (\mathcal{D}[0, 1], d_{sk})$ ,  $s > 0$  and  $\nu \in \mathcal{M}_s(B)$ . Furthermore, let  $(\mu_n)_{n \geq 0}$  be a sequence of probability measures from  $\mathcal{M}_s(\nu)$  which is a Cauchy sequence with respect to the  $\zeta_s$  metric. Then there exists a probability measure  $\mu$  on  $\mathbb{R}^{[0, 1]}$  such that, as  $n \rightarrow \infty$ ,*

$$\mu_n \xrightarrow{\text{fdd}} \mu. \quad (21)$$

*Proof.* Let  $\mathcal{L}(X_n) = \mu_n$  for all  $n \geq 0$ . According to (13),  $(X_n(t_1), \dots, X_n(t_k))_{n \geq 0}$  is a Cauchy sequence and hence it exists a random variable  $Y_{t_1, \dots, t_k}$  in  $\mathbb{R}^k$  with

$$(X_n(t_1), \dots, X_n(t_k)) \xrightarrow{d} Y_{t_1, \dots, t_k} \quad (n \rightarrow \infty).$$

The set of distributions of  $Y_{t_1, \dots, t_k}$  for  $0 \leq t_1 < \dots < t_k \leq 1$  and  $k \in \mathbb{N}$  is consistent so there exists a process  $Y$  on the product space  $\mathbb{R}^{[0, 1]}$  whose distribution satisfies (21).  $\square$

**Remark 14.** If the distribution  $\mu$  found in Proposition 13 has a version with continuous paths then condition (10) for  $\mu_n$  and  $\mu$  is satisfied.

We now come to the proofs of the theorems and corollaries. Theorem 9 essentially follows directly from Theorem 2 in [2], see also [3]. Nevertheless, we present a version of the proof given there so that we can deduce the variants and implications given in our other statements. A basic tool are Theorems 2.2, 2.3 and 2.4 in Billingsley [5]. The following lemma is a special case of Theorem 2.4 there.

**Lemma 15.** *Let  $(\mu_n)_{n \geq 0}, \mu$  be probability measures on a separable metric space  $(S, d)$ . For  $r > 0, x \in S$  let  $B_r(x) = \{y \in S : d(x, y) < r\}$ . If for any  $x_1, \dots, x_k \in S, \gamma_1, \dots, \gamma_k > 0$  with  $\mu(\partial B_{\gamma_i}(x_i)) = 0$  for  $i = 1, \dots, k$  it holds*

$$\mu_n\left(\bigcap_{i \in I} B_{\gamma_i}(x_i)\right) \rightarrow \mu\left(\bigcap_{i \in I} B_{\gamma_i}(x_i)\right),$$

where  $I = \{1, \dots, k\}$ , then  $\mu_n \rightarrow \mu$  weakly.

**Remark 16.** If  $(S, d) = (\mathcal{D}[0, 1], d_{sk})$  then the assertion of the latter lemma remains true when the balls  $B_{\gamma_i}(x_i)$  are still defined with respect to the uniform distance and  $\mu(\mathcal{C}[0, 1]) = 1$ . Technically, this follows from Theorem 2.3 in [5] upon choosing  $\mathcal{A}_P$  there to be the set of spheres (in the uniform distance) which are  $\mu$  continuous and whose centers lie in  $\mathcal{C}[0, 1]$ . The conditions of the theorem are then satisfied as the interior (with respect to the Skorokhod topology) of a sphere (in the uniform distance) is not empty if its center  $x$  is a continuous function. As the latter assertion is not true for arbitrary  $x \in \mathcal{D}[0, 1]$  — points which are close in the Skorokhod distance can have arbitrarily large uniform distance — our approach does not extend to the case of a non-continuous limiting distribution.

A main difficulty in deducing weak convergence from convergence in  $\zeta_s$  compared to the Hilbert space case is the non-differentiability of the norm function  $x \mapsto \|x\|_\infty$ , see [10, page 147]. We will instead use the smoother  $L_p$ -norm which approximates the supremum norm in the sense that

$$L_p(x) \rightarrow \|x\|_\infty, \quad (22)$$

for any fixed  $x \in \mathcal{C}[0, 1]$  as  $p \rightarrow \infty$ .

For the remaining part of this section,  $p$ , for fixed values or tending to infinity, is always to be understood as an even integer with  $p \geq 4$ . We use the Bachmann-Landau big- $O$  notation.

**Lemma 17.** *For  $x, y \in \mathcal{C}[0, 1]$  let*

$$L_p(x) = \left( \int_0^1 [x(t)]^p dt \right)^{1/p}, \quad \psi_{p,y}(x) = L_p \left( (1 + [x - y]^2)^{1/2} \right).$$

*Then  $L_p$  is smooth on  $\mathcal{C}[0, 1] \setminus \{\mathbf{0}\}$  where  $\mathbf{0}$  is the zero-function and  $\psi_{p,y}$  is smooth on  $\mathcal{C}[0, 1]$  for all  $y \in \mathcal{C}[0, 1]$ . Furthermore for  $k \in \{1, 2, 3\}$ , we have*

$$\|D^k L_p(x)\| = O(p^{k-1} L_p^{1-k}(x)),$$

*uniformly for  $p$  and  $x \in \mathcal{C}[0, 1] \setminus \{\mathbf{0}\}$ . Moreover, again for  $k \in \{1, 2, 3\}$ ,*

$$\|D^k \psi_{p,y}(x)\| = O(p^{k-1}) \quad (23)$$

*uniformly for  $p$  and  $x, y \in \mathcal{C}[0, 1]$ . All assertions remain valid when  $\mathcal{C}[0, 1]$  is replaced by  $\mathcal{D}[0, 1]$ , moreover both functions  $L_p$  and  $\psi_{p,y}$  are continuous with respect to the Skorokhod metric for all  $p$  and  $y \in \mathcal{D}[0, 1]$ .*

*Proof.* The smoothness properties are obvious. Differentiating  $L_p$  by the chain rule yields

$$DL_p(x)[h] = \left( \int_0^1 [x(t)]^p dt \right)^{1/p-1} \int_0^1 [x(t)]^{p-1} h(t) dt.$$

For  $h \in \mathcal{C}[0, 1]$  with  $\|h\| \leq 1$  by Jensen's inequality and  $L_p(h) \leq \|h\|$  we obtain that the right hand side of the latter display is uniformly bounded by 1. The bounds on the norms of the higher order derivatives follow along the same lines. Using the same ideas, it is easy to see that

$$\|D^k \psi_{p,y}(x)\| = O \left( \sum_{j=1}^k p^{j-1} L_p^{1-j}(\omega_y(x)) \right),$$

uniformly in  $p$  and  $x, y \in \mathcal{C}[0, 1]$  where  $\omega_y(x) = (1 + |x - y|^2)^{1/2}$ . This gives (23).  $\square$

Note that the convergence in (22) holds pointwise; it is easy to construct a sequence of continuous functions  $(x_p)_{p \geq 0}$  such that  $L_p(x_p) \rightarrow 0$  and  $\|x_p\|_\infty \rightarrow \infty$  as  $p \rightarrow \infty$ . Additionally to the obvious bound  $L_p(x) \leq \|x\|_\infty$  we will need the following simple lemma which contains sort of a converse of this inequality.

**Lemma 18.** *Let  $\lambda$  denote the Lebesgue measure on the unit interval and let  $\gamma > 0$  and  $0 < \vartheta < 1$ .*

*(a) For all  $f \in \mathcal{D}_r[0, 1]$  we have*

$$\|f\|_\infty \geq \gamma \Rightarrow \lambda(\{t : |f(t)| \geq (1 - \vartheta)\gamma\}) \geq r.$$

*Moreover, for any  $g \in \mathcal{C}[0, 1]$  there exists a  $\delta = \delta(g, \gamma, \vartheta) > 0$  such that*

$$\|f - g\|_\infty \geq \gamma \Rightarrow \lambda(\{t : |f(t) - g(t)| \geq (1 - \vartheta)\gamma\}) \geq \min(r, \delta).$$

(b) For all  $f \in \mathcal{C}_r[0, 1]$  we have

$$\|f\|_\infty \geq \gamma \Rightarrow \lambda(\{t : |f(t)| \geq (1 - \vartheta)\gamma\}) \geq \frac{\vartheta}{2}r.$$

Moreover, for  $g \in \mathcal{C}[0, 1]$  there exists a  $\delta = \delta(g, \gamma, \vartheta) > 0$  with

$$\|f - g\|_\infty \geq \gamma \Rightarrow \lambda(\{t : |f(t) - g(t)| \geq (1 - \vartheta)\gamma\}) \geq \frac{\vartheta}{4} \min(r, \delta).$$

*Proof.* Ad (a): The first assertion is trivial. The second one follows by choosing  $\delta > 0$  small enough such that  $|g(x) - g(y)| \leq \frac{\vartheta\gamma}{2}$  for all  $|x - y| < \delta$ .

Ad (b): For the first statement assume  $\|f\|_\infty \geq \gamma$  and let  $[e_0, e_1]$  be an interval where  $f$  attains its maximum. A geometric argument shows that the quantity  $\lambda(\{t \in [e_0, e_1] : |f(t)| \geq (1 - \vartheta)\gamma\})$  is minimized when  $f(e_0) = \gamma$  and  $f(e_1) = -(1 - \vartheta)\gamma$ . In this case, the quantity equals  $\frac{\vartheta}{2}r$  which implies assertion. Finally, the last statement follows from a combination of the latter argument and by choosing  $\delta > 0$  again such that  $|g(x) - g(y)| \leq \frac{\vartheta\gamma}{2}$  for all  $|x - y| < \delta$ .  $\square$

We start with the proofs of Theorem 9 and its corollaries in the continuous case.

*Proof.* (Theorem 9) For  $r > 0, x \in \mathcal{C}[0, 1]$  let  $B_r(x) = \{y \in \mathcal{C}[0, 1] : \|y - x\|_\infty < r\}$ . According to Lemma 15, we need to verify that

$$\mathbf{P}\left(X_n \in \bigcap_{i \in I} B_{\gamma_i}(x_i)\right) \rightarrow \mathbf{P}\left(X \in \bigcap_{i \in I} B_{\gamma_i}(x_i)\right), \quad (24)$$

for  $I = \{1, \dots, k\}$  and  $x_1, \dots, x_k \in S, \gamma_1, \dots, \gamma_k > 0$  such that  $\mathbf{P}(X \in (\partial B_{\gamma_i}(x_i))) = 0$ . The lack of uniformity in (22) leads us to find lower and upper bounds on the desired quantity. We will establish

$$\limsup_{n \rightarrow \infty} \mathbf{P}\left(X_n \in \bigcap_{i \in I} B_{\gamma_i}(x_i)\right) \leq \mathbf{P}\left(X \in \bigcap_{i \in I} B_{\gamma_i}(x_i)\right) \quad (25)$$

and

$$\liminf_{n \rightarrow \infty} \mathbf{P}\left(X_n \in \bigcap_{i \in I} B_{\gamma_i}(x_i)\right) \geq \mathbf{P}\left(X \in \bigcap_{i \in I} B_{\gamma_i}(x_i)\right) \quad (26)$$

separated from each other. To this end it is sufficient to construct functions  $g_{i,n}, \tilde{g}_{i,n} : \mathcal{C}[0, 1] \rightarrow [0, 1]$  satisfying

$$\tilde{g}_{i,n}(x) \leq \mathbf{1}_{\{B_{\gamma_i}(x_i)\}}(x) \leq g_{i,n}(x), \quad \text{for all } x \in \mathcal{C}_{r_n}[0, 1], \quad (27)$$

$$g_{i,n}(x), \tilde{g}_{i,n}(x) \rightarrow \mathbf{1}_{\{B_{\gamma_i}(x_i)\}}(x), \quad \text{for all } x \in \mathcal{C}[0, 1] \setminus \partial B_{\gamma_i}(x_i), \quad (28)$$

and such that  $a_n \prod_{i \in I} g_{i,n}, \tilde{a}_n \prod_{i \in I} \tilde{g}_{i,n} \in \mathcal{F}_s$  for appropriate constants  $a_n, \tilde{a}_n > 0$  such that  $a_n^{-1} \zeta_s(X_n, X) \rightarrow 0$  and  $\tilde{a}_n^{-1} \zeta_s(X_n, X) \rightarrow 0$  as  $n \rightarrow \infty$ . This is sufficient since we then may conclude

$$\mathbf{P}\left(X_n \in \bigcap_{i \in I} B_{\gamma_i}(x_i)\right) \leq \mathbf{E}\left[\prod_{i \in I} g_{i,n}(X_n)\right] \leq \mathbf{E}\left[\prod_{i \in I} g_{i,n}(X)\right] + a_n^{-1} \zeta_s(X_n, X) \quad (29)$$



and

$$\mathbf{P} \left( X_n \in \bigcap_{i \in I} B_{\gamma_i}(x_i) \right) \geq \mathbf{E} \left[ \prod_{i \in I} \tilde{g}_{i,n}(X_n) \right] \geq \mathbf{E} \left[ \prod_{i \in I} \tilde{g}_{i,n}(X) \right] - \tilde{a}_n^{-1} \zeta_s(X_n, X). \quad (30)$$

While this is the basic idea subsequently, the construction is slightly more involved.

We first give a motivation of how to construct the functions  $g_{i,n}$ : According to (28), asymptotically, the functions  $g_{i,n}$  have to separate points  $x \in \mathcal{C}[0, 1]$  which are in  $B_{\gamma_i}(x_i)$  from those which are not. This is why we use the  $L_p$  norm. Consider  $\psi_{p,x_i}$  as introduced in Lemma 17. If  $x \in \overline{B_{\gamma_i}(x_i)}$  then  $\psi_{p,x_i}(x) \leq (1 + \gamma_i^2)^{1/2}$  whereas if  $x \notin \overline{B_{\gamma_i}(x_i)}$  then  $\liminf_{p \rightarrow \infty} \psi_{p,x_i}(x) > (1 + \gamma_i^2)^{1/2}$ .

Let  $\varphi : \mathbb{R} \rightarrow [0, 1]$  be a three times continuously differentiable function with  $\varphi(u) = 1$  for  $u \leq 0$  and  $\varphi(u) = 0$  for  $u \geq 1$ . For  $\varrho \in \mathbb{R}$  and  $\eta > 0$  we denote  $\varphi_{\varrho,\eta} : \mathbb{R}^+ \rightarrow [0, 1]$  by  $\varphi_{\varrho,\eta}(u) = \varphi((u - \varrho)/\eta)$ .

Let  $g_i(x) = \varphi_{(1+\gamma_i^2)^{1/2}, \eta}(\psi_{p,x_i}(x))$ . Let  $g_{i,n} = g_i$  with  $\eta = \eta_n \downarrow 0$  and  $p = p_n \uparrow \infty$ . Then  $g_{i,n}$  has the properties in (27) and (28).

We do not know how to construct functions  $\tilde{g}_{i,n}$  with the properties (27) and (28). Instead, we construct functions  $\bar{g}_{i,n}$  satisfying related conditions: Let  $0 < \vartheta < 1$  and  $x \in \mathcal{C}_{r_n}[0, 1]$ . By Lemma 18 (b) we can find  $\delta = \delta(\vartheta)$  (also depending on  $x_1, \dots, x_k, \gamma_1, \dots, \gamma_k$  which are kept fixed) with

$$\begin{aligned} \{\|x - x_i\|_\infty \geq \gamma_i\} &\subseteq \left\{ \lambda(\{t : |x(t) - x_i(t)| \geq \gamma_i(1 - \vartheta)\}) \geq \frac{\vartheta}{4} \min(r_n, \delta) \right\} \\ &\subseteq \left\{ \psi_{p,x_i}(x) \geq (1 + \gamma_i^2(1 - \vartheta)^2)^{1/2} \left( \frac{\vartheta}{4} \min(r_n, \delta) \right)^{1/p} \right\} \\ &\subseteq \{\bar{g}_{i,n}(x) = 0\}, \end{aligned} \quad (31)$$

with  $\bar{g}_{i,n}(x) = \varphi_{(1+\gamma_i^2(1-\vartheta)^2)^{1/2}(\vartheta \min(r_n, \delta)/4)^{1/p-\eta}, \eta}(\psi_{p,x_i}(x))$ . This gives (27).  $\bar{g}_{i,n}$  does not fulfill (28), but we have

$$\bar{g}_{i,n}(x) \rightarrow \mathbf{1}_{\{B_{\gamma_i(1-\vartheta)}(x_i)\}}(x)$$

for  $x \in \mathcal{C}[0, 1] \setminus \partial B_{\gamma_i(1-\vartheta)}(x_i)$  and  $p = p_n \uparrow \infty, \eta = \eta_n \downarrow 0$  such that  $r_n^{1/p_n} \rightarrow 1$ . This gives for every  $0 < \vartheta < 1$  with  $\mathbf{P}(X \in \partial B_{\gamma_i(1-\vartheta)}(x_i)) = 0$  for all  $i \in I$

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[ \prod_{i \in I} \bar{g}_{i,n}(X) \right] = \mathbf{P} \left( X \in \bigcap_{i \in I} B_{\gamma_i(1-\vartheta)}(x_i) \right).$$

Assuming that  $\bar{a}_n \prod_{i \in I} \bar{g}_{i,n} \in \mathcal{F}_s$  and letting  $n$  tend to infinity (30) rewrites as

$$\liminf_{n \rightarrow \infty} \mathbf{P} \left( X_n \in \bigcap_{i \in I} B_{\gamma_i}(x_i) \right) \geq \mathbf{P} \left( X \in \bigcap_{i \in I} B_{\gamma_i(1-\vartheta)}(x_i) \right) - \limsup_{n \rightarrow \infty} \bar{a}_n^{-1} \zeta_s(X_n, X), \quad (32)$$

where  $\bar{a}_n$  may depend on  $\vartheta$  and  $\delta$ . Below, we will see that the error term on the righthand side of (32) vanishes as  $n \rightarrow \infty$  uniformly in  $\vartheta, \delta$ . So choosing  $\vartheta \downarrow 0$  such that  $\mathbf{P}(X \in \partial B_{\gamma_i(1-\vartheta)}(x_i)) = 0$  for all  $i \in I$  the assertion

$$\liminf_{n \rightarrow \infty} \mathbf{P} \left( X_n \in \bigcap_{i \in I} B_{\gamma_i}(x_i) \right) \geq \mathbf{P} \left( X \in \bigcap_{i \in I} B_{\gamma_i}(x_i) \right)$$

follows.

It remains to show that the error terms vanish in the limit. By Lemma 17  $g(x) = \varphi_{\varrho, \eta}(\psi_{p, y}(x))$  and using the mean value theorem we obtain for  $m = 0, 1, 2$

$$\|g^{(m)}(x+h) - g^{(m)}(x)\| \leq C_m p^m \eta^{-(m+1)} \|h\|_\infty^\alpha$$

for  $p \geq 4, \eta < 1$  and some constants  $C_m > 0$ . It is easy to check that the same is valid for products of functions of form  $g$  with different constants, independent of the parameters. It follows that both error terms in (29) and (32) are bounded by  $C'_m p_n^m \eta_n^{-(m+1)} \zeta_s(X_n, X)$  for all  $n$ , uniformly in  $\vartheta, \delta$ , where  $C'_m$  denotes a fixed constant for each  $m \in \{0, 1, 2\}$ . By (17) we can choose  $p_n \uparrow \infty$  and  $\eta_n \downarrow 0$  such that both  $r_n^{1/p_n} \rightarrow 1$  and the error terms vanish in the limit.  $\square$

*Proof.* (Corollary 10) Again, according to Lemma 15 we only have to verify (24), for which we modify the proof of Theorem 9: First note that the assumption of piecewise linearity of  $X_n$  and the convergence rate for  $\zeta_s(X_n, X)$  are not necessary for the upper bound

$$\limsup_{n \rightarrow \infty} \mathbf{P} \left( X_n \in \bigcap_{i \in I} B_{\gamma_i}(x_i) \right) \leq \mathbf{P} \left( X \in \bigcap_{i \in I} B_{\gamma_i}(x_i) \right).$$

For the lower bound let  $\varepsilon > 0$  and note that

$$\mathbf{P} \left( X_n \in \bigcap_{i \in I} B_{\gamma_i}(x_i) \right) \geq \mathbf{P} \left( X_n \in \bigcap_{i \in I} B_{\gamma_i}(x_i) \cap \{Y_n \in \mathcal{C}_{r_n}[0, 1]\} \right)$$

We modify the functions  $\bar{g}_{i,n}(x)$ . Let  $0 < \gamma_{K_i} < \gamma_i$  such that

$$\mathbf{P} \left( X \in \bigcap_{i \in I} B_{\gamma_{K_i}}(x_i) \right) \geq \mathbf{P} \left( X \in \bigcap_{i \in I} B_{\gamma_i}(x_i) \right) - \varepsilon.$$

and  $\mathbf{P} \left( X \in \partial B_{\gamma_{K_i}}(x_i) \right) = 0$  for all  $i$ . Let  $0 < \vartheta < 1$  and  $n_0$  be large enough such that  $\varrho_n = \|h_n - h\|_\infty < \min_i (\gamma_{K_i}(1 - \vartheta) \wedge \gamma - \gamma_{K_i})$  and  $\mathbf{P}(Y_n \notin \mathcal{C}_{r_n}[0, 1]) < \varepsilon$  for all  $n \geq n_0$ . By Lemma 18 (b) there exists  $\delta = \delta(\vartheta)$  such that for  $y \in \mathcal{C}_{r_n}[0, 1]$  with  $x = y + h_n$  and  $n \geq n_0$

$$\begin{aligned} & \{\|x - x_i\|_\infty \geq \gamma_i\} \subseteq \{\|y + h - x_i\|_\infty \geq \gamma_{K_i}\} \\ & \subseteq \left\{ \lambda(\{t : |y(t) + h(t) - x_i(t)| \geq \gamma_{K_i}(1 - \vartheta)\}) \geq \frac{\vartheta}{4} \min(r_n, \delta) \right\} \\ & \subseteq \left\{ \lambda(\{t : |x(t) - x_i(t)| \geq \gamma_{K_i}(1 - \vartheta) - \varrho_n\}) \geq \frac{\vartheta}{4} \min(r_n, \delta) \right\} \\ & \subseteq \left\{ \psi_{p, x_i}(x) \geq (1 + (\gamma_{K_i}(1 - \vartheta) - \varrho_n)^2)^{1/2} \left( \frac{\vartheta}{4} \min(r_n, \delta) \right)^{1/p} \right\} \\ & \subseteq \{\bar{g}_{i,n}(x) = 0\}, \end{aligned}$$

with  $\bar{g}_{i,n}(x) = \varphi_{(1 + (\gamma_{K_i}(1 - \vartheta) - \varrho_n)^2)^{1/2} (\vartheta \min(r_n, \delta) / 4)^{1/p} - \eta, \eta}(\psi_{p, x_i}(x))$ . Hence,

$$\begin{aligned} \mathbf{P} \left( X_n \in \bigcap_{i \in I} B_{\gamma_i}(x_i) \right) & \geq \mathbf{E} \left[ \prod_{i \in I} \bar{g}_{i,n}(X_n) \mathbf{1}_{\{Y_n \in \mathcal{C}_{r_n}[0, 1]\}} \right] \\ & \geq \mathbf{E} \left[ \prod_{i \in I} \bar{g}_{i,n}(X_n) \right] - \varepsilon \end{aligned}$$

for  $n \geq n_0$ . The upper bound of the error term  $\bar{a}_n^{-1} \zeta_s(X_n, X)$  is a function of  $p$  and  $\eta$  so it is uniform in  $\varrho_n, \vartheta, \delta$ . Following the same lines as in the proof of Theorem 9 gives

$$\begin{aligned} \liminf_{n \rightarrow \infty} \mathbf{P} \left( X_n \in \bigcap_{i \in I} B_{\gamma_i}(x_i) \right) &\geq \mathbf{P} \left( X \in \bigcap_{i \in I} B_{\gamma_{K_i}}(x_i) \right) - \varepsilon \\ &\geq \mathbf{P} \left( X \in \bigcap_{i \in I} B_{\gamma_i}(x_i) \right) - 2\varepsilon. \end{aligned}$$

Since  $\varepsilon > 0$  was arbitrary, the result follows.  $\square$

*Proof.* (Corollary 11) In the setting of the proof of Theorem 9, (29) rewrites as

$$\begin{aligned} &\mathbf{P} \left( X_n \in \bigcap_{i \in I} B_{\gamma_i}(x_i) \right) \\ &\leq \mathbf{E} \left[ \prod_{i \in I} g_{i,n}(X_n) \right] \leq \mathbf{E} \left[ \prod_{i \in I} g_{i,n}(Y_n) \right] + a_n^{-1} \zeta_s(X_n, Y_n) \\ &= \mathbf{E} \left[ \prod_{i \in I} g_{i,n}(Y_n) \right] - \mathbf{E} \left[ \prod_{i \in I} g_{i,n}(X) \right] + \mathbf{E} \left[ \prod_{i \in I} g_{i,n}(X) \right] + a_n^{-1} \zeta_s(X_n, Y_n) \end{aligned}$$

We may choose  $Y_n \rightarrow X$  almost surely. On the event  $\{X \in B_{\gamma_i}(x_i)\}$  we have  $\lim_n g_{i,n}(Y_n) = \lim_n g_{i,n}(X) = 1$  and on  $\{X \notin \overline{B_{\gamma_i}(x_i)}\}$  we have  $\lim_n g_{i,n}(Y_n) = \lim_n g_{i,n}(X) = 0$ . Since  $\mathbf{P}(X \in \partial B_{\gamma_i}(x_i)) = 0$  it follows

$$\prod_{i \in I} g_{i,n}(Y_n) - \prod_{i \in I} g_{i,n}(X) \rightarrow 0$$

for  $n \rightarrow \infty$  almost surely and dominated convergence yields

$$\limsup_{n \rightarrow \infty} \mathbf{P} \left( X_n \in \bigcap_{i \in I} B_{\gamma_i}(x_i) \right) \leq \mathbf{P} \left( X \in \bigcap_{i \in I} B_{\gamma_i}(x_i) \right),$$

just like in the proof of Theorem 9. The lower bound follows similarly.  $\square$

We now head over to the case of càdlàg functions. We only discuss the approach in the proof of Theorem 9. Following exactly the same arguments as in the continuous case and using the additional statements of Lemma 17 and Lemma 18 (a), it is easy to see that we also obtain (24) if the balls  $B_{\gamma_i}(x_i)$  are defined with the uniform metric in  $\mathcal{D}[0, 1]$ . Remember that we still have  $x_i \in \mathcal{C}[0, 1]$ . Thus, Remark 16 yields the assertion.

The proof of Theorem 12 is close to the one of Lemma 5.3 in [12]. The  $L_p$  approximation of the supremum norm complicates the argument slightly. We only give all details in the continuous case.

*Proof.* (Theorem 12) Suppose  $0 \leq s \leq 3$  and that the first assumption of Theorem 12 is satisfied. Let  $\kappa : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$  be a smooth, monotonic function with  $\kappa(u) = 0$  for  $|u| \leq \frac{1}{2}$  and  $\kappa(u) = |u|^s$  for  $|u| \geq 1$ . We may as well assume that the interpolation for  $\frac{1}{2} \leq u \leq 1$  is done smoothly such that

we have  $\kappa(u) \leq |u|^s$  for  $\frac{1}{2} \leq u \leq 1$ , thus  $\kappa(u) \leq |u|^s$  for all  $u \in \mathbb{R}_0^+$ . Let  $f, f^{(p)} : \mathcal{C}[0, 1] \rightarrow \mathbb{R}$  be given by

$$\begin{aligned} f(x) &= \kappa(\|x\|_\infty), \\ f^{(p)}(x) &= \kappa(L_p(x)). \end{aligned}$$

By Lemma 17, the restrictions of  $L_p$  and  $f^{(p)}$  to  $\mathcal{C}[0, 1] \setminus \{\mathbf{0}\}$  are smooth. Furthermore, all derivatives of  $f^{(p)}$  vanish for  $\|x\|_\infty < 1/2$  which implies that  $f^{(p)}$  is smooth on  $\mathcal{C}[0, 1]$ . Again, by Lemma 17 it is easy to check that for any  $k \in \{1, \dots, m+1\}$ ,

$$\|D^k f^{(p)}(x)\| = O(p^{k-1} L_p^{s-k}(x)),$$

uniformly in  $p$  and  $x \in \mathcal{C}[0, 1]$ . Let  $x, y \in \mathcal{C}[0, 1]$  with  $L_p(x), L_p(y) \leq 2\|x - y\|_\infty$ . Then

$$\|D^m f^{(p)}(x) - D^m f^{(p)}(y)\| \leq \|D^m f^{(p)}(x)\| + \|D^m f^{(p)}(y)\| = O(p^{m-1} \|x - y\|_\infty^\alpha).$$

Conversely let  $2\|x - y\|_\infty \leq L_p(x)$  (the case  $2\|x - y\|_\infty \leq L_p(y)$  being analogous). Then, by the mean value theorem, there exists  $z \in [x, y] := \{\lambda x + (1 - \lambda)y \mid \lambda \in [0, 1]\}$ , such that

$$\begin{aligned} \|D^m f^{(p)}(x) - D^m f^{(p)}(y)\| &= \|D^{m+1} f^{(p)}(z)\| \cdot \|x - y\|_\infty \\ &= O(p^m L_p^{\alpha-1}(x)) \cdot \|x - y\|_\infty = O(p^m \|x - y\|_\infty^\alpha). \end{aligned}$$

Hence, there is a constant  $c > 0$  such that  $cp^{-m} f^{(p)} \in \mathcal{F}_s$  for all  $p \geq 4$ . We define, for  $r > 0$ ,

$$\begin{aligned} f_r(x) &:= cr^s f(x/r), \\ f_r^{(p)}(x) &:= cr^s f^{(p)}(x/r). \end{aligned}$$

Then  $p^{-m} f_r^{(p)} \in \mathcal{F}_s$ . Furthermore,  $f_r(x)$  and  $f_r^{(p)}(x)$  are bounded by  $c\|x\|^s$  for all  $x \in \mathcal{C}[0, 1]$ , uniformly in  $p$ . For any fixed  $x$  we have  $f_r(x) \rightarrow 0$  and  $\sup_{p \geq 4} f_r^{(p)}(x) \rightarrow 0$  as  $r \rightarrow \infty$ . Hence, by  $\mathbf{E}[\|X\|^s] < \infty$  and dominated convergence this implies

$$\mathbf{E} \left[ \sup_{p \geq 4} f_r^{(p)}(X) \right] \rightarrow 0, \quad (r \rightarrow \infty). \quad (33)$$

By definition of  $\zeta_s$  we have

$$\mathbf{E} \left[ f_r^{(p)}(X_n) \right] \leq \mathbf{E} \left[ f_r^{(p)}(X) \right] + p^m \zeta_s(X_n, X).$$

By definition of  $f_r$ , for  $\|x\| > r$  we have  $\|x\|^s = c^{-1} f_r(x)$ . Hence,

$$\begin{aligned} \mathbf{E} \left[ \|X_n\|_\infty^s \mathbf{1}_{\{\|X_n\|_\infty \geq 2r\}} \right] &= c^{-1} \mathbf{E} \left[ f_r(X_n) \mathbf{1}_{\{\|X_n\|_\infty \geq 2r\}} \right] \\ &\leq c^{-1} \mathbf{E} \left[ f_r^{(p)}(X_n) \right] + c^{-1} \left( \mathbf{E} \left[ (f_r(X_n) - f_r^{(p)}(X_n)) \mathbf{1}_{\{\|X_n\|_\infty \geq 2r\}} \right] \right) \\ &\leq c^{-1} \mathbf{E} \left[ f_r^{(p)}(X) \right] + c^{-1} p^m \zeta_s(X_n, X) \\ &\quad + c^{-1} \left( \mathbf{E} \left[ (f_r(X_n) - f_r^{(p)}(X_n)) \mathbf{1}_{\{\|X_n\|_\infty \geq 2r\}} \right] \right). \end{aligned} \quad (34)$$

Now, let  $\varepsilon > 0$  be arbitrary. By (33) fix  $r > 0$  such that  $\mathbf{E} \left[ f_r^{(p)}(X) \right] < \varepsilon$  for all  $p \geq 4$ . Additionally, by the given assumptions there exists a sequence  $p_n \uparrow \infty$  such that

$$\frac{\log r_n}{p_n} \rightarrow 0, \quad p_n^m \zeta_s(X_n, X) \rightarrow 0, \quad (n \rightarrow \infty).$$

Therefore, let  $N_0$  be large enough such that  $p_n^m \zeta_s(X_n, X) < \varepsilon$  for all  $n \geq N_0$ . It remains to bound the third summand in (34). Using Lemma 18 (a), piecewise linearity of  $X_n$  implies that for all  $0 < \vartheta < 1$ ,

$$L_p(X_n) \geq \|X_n\|_\infty (1 - \vartheta) \left( \frac{\vartheta r_n}{2} \right)^{1/p_n}.$$

In particular, we have  $L_p(X_n) \geq \frac{\|X_n\|_\infty}{2}$  for all  $n$  sufficiently large. For those  $n$  and  $\|X_n\| > 2r$  we also have  $f_r^{(p)}(X_n) = cL_p^s(X_n)$ . This yields

$$\mathbf{E} \left[ (f_r(X_n) - f_r^{(p)}(X_n)) \mathbf{1}_{\{\|X_n\|_\infty \geq 2r\}} \right] = c \mathbf{E} \left[ (\|X_n\|_\infty^s - L_p^s(X_n)) \mathbf{1}_{\{\|X_n\|_\infty \geq 2r\}} \right] \quad (35)$$

$$\leq c(1 - 2^{-s}) \mathbf{E} \left[ \|X_n\|_\infty^s \mathbf{1}_{\{\|X_n\|_\infty \geq 2r\}} \right]. \quad (36)$$

for all  $n$  sufficiently large. Increasing  $N_0$  if necessary, inserting (36) into (34) and rearranging terms implies

$$\mathbf{E} \left[ \|X_n\|_\infty^s \mathbf{1}_{\{\|X_n\|_\infty \geq 2r\}} \right] \leq 2^{1+s} c^{-1} \varepsilon.$$

for all  $n \geq N_0$ . Since  $\varepsilon$  was arbitrary, the assertion follows.

Now, suppose the second assumption to be satisfied. Then, we have to modify the last part of the proof. In (35) we can decompose

$$L_p^s(X_n) = L_p^s(X_n) \mathbf{1}_{\{Y_n \in \mathcal{C}_{r_n}[0,1]\}} + L_p^s(X_n) \mathbf{1}_{\{Y_n \notin \mathcal{C}_{r_n}[0,1]\}}.$$

Using  $L_p^s(X_n) \leq \|X_n\|_\infty^s$ , the assumptions guarantee the expectation of the second term to be small in the limit  $n \rightarrow \infty$ . For the first one, using similar arguments as above, given  $\{Y_n \in \mathcal{C}_{r_n}[0,1]\}$ , we find

$$L_p(X_n) \geq \frac{\|X_n\|_\infty}{2} - 2\varrho_n$$

with  $\varrho_n = \|h_n - h\|_\infty$  for all  $n$  sufficiently large. Proceeding as in the first part, we obtain the result. Given the third assumption, it only remains to bound  $\mathbf{E} \left[ f_r^{(p)}(Y_n) \right]$  which appears instead of  $\mathbf{E} \left[ f_r^{(p)}(X) \right]$  by  $\mathbf{E} \left[ f_r^{(p)}(Z) \right]$  in (34).  $\square$

### 3 The Contraction Method

In this section the contraction method is developed first for a general separable Banach space  $B$ . Then the framework is specialized to the cases  $(\mathcal{C}[0,1], \|\cdot\|_\infty)$  and  $(\mathcal{D}[0,1], d_{sk})$ . For this section  $B$  will always denote a separable Banach space or  $(\mathcal{D}[0,1], d_{sk})$ .

We recall the recursive equation (2). We have

$$X_n \stackrel{d}{=} \sum_{r=1}^K A_r^{(n)} X_{I_r^{(n)}}^{(r)} + b^{(n)}, \quad n \geq n_0. \quad (37)$$

where  $A_1^{(n)}, \dots, A_K^{(n)}$  are random continuous linear operators,  $b^{(n)}$  is a  $B$ -valued random variable,  $(X_n^{(1)})_{n \geq 0}, \dots, (X_n^{(K)})_{n \geq 0}$  are distributed like  $(X_n)_{n \geq 0}$ , and  $I^{(n)} = (I_1^{(n)}, \dots, I_K^{(n)})$  is a vector of random integers in  $\{0, \dots, n\}$ . Moreover  $(A_1^{(n)}, \dots, A_K^{(n)}, b^{(n)}, I^{(n)}), (X_n^{(1)})_{n \geq 0}, \dots, (X_n^{(K)})_{n \geq 0}$  are independent and  $n_0 \in \mathbb{N}$ .

Recall that in order to be a random continuous linear operator,  $A$  has to take values in the set of continuous endomorphisms on  $\mathcal{C}[0,1]$  respectively the set of norm-continuous endomorphisms that

are continuous with respect to  $d_{sk}$  on  $\mathcal{D}[0, 1]$  such that  $A(x)(t)$  is a real-valued random variable for all  $x \in \mathcal{C}[0, 1]$  respectively  $x \in \mathcal{D}[0, 1]$  and  $t \in [0, 1]$ . In  $\mathcal{D}[0, 1]$  we additionally have to guarantee  $\|A\|_{\text{op}}$  to be a real-valued random variable, see section 2.2.

We make assumptions about the moments and the asymptotic behavior of the coefficients  $A_1^{(n)}, \dots, A_K^{(n)}, b^{(n)}$ . For a random continuous linear operator  $A$  we write

$$\|A\|_s := \mathbf{E} [\|A\|_{\text{op}}^s]^{1 \wedge (1/s)}.$$

We consider the following conditions with an  $s > 0$ :

**C1.** We have  $\|X_0\|_s, \dots, \|X_{n_0-1}\|_s, \|A_r^{(n)}\|_s, \|b^{(n)}\|_s < \infty$  for all  $r = 1, \dots, K$  and  $n \geq 0$  and there exist random continuous linear operators  $A_1, \dots, A_K$  on  $B$  and a  $B$ -valued random variable  $b$  such that, as  $n \rightarrow \infty$ ,

$$\gamma(n) := \|b^{(n)} - b\|_s + \sum_{r=1}^K \left( \|A_r^{(n)} - A_r\|_s + \left\| \mathbf{1}_{\{I_r^{(n)} \leq n_0\}} A_r^{(n)} \right\|_s \right) \rightarrow 0. \quad (38)$$

and for all  $\ell \in \mathbb{N}$ ,

$$\mathbf{E} \left[ \mathbf{1}_{\{I_r^{(n)} \in \{0, \dots, \ell\} \cup \{n\}\}} \|A_r^{(n)}\|_{\text{op}}^s \right] \rightarrow 0. \quad (39)$$

**C2.** We have

$$L := \sum_{r=1}^K \mathbf{E} [\|A_r\|_{\text{op}}^s] < 1.$$

The limits of the coefficients determine the limiting operator  $T$  from (5):

$$\begin{aligned} T : \mathcal{M}(B) &\rightarrow \mathcal{M}(B) \\ \mu &\mapsto \mathcal{L} \left( \sum_{r=1}^K A_r Z^{(r)} + b \right) \end{aligned} \quad (40)$$

where  $(A_1, \dots, A_K, b), Z^{(1)}, \dots, Z^{(K)}$  are independent and  $Z^{(1)}, \dots, Z^{(K)}$  have distribution  $\mu$ .

**C3.** The map  $T$  has a fixed-point  $\eta \in \mathcal{M}_s(B)$ , such that  $\mathcal{L}(X_n) \in \mathcal{M}_s(\eta)$  for all  $n \geq n_0$ .

The existence of a fixed-point is not in general implied by contraction properties of  $T$  with respect to a Zolotarev metric due to the lack of knowledge of completeness of the metric on the space  $B$ . However, we can argue that there is at most one fixed-point of  $T$  in  $\mathcal{M}_s(\eta)$ :

**Lemma 19.** *Assume the sequence  $(X_n)_{n \geq 0}$  satisfies (37). Under conditions C1–C3 we have  $T(\mathcal{M}_s(\eta)) \subseteq \mathcal{M}_s(\eta)$  and*

$$\zeta_s(T(\mu), T(\lambda)) \leq L \zeta_s(\mu, \lambda) \quad \text{for all } \mu, \lambda \in \mathcal{M}_s(\eta).$$

*In particular, the restriction of  $T$  to  $\mathcal{M}_s(\eta)$  is a contraction and has the unique fixed-point  $\eta$ .*

*Proof.* Let  $\mu \in \mathcal{M}_s(\eta)$ . Recall that we have  $s = m + \alpha$  with  $m \in \mathbb{N}_0$  and  $\alpha \in (0, 1]$ . We introduce an accompanying sequence

$$Q_n := \sum_{r=1}^K A_r^{(n)} \left( \mathbf{1}_{\{I_r^{(n)} < n_0\}} X_{I_r^{(n)}}^{(r)} + \mathbf{1}_{\{I_r^{(n)} \geq n_0\}} Z^{(r)} \right) + b^{(n)}, \quad n \geq n_0, \quad (41)$$



where  $(A_1^{(n)}, \dots, A_K^{(n)}, b^{(n)})$ ,  $Z^{(1)}, \dots, Z^{(K)}$  are independent and  $Z^{(1)}, \dots, Z^{(K)}$  have distribution  $\mu$ .

We first show that  $\mathcal{L}(Q_n) \in \mathcal{M}_s(\eta)$  for all  $n \geq n_0$ . Condition **C1**, conditioning on the coefficients and Minkowski's inequality imply  $\mathbf{E}[\|Q_n\|_\infty^s] < \infty$  for all  $n$ . For  $s \leq 1$  we already obtain  $\mathcal{L}(Q_n) \in \mathcal{M}_s(\eta)$ .

For  $s > 1$  we choose arbitrary  $1 \leq k \leq m$  and multilinear and bounded  $f : B^k \rightarrow \mathbb{R}$ . We have

$$\begin{aligned} \mathbf{E}[f(Z, \dots, Z)] &= \mathbf{E}[f(X_n, \dots, X_n)] \\ &= \mathbf{E}\left[f\left(\sum_{r=1}^K A_r^{(n)} X_{I_r^{(n)}}^{(r)} + b^{(n)}, \dots, \sum_{r=1}^K A_r^{(n)} X_{I_r^{(n)}}^{(r)} + b^{(n)}\right)\right]. \end{aligned}$$

To show  $\mathcal{L}(Q_n) \in \mathcal{M}_s(\eta)$  we need to verify that the latter display is equal to  $\mathbf{E}[f(Q_n, \dots, Q_n)]$ : Since  $f$  is multilinear, both terms can be expanded as a sum and it suffices to show that the corresponding summands are equal:

$$\mathbf{E}\left[f\left(C_{j_1}^{(n)}, \dots, C_{j_k}^{(n)}\right)\right] = \mathbf{E}\left[f\left(D_{j_1}^{(n)}, \dots, D_{j_k}^{(n)}\right)\right], \quad (42)$$

where  $j_1, \dots, j_k \in \{1, \dots, K\}$  and for each  $i \in \{1, \dots, k\}$  we either have

$$C_{j_i}^{(n)} = A_{j_i}^{(n)} X_{I_{j_i}^{(n)}}^{(j_i)} \quad \text{and} \quad D_{j_i}^{(n)} = A_{j_i}^{(n)} \left( \mathbf{1}_{\{I_{j_i}^{(n)} < n_0\}} X_{I_{j_i}^{(n)}}^{(j_i)} + \mathbf{1}_{\{I_{j_i}^{(n)} \geq n_0\}} Z^{(j_i)} \right) \quad (43)$$

or

$$C_{j_i}^{(n)} = b^{(n)} \quad \text{and} \quad D_{j_i}^{(n)} = b^{(n)}. \quad (44)$$

The equality in (42) is obvious for the case where we have (44) for all  $i = 1, \dots, k$ . For the other cases we have (43) for at least  $1 \leq \ell \leq k$  arguments of  $f$ , say, for simplicity of presentation, for the first  $\ell$  with  $1 \leq \ell_1 < \dots < \ell_d = \ell$  such that  $j_s = j_{\ell_i}$  for all  $s = \ell_{i-1} + 1, \dots, \ell_i, i = 1, \dots, d$  and  $j_{\ell_i}$  pairwise different for  $i = 1, \dots, d$  (by convention  $\ell_0 := 0$ ). The claim in (42) reduces to

$$\begin{aligned} &\mathbf{E}\left[f(C_{j_{\ell_1}}^{(n)}, \dots, C_{j_{\ell_1}}^{(n)}, C_{j_{\ell_2}}^{(n)}, \dots, C_{j_{\ell_d}}^{(n)}, b^{(n)}, \dots, b^{(n)})\right] \\ &= \mathbf{E}\left[f(D_{j_{\ell_1}}^{(n)}, \dots, D_{j_{\ell_1}}^{(n)}, D_{j_{\ell_2}}^{(n)}, \dots, D_{j_{\ell_d}}^{(n)}, b^{(n)}, \dots, b^{(n)})\right] \end{aligned} \quad (45)$$

We will prove that, for each  $p \in \{1, \dots, d\}$ ,

$$\mathbf{E}\left[f(C_{j_{\ell_1}}^{(n)}, \dots, C_{j_{\ell_{p-1}}}^{(n)}, C_{j_{\ell_p}}^{(n)}, \dots, C_{j_{\ell_p}}^{(n)}, D_{j_{\ell_{p+1}}}^{(n)}, \dots, D_{j_{\ell_d}}^{(n)}, b^{(n)}, \dots, b^{(n)})\right] \quad (46)$$

$$= \mathbf{E}\left[f(C_{j_{\ell_1}}^{(n)}, \dots, C_{j_{\ell_{p-1}}}^{(n)}, D_{j_{\ell_p}}^{(n)}, \dots, D_{j_{\ell_p}}^{(n)}, D_{j_{\ell_{p+1}}}^{(n)}, \dots, D_{j_{\ell_d}}^{(n)}, b^{(n)}, \dots, b^{(n)})\right], \quad (47)$$

which in turn implies (45). Abbreviating  $Y_i^{(r)} = \left(\mathbf{1}_{\{i < n_0\}} X_i^{(r)} + \mathbf{1}_{\{i \geq n_0\}} Z^{(r)}\right)$  and denoting by  $\Upsilon$  the joint distribution of  $(A_{j_{\ell_1}}^{(n)}, \dots, A_{j_{\ell_d}}^{(n)}, I_{j_{\ell_1}}^{(n)}, \dots, I_{j_{\ell_d}}^{(n)}, b^{(n)})$  we have

$$\begin{aligned} &\mathbf{E}\left[f(C_{j_{\ell_1}}^{(n)}, \dots, C_{j_{\ell_{i-1}}}^{(n)}, C_{j_{\ell_i}}^{(n)}, \dots, C_{j_{\ell_i}}^{(n)}, D_{j_{\ell_{i+1}}}^{(n)}, \dots, D_{j_{\ell_d}}^{(n)}, b^{(n)}, \dots, b^{(n)})\right] \\ &= \int f(\alpha_1 x_1, \dots, \alpha_{p-1} x_{p-1}, \alpha_p x_p, \dots, \alpha_p x_p, \alpha_{p+1} x_{p+1}, \dots, \alpha_d x_d, b, \dots, b) \\ &\quad d\mathbb{P}_{X_{i_1}}(x_1) \cdots d\mathbb{P}_{X_{i_p}}(x_p) d\mathbb{P}_{Y_{i_{p+1}}}(x_{p+1}) \cdots d\mathbb{P}_{Y_{i_d}}(x_d) d\Upsilon(\alpha_1, \dots, \alpha_d, i_1, \dots, i_d, b). \\ &= \int \mathbf{E}[g(X_{i_p}, \dots, X_{i_p})] d\mathbb{P}_{X_{i_1}} \cdots d\mathbb{P}_{X_{i_{p-1}}} d\mathbb{P}_{Y_{i_{p+1}}} \cdots d\mathbb{P}_{Y_{i_d}} d\Upsilon, \end{aligned}$$

where, for all fixed  $\alpha_1, \dots, \alpha_d, i_1, \dots, i_d, b, x_1, \dots, x_{p-1}, x_{p+1}, \dots, x_d$ , we use the bounded and multilinear function  $g : B^{\ell_p - \ell_{p-1}} \rightarrow \mathbb{R}$ ,

$$\begin{aligned} & g(y_1, \dots, y_{\ell_p - \ell_{p-1}}) \\ & := f(\alpha_1 x_1, \dots, \alpha_{p-1} x_{p-1}, \alpha_p y_1, \dots, \alpha_p y_{\ell_p - \ell_{p-1}}, \alpha_{p+1} x_{p+1}, \dots, \alpha_d x_d, b, \dots, b). \end{aligned}$$

Since  $\mathcal{L}(X_m), \mathcal{L}(Z) \in \mathcal{M}_s(\eta)$  for all  $m \geq n_0$  we can replace  $X_{i_p}$  by  $Y_{i_p}$ . This shows the equality (46), hence (42). Altogether, we obtain  $\mathcal{L}(Q_n) \in \mathcal{M}_s(\eta)$  for all  $n \geq n_0$ .

Now, we show  $T(\mu) \in \mathcal{M}_s(\eta)$ . Let  $W$  be a random variable with distribution  $T(\mu)$ . By **C2**, in particular  $\|A_r\|_s < \infty$  for  $r = 1, \dots, K$ , by **C1** we have  $\|b\|_s < \infty$ . Thus, as for  $Q_n$ , from Minkowski's inequality we obtain  $\mathbf{E}[\|W\|_\infty^s] < \infty$ , hence  $T(\mu) \in \mathcal{M}_s(\eta)$  for  $s \leq 1$ . For the case  $s > 1$  we consider again arbitrary  $1 \leq k \leq m$  and multilinear and bounded  $f : B^k \rightarrow \mathbb{R}$ . It suffices to show  $\mathbf{E}[f(Q_n, \dots, Q_n)] = \mathbf{E}[f(W, \dots, W)]$  for some  $n \geq n_0$ . In fact, we will show that  $\lim_{n \rightarrow \infty} \mathbf{E}[f(Q_n, \dots, Q_n)] = \mathbf{E}[f(W, \dots, W)]$ . For this we expand

$$\mathbf{E}[f(W, \dots, W)] = \mathbf{E}\left[f\left(\sum_{r=1}^K A_r Z^{(r)} + b, \dots, \sum_{r=1}^K A_r Z^{(r)} + b\right)\right]$$

into summands corresponding to (42) and have to show that

$$\lim_{n \rightarrow \infty} \mathbf{E}\left[f\left(D_{j_1}^{(n)}, \dots, D_{j_k}^{(n)}\right)\right] = \mathbf{E}[f(E_{j_1}, \dots, E_{j_k})], \quad (48)$$

where  $j_1, \dots, j_k \in \{1, \dots, K\}$ . For each  $i \in \{1, \dots, k\}$  we have in case (43) that  $E_{j_i} = A_{j_i} Z^{(j_i)}$ , in case (44) that  $E_{j_i} = b$ . We obtain, introducing a telescoping sum and using Hölder's inequality,

$$\begin{aligned} & \left| \mathbf{E}\left[f\left(D_{j_1}^{(n)}, \dots, D_{j_k}^{(n)}\right)\right] - \mathbf{E}[f(E_{j_1}, \dots, E_{j_k})] \right| \\ &= \left| \sum_{q=1}^k \mathbf{E}\left[f\left(E_{j_1}, \dots, E_{j_{q-1}}, D_{j_q}^{(n)}, \dots, D_{j_k}^{(n)}\right) - f\left(E_{j_1}, \dots, E_{j_q}, D_{j_{q+1}}^{(n)}, \dots, D_{j_k}^{(n)}\right)\right] \right| \\ &\leq \sum_{q=1}^k \left| \mathbf{E}\left[f\left(E_{j_1}, \dots, E_{j_{q-1}}, D_{j_q}^{(n)} - E_{j_q}, D_{j_{q+1}}^{(n)}, \dots, D_{j_k}^{(n)}\right)\right] \right| \\ &\leq \sum_{q=1}^k \|f\| \|D_{j_q}^{(n)} - E_{j_q}\|_k \prod_{v=1}^{q-1} \|E_{j_v}\|_k \prod_{v=q+1}^k \|D_{j_v}^{(n)}\|_k. \end{aligned}$$

Note that the  $\|E_{j_v}\|_k$  and  $\|D_{j_v}^{(n)}\|_k$  are all uniformly bounded by independence, **C1**, and  $\|X_0\|_s, \dots, \|X_{n_0-1}\|_s, \|Z\|_s < \infty$ . Hence it suffices to show that  $\|D_{j_v}^{(n)} - E_{j_v}\|_k \rightarrow 0$  for all  $j_v$ . In case (44) this is  $\|b^{(n)} - b\|_k \rightarrow 0$  by condition **C1**. In case (44) we have, abbreviating  $r = j_i$ ,

$$\begin{aligned} & \left\| A_r^{(n)} \left( \mathbf{1}_{\{I_r^{(n)} < n_0\}} X_{I_r^{(n)}}^{(r)} + \mathbf{1}_{\{I_r^{(n)} \geq n_0\}} Z^{(r)} \right) - A_r Z^{(r)} \right\|_k \\ & \leq \left\| (A_r^{(n)} - A_r) Z^{(r)} \right\|_k + \left\| A_r^{(n)} \left( \mathbf{1}_{\{I_r^{(n)} < n_0\}} \left( X_{I_r^{(n)}}^{(r)} - Z^{(r)} \right) \right) \right\|_k \end{aligned}$$

The first summand of the latter display tends to zero by independence,  $\|Z\|_s < \infty$  and condition **C1**. The second summand tends to zero applying Hölder's inequality, condition **C1**, which implies that  $\|A_r^{(n)}\|_s$  is uniformly bounded,  $\|X_0\|_s, \dots, \|X_{n_0-1}\|_s, \|Z\|_s < \infty$  and conditions **C1** and **C3**. Altogether we obtain  $T(\mu) \in \mathcal{M}_s(\eta)$ .

Let  $\mu, \lambda \in \mathcal{M}_s(\eta)$ . Conditioning on the coefficients, using Lemma 1 and (11) it follows that

$$\zeta_s(T(\mu), T(\lambda)) \leq \left( \sum_{r=1}^K \mathbf{E} [\|A_r\|_{\text{op}}^s] \right) \zeta_s(\mu, \lambda).$$

Thus, by condition **C2**, the restriction of  $T$  to  $\mathcal{M}_s(\eta)$  is a contraction with respect to  $\zeta_s$ .

Assume,  $\mu$  was a fixed-point of  $T$  as well. Then the contraction property implies

$$\zeta_s(\mu, \eta) = \zeta_s(T(\mu), T(\eta)) \leq L\zeta_s(\mu, \eta),$$

hence  $\zeta_s(\mu, \eta) = 0$ . Since the  $\zeta_s$ -distance is a metric on  $\mathcal{M}_s(\eta)$  it follows  $\mu = \eta$ .  $\square$

We now turn to the problem of convergence of the sequence  $(X_n)_{n \geq 0}$  to the fixed-point  $\eta$ .

Aiming to proof  $X_n \rightarrow X$  condition **C1** is natural in the context of contraction method. Condition **C2** is necessary if working with  $\zeta_s$  metrics. We will discuss this in detail for the cases  $\mathcal{C}[0, 1]$  and  $\mathcal{D}[0, 1]$  below. The existence of a solution of the fixed-point equation in condition **C3** is required since we miss knowledge about completeness of the  $\zeta_s$  metrics. If  $\mu \in \mathcal{M}_s(B)$  then  $(T^n(\mu))_{n \geq 0}$  is a Cauchy sequence with respect to  $\zeta_s$ , the proof being similar to the one of the previous lemma. Then, for  $B = \mathcal{C}[0, 1]$  or  $B = \mathcal{D}[0, 1]$ , by Proposition 13, all finite dimensional marginals of  $T^n(\mu)$  converge to the corresponding marginals of some measure  $\nu$  on  $\mathbb{R}^{[0,1]}$ , the natural candidate for a fixed-point of (40). In the application discussed in section 4.2, the solution of the fixed-point equation (68) is constructed via a sequence  $(Z_n)_{n \geq 0}$  of random variables that satisfy  $\mathcal{L}(Z_n) = T^n(\mu)$  and converge uniformly almost surely (cf. [6] for details). The starting point is the Dirac measure  $\mu = \delta_f$  with a specific function  $f \in \mathcal{C}[0, 1]$ .

The following proposition uses the ideas developed so far to infer convergence of  $X_n$  to  $X$  in the  $\zeta_s$  distance. The proof extends a similar proof for the case  $B = \mathbb{R}^d$ , see [25, Theorem 4.1]. We draw further implications from this proof, see Corollary 22.

**Proposition 20.** *Let  $(X_n)_{n \geq 0}$  satisfy recurrence (37) with conditions **C1** – **C3**. Then for the fixed-point  $\eta = \mathcal{L}(X)$  of  $T$  in (40) we have, as  $n \rightarrow \infty$ ,*

$$\zeta_s(X_n, X) \rightarrow 0.$$

*Proof.* We use the accompanying sequence defined in (41). Throughout the proof let  $n \geq n_0$ . Again since the  $\zeta_s$ -distance is a metric we have

$$\zeta_s(X_n, X) \leq \zeta_s(X_n, Q_n) + \zeta_s(Q_n, X). \quad (49)$$

First, we consider the second term. By **C1** and Minkowski's inequality, absolute moments of order  $s$  of the sequence  $(Q_n)_{n \geq n_0}$  are bounded, hence using Lemma 5 it suffices to show

$$\ell_s(Q_n, X) \rightarrow 0.$$

Using the same set of independent random variables  $X^{(1)}, \dots, X^{(K)}$  for  $Q_n$  and in the recurrence of  $X$ , we obtain

$$\begin{aligned} \ell_s(Q_n, X) &\leq \left\| \sum_{r=1}^K \left( A_r - \mathbf{1}_{\{I_r^{(n)} \geq n_0\}} A_r^{(n)} \right) X^{(r)} \right\|_s + \left\| \sum_{r=1}^K \mathbf{1}_{\{I_r^{(n)} < n_0\}} A_r^{(n)} X_{I_r^{(n)}}^{(r)} \right\|_s + \|b^{(n)} - b\|_s \\ &\leq \sum_{r=1}^K \left( \|A_r - A_r^{(n)}\|_s + \left\| \mathbf{1}_{\{I_r^{(n)} < n_0\}} \|A_r^{(n)}\|_{\text{op}} \right\|_s \right) \|X\|_s + \|b^{(n)} - b\|_s \\ &\quad + \left\| \sum_{r=1}^K \mathbf{1}_{\{I_r^{(n)} < n_0\}} A_r^{(n)} X_{I_r^{(n)}}^{(r)} \right\|_s \end{aligned}$$

By **C1** the first two summands tend to zero. Also, the third one converges to zero using **C1** and

$$\left\| \mathbf{1}_{\{I_r^{(n)} < n_0\}} \|A_r^{(n)}\|_{\text{op}} X_{I_r^{(n)}}^{(r)} \right\|_s \leq \left\| \mathbf{1}_{\{I_r^{(n)} < n_0\}} \|A_r^{(n)}\|_{\text{op}} \right\|_s \left\| \sup_{j < n_0} \|X_j\| \right\|_s$$

Furthermore, conditioning on the coefficients and using that  $\zeta_s$  is  $(s, +)$  ideal and Lemma 1, it is easy to see that

$$\zeta_s(Q_n, X_n) \leq p_n \zeta_s(X_n, X) + \mathbf{E} \left[ \sum_{r=1}^K \mathbf{1}_{\{n_0 \leq I_r^{(n)} \leq n-1\}} \|A_r^{(n)}\|_{\text{op}}^s \zeta_s(X_{I_r^{(n)}}, X) \right] \quad (50)$$

$$\leq p_n \zeta_s(X_n, X) + \left( \sum_{r=1}^K \mathbf{E} [\|A_r^{(n)}\|_{\text{op}}^s] \right) \sup_{n_0 \leq i \leq n-1} \zeta_s(X_i, X), \quad (51)$$

where

$$p_n = \mathbf{E} \left[ \sum_{r=1}^K \mathbf{1}_{\{I_r^{(n)} = n\}} \|A_r^{(n)}\|_{\text{op}}^s \right] \rightarrow 0, \quad n \rightarrow \infty.$$

Combining (49) and (51) implies

$$\zeta_s(X_n, X) \leq \frac{1}{1 - p_n} \left[ \sum_{r=1}^K \mathbf{E} [\|A_r^{(n)}\|_{\text{op}}^s] \sup_{n_0 \leq i \leq n-1} \zeta_s(X_i, X) + o(1) \right].$$

From this it follows that  $\zeta_s(X_n, X)$  is bounded. Let

$$\bar{\eta} := \sup_{n \geq n_0} \zeta_s(X_n, X), \quad \eta := \limsup_{n \rightarrow \infty} \zeta_s(X_n, X)$$

and  $\varepsilon > 0$  arbitrary. Then, there exists  $\ell > 0$  with  $\zeta_s(X_n, X) \leq \eta + \varepsilon$  for all  $n \geq \ell$ . Using (49), (50) and splitting  $\{n_0 \leq I_r^{(n)} \leq n-1\}$  into  $\{n_0 \leq I_r^{(n)} \leq \ell\}$  and  $\{\ell < I_r^{(n)} \leq n-1\}$ , we obtain

$$\zeta_s(X_n, X) \leq \frac{\bar{\eta}}{1 - p_n} \mathbf{E} \left[ \sum_{r=1}^K \mathbf{1}_{\{n_0 \leq I_r^{(n)} \leq \ell\}} \|A_r^{(n)}\|_{\text{op}}^s \right] + \frac{\eta + \varepsilon}{1 - p_n} \mathbf{E} \left[ \sum_{r=1}^K \|A_r^{(n)}\|_{\text{op}}^s \right] + o(1)$$

which, by **C1**, finally implies

$$\eta \leq \mathbf{E} \left[ \sum_{r=1}^K \|A_r\|_{\text{op}}^s \right] (\eta + \varepsilon).$$

Since  $\varepsilon > 0$  is arbitrary and by condition **C2**, we obtain  $\eta = 0$ .  $\square$

**Remark 21.** As pointed out in [13] for a related convergence result, the statements of Lemma 19 and Proposition 20 remain true if condition **C1** is weakened by replacing

$$\sum_{r=1}^K \|A_r^{(n)} - A_r\|_s \rightarrow 0$$

by

$$\sum_{r=1}^K \|(A_r^{(n)} - A_r)f\|_s \rightarrow 0, \quad \|A_r^{(n)}\| \rightarrow \|A_r^{(n)}\|_s,$$

for all  $f \in \mathcal{C}[0, 1]$  and uniform boundedness of  $\|A_r^{(n)}\|_s$  for all  $n \geq 0$  and all  $r = 1, \dots, K$ . This follows from the given independence structure and the dominated convergence theorem.

To be able to apply the results of the previous section to deduce weak convergence from convergence in  $\zeta_s$  for the special cases  $\mathcal{C}[0, 1]$  and  $\mathcal{D}[0, 1]$ , rates of convergence for  $\zeta_s$  are required. We impose a further assumption on the convergence rate of the coefficients to establish a rate of convergence for the process that strengthens condition **C2**. We use the Bachmann-Landau big-O notation for sequences of numbers.

**C4.** The sequence  $(\gamma(n))_{n \geq n_0}$  from condition **C1** satisfies  $\gamma(n) = O(R(n))$  as  $n \rightarrow \infty$  for some positive sequence  $R(n) \downarrow 0$  such that

$$L^* = \limsup_{n \rightarrow \infty} \mathbf{E} \left[ \sum_{r=1}^K \|A_r^{(n)}\|_{\text{op}}^s \frac{R(I_r^{(n)})}{R(n)} \right] < 1.$$

**Corollary 22.** Let  $(X_n)_{n \geq 0}$  satisfy recurrence (37) with conditions **C1**, **C3** and **C4**. Then for the fixed-point  $\eta = \mathcal{L}(X)$  of  $T$  in (40) we have, as  $n \rightarrow \infty$ ,

$$\zeta_s(X_n, X) = O(R(n)).$$

*Proof.* We consider the quantities introduced in the proof of Proposition 20 again. By condition **C4** we have  $\zeta_s(Q_n, X) \leq CR(n)$  for some  $C > 0$  and all  $n$ . Furthermore, we can choose  $\gamma > 0$  and  $n_1 > 0$  such that

$$\mathbf{E} \left[ \sum_{r=1}^K \|A_r^{(n)}\|_{\text{op}}^s \frac{R(I_r^{(n)})}{R(n)} \right] \leq 1 - \gamma, \quad p_n \leq \frac{\gamma}{2}$$

for  $n \geq n_1$ . Obviously, for any  $n_2 \geq n_1$ , we can choose  $K \geq 2C/\gamma$  such that  $d(n) := \zeta_s(X_n, X) \leq KR(n)$  for all  $n < n_2$ . Using (50), this implies

$$d(n_2) \leq p_{n_2} d(n_2) + \mathbf{E} \left[ \sum_{r=1}^K \mathbf{1}_{\{I_r^{(n_2)} \leq n_2-1\}} \|A_r^{(n_2)}\|_{\text{op}}^s d(I_r^{(n_2)}) \right] + CR(n_2)$$

hence

$$\begin{aligned} d(n_2) &\leq \frac{1}{1 - p_{n_2}} \left( \mathbf{E} \left[ \sum_{r=1}^K \|A_r^{(n_2)}\|_{\text{op}}^s KR(I_r^{(n_2)}) \right] + CR(n_2) \right) \\ &= \frac{1}{1 - p_{n_2}} \left( KR(n_2) \mathbf{E} \left[ \sum_{r=1}^K \|A_r^{(n_2)}\|_{\text{op}}^s \frac{R(I_r^{(n_2)})}{R(n_2)} \right] + CR(n_2) \right) \\ &\leq \frac{1}{1 - p_{n_2}} ((1 - \gamma)K + C)R(n_2) \leq KR(n_2). \end{aligned}$$

Inductively,  $d(n) \leq KR(n)$  for all  $n$ . □

We now consider the special cases  $\mathcal{C}[0, 1]$  and  $\mathcal{D}[0, 1]$ . Related to Corollary 10 we consider the following additional assumption, where the notation  $\mathcal{C}_r[0, 1]$  defined in (15) is used:

**C5** Case  $(\mathcal{C}[0, 1], \|\cdot\|_\infty)$ : We have  $X_n = Y_n + h_n$  for all  $n \geq 0$ , where  $\|h_n - h\|_\infty \rightarrow 0$  with  $h_n, h \in \mathcal{C}[0, 1]$ , and there exists a positive sequence  $(r_n)_{n \geq 0}$  such that

$$\mathbf{P}(Y_n \notin \mathcal{C}_{r_n}[0, 1]) \rightarrow 0.$$

Case  $(\mathcal{D}[0, 1], d_{sk})$ : We have  $X_n = Y_n + h_n$  for all  $n \geq 0$ , where  $\|h_n - h\|_\infty \rightarrow 0$  with  $h_n \in \mathcal{D}[0, 1], h \in \mathcal{C}[0, 1]$ , and there exists a positive sequence  $(r_n)_{n \geq 0}$  such that

$$\mathbf{P}(Y_n \notin \mathcal{D}_{r_n}[0, 1]) \rightarrow 0.$$

We now state the main theorem of this section. It follows immediately from Proposition 8, Corollary 10, Proposition 20 and Corollary 22.

**Theorem 23.** *Let  $(X_n)_{n \geq 0}$  be a sequence of random variables in  $(\mathcal{C}[0, 1], \|\cdot\|_\infty)$  or  $(\mathcal{D}[0, 1], d_{sk})$  satisfying recurrence (37) with conditions **C1**, **C2**, **C3** being satisfied. Then, for  $\mathcal{L}(X) = \eta$  we have for all  $t \in [0, 1]$*

$$X_n(t) \xrightarrow{d} X(t), \quad \mathbf{E}[|X_n(t)|^s] \rightarrow \mathbf{E}[|X(t)|^s]. \quad (52)$$

*If  $Z$  is distributed on  $[0, 1]$  and independent of  $(X_n)$  and  $X$  then*

$$X_n(Z) \xrightarrow{d} X(Z), \quad \mathbf{E}[|X_n(Z)|^s] \rightarrow \mathbf{E}[|X(Z)|^s]. \quad (53)$$

*If moreover conditions **C4** and **C5** are satisfied, where  $R(n)$  in **C4** and  $r_n$  in **C5** can be chosen with*

$$R(n) = o\left(\frac{1}{\log^m(1/r_n)}\right), \quad n \rightarrow \infty, \quad (54)$$

*then we have convergence in distribution:*

$$X_n \xrightarrow{d} X.$$

Finally, we give sufficient criteria to verify condition **C3** for the cases  $\mathcal{C}[0, 1]$  and  $\mathcal{D}[0, 1]$ . First, consider the general case where  $\mathcal{L}(Y) = \nu$  is a probability distribution on a separable Banach space  $(B, \|\cdot\|)$  with  $\mathbf{E}[\|Y\|^s] < \infty$ . If  $B$  is a Hilbert space, it is easy to see (and already indicated in [39] for  $m = 2$ ) that for a probability measure  $\mathcal{L}(X) = \mu$  on  $B$  to be in  $\mathcal{M}_s(\nu)$  the defining properties (9) and (10) are equivalent to  $\mathbf{E}[\|X\|^s] < \infty$  and

$$\mathbf{E}[\varphi_1(X) \cdots \varphi_k(X)] = \mathbf{E}[\varphi_1(Y) \cdots \varphi_k(Y)]$$

for all  $0 < k \leq m$  and continuous linear forms  $\varphi_1, \dots, \varphi_n$  on  $B$ . A generalization of this equivalence to Banach spaces does not hold in general, a counterexample is constructed in Janson and Kaijser [19]. However, with deeper arguments from functional analysis Janson and Kaijser [19] proved that this equivalence does hold for separable Banach spaces having the approximation property, such as  $\mathcal{C}[0, 1]$ . The case  $\mathcal{D}[0, 1]$  is also treated in [19]. Combining (9), (10) and Theorems 1.3 and 16.13 in [19] implies the following lemma.

**Lemma 24.** *Let  $\mathcal{L}(Y) = \mathcal{L}((Y_t)_{t \in [0, 1]}) = \nu$  and  $\mathcal{L}(X) = \mathcal{L}((X_t)_{t \in [0, 1]}) = \mu$  be probability measures on  $\mathcal{C}[0, 1]$ . For  $0 < s \leq 1$  we have  $\mu \in \mathcal{M}_s(\nu)$  if*

$$\mathbf{E}[\|X\|_\infty^s], \mathbf{E}[\|Y\|_\infty^s] < \infty. \quad (55)$$

*For  $1 < s \leq 2$  we obtain  $\mu \in \mathcal{M}_s(\nu)$  if we have condition (55) and*

$$\mathbf{E}[X_t] = \mathbf{E}[Y_t] \text{ for all } 0 \leq t \leq 1. \quad (56)$$

*For  $2 < s \leq 3$  we obtain  $\mu \in \mathcal{M}_s(\nu)$  if we have conditions (55), (56) and*

$$\text{Cov}(X_t, X_u) = \text{Cov}(Y_t, Y_u) \text{ for all } 0 \leq t, u \leq 1. \quad (57)$$

*The assertions remain true if  $\mathcal{C}[0, 1]$  is replaced by  $\mathcal{D}[0, 1]$ .*



**Remark 25.** Interpreting  $\mathbf{E}[X]$  as a Bochner integral in the continuous case, condition (56) is equivalent to  $\mathbf{E}[X] = \mathbf{E}[Y]$ . This is due to the fact that  $\mathbf{E}[X]$  is a continuous function with  $\mathbf{E}[X](t) = \mathbf{E}[X(t)]$  and  $\varphi(\mathbf{E}[X]) = \mathbf{E}[\varphi(X)]$  for all continuous linear forms  $\varphi$  on  $\mathcal{C}[0, 1]$ . Also the higher moments can be interpreted similarly as expectations of corresponding tensor products, see [12] or, for an elaborate account, [19].

**Remark 26.** Note that condition (57) typically cannot be achieved for a sequence  $(X_n)_{n \geq 0}$  that arises as in (2) by an affine scaling from a sequence  $(Y_n)_{n \geq 0}$  as in (1). This fundamental problem for developing a functional contraction method on the basis of the Zolotarev metrics  $\zeta_s$  with  $2 < s \leq 3$  was already mentioned in [12, Remark 6.2]. We describe a way to circumvent this problem in our application to Donsker's invariance principle by a perturbation argument, see section 4.1.

## 4 Applications

As applications we first give as a toy example a short proof of Donsker's invariance principle in section 4.1. In section 4.2 we discuss further examples from the probabilistic analysis of algorithms on partial match queries which requires the full generality of our abstract setting. This allows to settle various long standing open questions about asymptotics of the complexity of such queries.

### 4.1 Donsker's invariance principle

Let  $(V_n)_{n \in \mathbb{N}}$  be a sequence of independent, identically distributed real valued random variables with  $\mathbf{E}[V_1] = 0$ ,  $\text{Var}(V_1) = 1$  (for simplicity) and  $\mathbf{E}[|V_1|^{2+\varepsilon}] < \infty$  for some  $\varepsilon > 0$ . We consider the properly scaled and linearized random walk  $S^n = (S_t^n)_{t \in [0, 1]}$ ,  $n \geq 1$ , defined by

$$S_t^n = \frac{1}{\sqrt{n}} \left( \sum_{k=1}^{\lfloor nt \rfloor} V_k + (nt - \lfloor nt \rfloor) V_{\lfloor nt \rfloor + 1} \right), \quad t \in [0, 1].$$

With  $W = (W_t)_{t \in [0, 1]}$  a standard Brownian motion Donsker's function limit law states:

**Theorem 27.** (Donsker, 1951) *We have  $S^n \xrightarrow{d} W$  as  $n \rightarrow \infty$  in  $(\mathcal{C}[0, 1], \|\cdot\|_\infty)$ .*

#### 4.1.1 A contraction proof

In this section we apply the general methodology of sections 2 and 3 to give a short proof of Theorem 27. For a recursive decomposition of  $S^n$  and  $W$  we define operators for  $\beta > 1$ ,

$$\begin{aligned} \varphi_\beta : \mathcal{C}[0, 1] &\rightarrow \mathcal{C}[0, 1], & \varphi_\beta(f)(t) &= \mathbf{1}_{\{t \leq 1/\beta\}} f(\beta t) + \mathbf{1}_{\{t > 1/\beta\}} f(1), \\ \psi_\beta : \mathcal{C}[0, 1] &\rightarrow \mathcal{C}[0, 1], & \psi_\beta(f)(t) &= \mathbf{1}_{\{t \leq 1/\beta\}} f(0) + \mathbf{1}_{\{t > 1/\beta\}} f\left(\frac{\beta t - 1}{\beta - 1}\right). \end{aligned}$$

Note that both  $\varphi_\beta$  and  $\psi_\beta$  are linear, continuous and  $\|\varphi_\beta(f)\|_\infty = \|\psi_\beta(f)\|_\infty = \|f\|_\infty$  for all  $f \in \mathcal{C}[0, 1]$ , hence we have  $\|\varphi_\beta\|_{\text{op}} = \|\psi_\beta\|_{\text{op}} = 1$ . By construction we have

$$S^n \stackrel{d}{=} \sqrt{\frac{\lceil n/2 \rceil}{n}} \varphi_{\frac{n}{\lceil n/2 \rceil}} \left( S^{\lceil n/2 \rceil} \right) + \sqrt{\frac{\lfloor n/2 \rfloor}{n}} \psi_{\frac{n}{\lfloor n/2 \rfloor}} \left( \hat{S}^{\lfloor n/2 \rfloor} \right), \quad n \geq 2, \quad (58)$$

where  $(S^1, \dots, S^n)$  and  $(\widehat{S}^1, \dots, \widehat{S}^n)$  are independent and  $S^j$  and  $\widehat{S}^j$  are identically distributed for all  $j \geq 1$ . Therefore  $(S^n)_{n \geq 1}$  satisfies recurrence (37) choosing

$$K = 2, \quad I_1^{(n)} = \lceil n/2 \rceil, \quad I_2^{(n)} = \lfloor n/2 \rfloor, \quad n_0 = 2, \\ A_1^{(n)} = \sqrt{\frac{\lceil n/2 \rceil}{n}} \varphi_{\frac{n}{\lceil n/2 \rceil}}, \quad A_2^{(n)} = \sqrt{\frac{\lfloor n/2 \rfloor}{n}} \psi_{\frac{n}{\lfloor n/2 \rfloor}}, \quad b^{(n)} = 0.$$

In the following let  $\widehat{W} = (\widehat{W}_t)_{t \in [0,1]}$  be a standard Brownian motion, independent of  $W$ . Properties of Brownian motion imply

$$W \stackrel{d}{=} \sqrt{\frac{1}{\beta}} \varphi_\beta(W) + \sqrt{\frac{\beta-1}{\beta}} \psi_\beta(\widehat{W}), \quad (59)$$

for any  $\beta > 1$ . Hence, the Wiener measure  $\mathcal{L}(W)$  is a fixed-point of the operator  $T$  in (40) with

$$K = 2, \quad A_1 = \sqrt{\frac{1}{\beta}} \varphi_\beta, \quad A_2 = \sqrt{\frac{\beta-1}{\beta}} \psi_\beta, \quad b = 0. \quad (60)$$

For  $\beta = 2$  the coefficients in (58) converge to the ones in (59), i.e., as  $n \rightarrow \infty$ ,

$$\sqrt{\frac{\lceil n/2 \rceil}{n}} \rightarrow \frac{1}{\sqrt{2}}, \quad \sqrt{\frac{\lfloor n/2 \rfloor}{n}} \rightarrow \frac{1}{\sqrt{2}},$$

but the coefficients  $A_1^{(n)}, A_2^{(n)}$  only converge to  $A_1, A_2$  in the operator norm for  $n$  even. Nevertheless, from the point of view of the contraction method this suggests weak convergence of  $S^n$  to  $W$ .

Note that the operator  $T$  associated with the fixed-point equation (59), i.e., with the coefficients in (60), satisfies condition **C2** only with  $s > 2$ . In view of condition **C3** and Lemma 24 we need to match the mean and covariance structure. We have  $\mathbf{E}[S_t^n] = 0$  for all  $0 \leq t \leq 1$  and a direct computation yields

$$\text{Cov}(S_s^n, S_t^n) = \begin{cases} s, & \text{for } \lfloor ns \rfloor < \lfloor nt \rfloor, \\ \frac{1}{n} (\lfloor ns \rfloor + (ns - \lfloor ns \rfloor)(nt - \lfloor nt \rfloor)), & \text{for } \lfloor ns \rfloor = \lfloor nt \rfloor. \end{cases} \quad (61)$$

Hence, we do not have finite  $\zeta_{2+\varepsilon}$ -distance between  $S^n$  and  $W$  since they do not share their covariance functions. To surmount this problem we consider a linearized version of the Brownian motion  $W$ . For fixed  $n \in \mathbb{N}$  we divide the unit interval into pieces of length  $1/n$  and interpolate  $W$  linearly between the points  $0, 1/n, 2/n, \dots, (n-1)/n, 1$ . The interpolated process  $W^n = (W_t^n)_{t \in [0,1]}$  is given by

$$W_t^n := W_{\frac{\lfloor nt \rfloor}{n}} + (nt - \lfloor nt \rfloor) \left( W_{\frac{\lfloor nt \rfloor + 1}{n}} - W_{\frac{\lfloor nt \rfloor}{n}} \right), \quad t \in [0, 1].$$

We have  $\mathbf{E}[W_t^n] = 0$  and  $W^n$  and  $S^n$  have the same covariance function (61) for all  $n \in \mathbb{N}$ . Furthermore  $W^n$  has the same distributional recursive decomposition (58) as  $S^n$ .

Note that the linearized Brownian motion does not differ much from the original one:

**Lemma 28.** *We have  $\|W^n - W\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$  almost surely.*

*Proof.* This directly follows from the uniform continuity of  $W$ . For  $\varepsilon > 0$  there exists a random  $\delta > 0$  such that  $|W(t) - W(s)| < \varepsilon$  for any  $s, t \in [0, 1]$  with  $|t - s| < \delta$ . The triangle inequality implies  $\|W^n - W\|_\infty < 2\varepsilon$  for any  $n > 1/\delta$ .  $\square$

In view of Corollary 11 it suffices to prove that  $S^n$  and  $W^n$  are close with respect to  $\zeta_{2+\varepsilon}$ . The proof of this runs along the same lines as the one for Proposition 20, resp. Corollary 22, in fact it is much shorter due to the simple form of the recurrence:

**Proposition 29.** *For any  $\delta < \varepsilon/2$  we have  $\zeta_{2+\varepsilon}(S^n, W^n) = O(n^{-\delta})$  as  $n \rightarrow \infty$ .*

*Proof.* We have

$$\begin{aligned} \zeta_{2+\varepsilon}(S^n, W^n) &= \zeta_{2+\varepsilon} \left( \sqrt{\frac{\lceil n/2 \rceil}{n}} \varphi_{\frac{n}{\lceil n/2 \rceil}}(S^{\lceil n/2 \rceil}) + \sqrt{\frac{\lfloor n/2 \rfloor}{n}} \psi_{\frac{n}{\lfloor n/2 \rfloor}}(\bar{S}^{\lfloor n/2 \rfloor}), \right. \\ &\quad \left. \sqrt{\frac{\lceil n/2 \rceil}{n}} \varphi_{\frac{n}{\lceil n/2 \rceil}}(W^{\lceil n/2 \rceil}) + \sqrt{\frac{\lfloor n/2 \rfloor}{n}} \psi_{\frac{n}{\lfloor n/2 \rfloor}}(\bar{W}^{\lfloor n/2 \rfloor}) \right) \\ &\leq \left( \frac{\lceil n/2 \rceil}{n} \right)^{1+\varepsilon/2} \zeta_{2+\varepsilon}(S^{\lceil n/2 \rceil}, W^{\lceil n/2 \rceil}) \\ &\quad + \left( \frac{\lfloor n/2 \rfloor}{n} \right)^{1+\varepsilon/2} \zeta_{2+\varepsilon}(S^{\lfloor n/2 \rfloor}, W^{\lfloor n/2 \rfloor}). \end{aligned}$$

We abbreviate

$$d_n := \zeta_{2+\varepsilon}(S^n, W^n), \quad a_n := \left( \frac{\lceil n/2 \rceil}{n} \right)^{1+\varepsilon/2}, \quad b_n := \left( \frac{\lfloor n/2 \rfloor}{n} \right)^{1+\varepsilon/2}$$

and note that we have  $a_n + b_n \leq 2^{-\varepsilon/2} + C'/n$  for some constant  $C' > 0$  and all  $n \in \mathbb{N}$ . For arbitrary  $\delta < \varepsilon/2$  we prove the assertion by induction: Fix  $\delta < \delta' < \varepsilon/2$  and choose  $m_0 \in \mathbb{N}$  such that  $\lfloor n/2 \rfloor^{-\delta} \leq (n/2)^{-\delta} 2^{\varepsilon/2-\delta'}$  and  $1 + 2^{\varepsilon/2} C'/n \leq 2^{\delta'-\delta}$  for all  $n \geq m_0$ . Furthermore, let  $C > 0$  be large enough such that  $d_n \leq Cn^{-\delta}$  for all  $1 \leq n \leq m_0$ . Then, for  $n > m_0$ , assuming the claim to be verified for all smaller indices,

$$\begin{aligned} d_n &\leq a_n d_{\lceil n/2 \rceil} + b_n d_{\lfloor n/2 \rfloor} \\ &\leq C \left( a_n (n/2)^{-\delta} + b_n (n/2)^{-\delta} 2^{\varepsilon/2-\delta'} \right) \\ &\leq Cn^{-\delta} 2^{\delta} 2^{\varepsilon/2-\delta'} (a_n + b_n) \\ &\leq Cn^{-\delta}. \end{aligned}$$

The assertion follows.  $\square$

Now Donsker's theorem (Theorem 27) follows from Proposition 29, Lemma 28 and Corollary 11.

Note, that our approach requires the assumption  $\mathbf{E} [|V_1|^{2+\varepsilon}] < \infty$  for some  $\varepsilon > 0$ , which, in Donsker's theorem can be weakened to  $\mathbf{E} [V_1^2] < \infty$ .

By Theorem 12 we directly obtain convergence of moments of the supremum:

**Corollary 30.** *Suppose  $\mathbf{E} [|V_1|^{2+\alpha}] < \infty$  with  $0 < \alpha \leq 1$ . Then  $\|S^n\|_\infty^{2+\alpha}$  is uniformly integrable. Thus,  $\mathbf{E} [\|S^n\|_\infty^\kappa]$  converges to  $\mathbf{E} [\|W\|_\infty^\kappa]$  for any  $0 < \kappa \leq 2 + \alpha$ .*

**Remark 31.** Based on the recursion (58), it is easy to show that  $\mathbf{E} [\|S^n\|_\infty^k]$  is bounded uniformly in  $n$  for integer valued  $k \geq 3$  if the increment  $V_1$  has finite absolute moment of order  $k$ . In this case, we have  $\mathbf{E} [\|S^n\|_\infty^\kappa] \rightarrow \mathbf{E} [\|W\|_\infty^\kappa]$  for any real  $0 < \kappa < k$ .

#### 4.1.2 Characterizing the Wiener measure by a fixed-point property

We reconsider the map  $T$  corresponding to the fixed-point equation (59) for the case  $\beta = 2$ :

$$\begin{aligned} T : \mathcal{M}(\mathcal{C}[0, 1]) &\rightarrow \mathcal{M}(\mathcal{C}[0, 1]) \\ T(\mu) &= \mathcal{L} \left( \frac{1}{\sqrt{2}} \varphi_2(Z) + \frac{1}{\sqrt{2}} \psi_2(\bar{Z}) \right), \end{aligned} \quad (62)$$

where  $Z, \bar{Z}$  are independent with distribution  $\mathcal{L}(Z) = \mathcal{L}(\bar{Z}) = \mu$ . Our discussion above implies that the Wiener measure  $\mathcal{L}(W)$  is the unique fixed-point of  $T$  restricted to  $\mathcal{M}_{2+\varepsilon}(\mathcal{L}(W))$  for any  $\varepsilon > 0$ . Note that  $\mathcal{M}_{2+\varepsilon}(\mathcal{L}(W))$  is the space of the distributions of all continuous stochastic processes  $V = (V_t)_{t \in [0, 1]}$  with  $\mathbf{E} [\|V\|_\infty^{2+\varepsilon}] < \infty$ ,  $\mathbf{E} [V_t] = 0$  and  $\text{Cov}(V_t, V_u) = t \wedge u$  for all  $0 \leq t, u \leq 1$ . Note, that one easily verifies that  $T(\mathcal{M}_{2+\varepsilon}(\mathcal{L}(W))) \subset \mathcal{M}_{2+\varepsilon}(\mathcal{L}(W))$  and the last part of the proof of Lemma 19 implies that  $T$  restricted to  $\mathcal{M}_{2+\varepsilon}(\mathcal{L}(W))$  is Lipschitz-continuous with Lipschitz constant at most  $L = 2^{-\varepsilon/2} < 1$ , hence  $\mathcal{L}(W)$  is the unique fixed-point of  $T$  in  $\mathcal{M}_{2+\varepsilon}(\mathcal{L}(W))$ .

We now show that a more general statement is true, the Wiener measure is also, up to multiplicative scaling, the unique fixed-point of  $T$  in the larger space of probability measures  $\mathcal{L}(V) \in \mathcal{M}(\mathcal{C}[0, 1])$  with  $V_0 = 0$ . For a related statement, see also Aldous [1, page 528]. The subsequent proof is based on the fact that the centered normal distributions are the only solutions of the fixed-point equation

$$X \stackrel{d}{=} \frac{X + \bar{X}}{\sqrt{2}} \quad (63)$$

where  $X, \bar{X}$  are independent, identically distributed real-valued random variables, see Theorem 7.2.1 in [22].

**Theorem 32.** *Let  $X = (X_t)_{t \in [0, 1]}$  be a continuous process with  $X_0 = 0$ . Then  $\mathcal{L}(X)$  is a fixed-point of (62) if and only if either  $X = \mathbf{0}$  a.s. or there exists a constant  $\sigma > 0$ , such that  $(\sigma^{-1} X_t)_{t \in [0, 1]}$  is a standard Brownian Motion.*

*Proof.* Let  $\mathcal{L}(X)$  be a fixed-point of (62) and  $\bar{X} = (\bar{X}_t)_{t \in [0, 1]}$  be independent of  $X$  with the same distribution. The fixed-point property implies

$$X_1 \stackrel{d}{=} \frac{X_1 + \bar{X}_1}{\sqrt{2}},$$

hence  $\mathcal{L}(X_1) = \mathcal{N}(0, \sigma^2)$  for some  $\sigma^2 \geq 0$ , where  $\mathcal{N}(0, \sigma^2)$  denotes the centered normal distribution with variance  $\sigma^2$ . This implies

$$X_{1/2} \stackrel{d}{=} \frac{X_1}{\sqrt{2}},$$

hence  $\mathcal{L}(X_{1/2}) = \mathcal{N}(0, \sigma^2/2)$ . Let  $\mathcal{D} = \{m2^{-n} : m, n \in \mathbb{N}_0, m \leq 2^n\}$  be the set of dyadic numbers in  $[0, 1]$ . By induction, we obtain  $\mathcal{L}(X_t) = \mathcal{N}(0, \sigma^2 t)$  for all  $t \in \mathcal{D}$ . For the distribution of the increments we first obtain

$$X_1 - X_{1/2} \stackrel{d}{=} \frac{X_1}{\sqrt{2}},$$

hence  $\mathcal{L}(X_1 - X_{1/2}) = \mathcal{N}(0, \sigma^2/2)$ . Again inductively, we obtain  $\mathcal{L}(X_1 - X_t) = \mathcal{N}(0, (1-t)\sigma^2)$  for all  $t \in \mathcal{D}$ . Also by induction, it follows  $\mathcal{L}(X_t - X_s) = \mathcal{N}(0, (t-s)\sigma^2)$  for all  $s, t \in \mathcal{D}$  with  $s < t$ . Finally, continuity of  $X$  implies the same property for all  $s, t \in [0, 1]$ . It remains to prove

independence of increments. Denoting by  $X^{(1)}, X^{(2)}, \dots$  independent distributional copies of  $X$ , we obtain from iterating the fixed-point property

$$(X_t)_{t \in [0,1]} \stackrel{d}{=} \left( 2^{-n/2} \sum_{m=1}^{2^n} \mathbf{1}_{\{(m-1)2^{-n} < t \leq m2^{-n}\}} X_{2^{-n}t-m+1}^{(m)} + \mathbf{1}_{\{m2^{-n} < t\}} X_1^{(m)} \right)_{t \in [0,1]}$$

for all  $n \in \mathbb{N}$ . Hence, for any dyadic points  $0 \leq t_1 < t_2 < \dots < t_k \leq 1$ , choosing  $n$  large enough, each  $X_{t_{i+1}} - X_{t_i}$  can be expressed as a function of a subset of  $X^{(1)}, \dots, X^{(2^n)}$  these subsets being pairwise disjoint for  $i = 0, \dots, n-1$ . Since,  $\mathcal{D}$  is dense in  $[0, 1]$ , this shows that  $X$  has independent increments. For  $\sigma = 0$  we have  $X = \mathbf{0}$  a.s., otherwise  $\sigma^{-1}X$  is a standard Brownian motion.

The converse direction of the theorem is trivial.  $\square$

**Remark 33.** Note that we cannot cancel the assumption on continuity of  $X$  without replacement, e.g., the process

$$Y_t = \begin{cases} W_t & : t \notin \mathcal{D} \\ 0 & : t \in \mathcal{D} \end{cases}$$

also solves (59) and is not a multiple of Brownian motion. However, it would be sufficient to require càdlàg paths, so  $\mathcal{C}[0, 1]$  could be replaced by  $\mathcal{D}[0, 1]$  in our statement.

**Remark 34.** Our decomposition of Brownian motion in (59) is in time. However, equation (63) suggests to also investigate a decomposition in space

$$(X_t)_{t \in [0,1]} \stackrel{d}{=} \left( \frac{X_t + \bar{X}_t}{\sqrt{2}} \right)_{t \in [0,1]} \quad (64)$$

where  $(X_t)_{t \in [0,1]}$  and  $(\bar{X}_t)_{t \in [0,1]}$  are independent and identically distributed. Again, equation (64) induces a map on  $\mathcal{M}(\mathcal{C}[0, 1])$  that is a contraction in  $\zeta_{2+\varepsilon}$  on the subspace  $\mathcal{M}_{2+\varepsilon}(\mathcal{L}(W))$ , so the Wiener measure is the only solution in  $\mathcal{M}_{2+\varepsilon}(\mathcal{L}(W))$ . In this case, we cannot remove the moment assumption as in Theorem 32 since any centered, continuous Gaussian process solves equation (64). Using (63), it is not hard to see that there are no further solutions of (64).

## 4.2 Partial match queries in quad trees

In this section we outline recurrences coming up in the probabilistic analysis of the performance of data structures and discuss in detail the use and verification of our conditions **C1–C5** and Theorem 23. In this example the full generality of our setup is needed.

For preprocessing and supporting search queries in multidimensional data various types of search trees are in use, most prominently quad trees and  $k$ -d trees. Among various other fundamental search operations in multivariate data so called partial match queries are of particular importance. For a partial match query one specifies some of the components of the data and asks to report all data in the given set that match the specified components and are arbitrary in the remaining components. We will subsequently not need to introduce these data structures and the partial match queries since there is a geometric reformulation that is discussed and used below. For details about the Computer Science background and precise definition of the structures and queries, see [6].

Consider a sequence  $(U_i, V_i)_{i \geq 1}$  of independent and identically distributed random vectors all with the uniform distribution on the unit square  $[0, 1]^2$ . We iteratively construct a decomposition of  $[0, 1]^2$  as follows. The first point  $(U_1, V_1)$  decomposes the square into four rectangles by drawing the two lines through  $(U_1, V_1)$  in  $[0, 1]^2$  that are perpendicular to its sides. We call these line

segments the horizontal and vertical lines. The second point  $(U_2, V_2)$  almost surely falls into the interior of one of the four rectangles. We recursively draw the horizontal and vertical lines through  $(U_2, V_2)$  within the rectangle. Hence, we then have a decomposition of the original square  $[0, 1]^2$  into seven rectangles. Now we iterate this process. After  $n - 1$  steps we have  $3(n - 1) + 1$  rectangles and the  $n$ -th point is used to decompose the rectangle it falls in into four new rectangles by the horizontal and vertical lines through it, see Figure 1. We identify this decomposition of the unit square with all the line segments drawn and call it the *decomposition after  $n$  steps*.

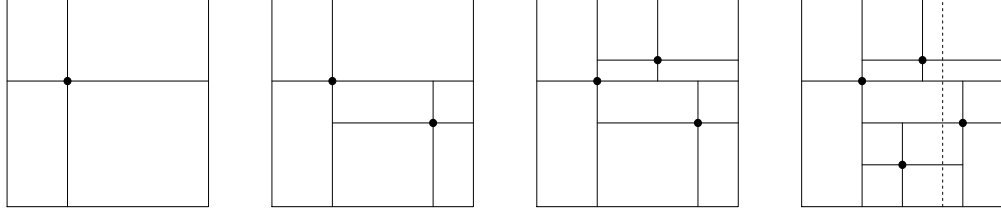


Figure 1: The construction of a quad tree at times  $n = 1, 2, 3, 4$ . The dashed line in the right most square indicates the query line  $x_1 = t$ .

Now fix  $t \in [0, 1]$  and denote the number of horizontal lines in the decomposition after  $n$  steps that are cut by the vertical line  $x_1 = t$  by  $C_n(t)$ , see Figures 1 and 2.

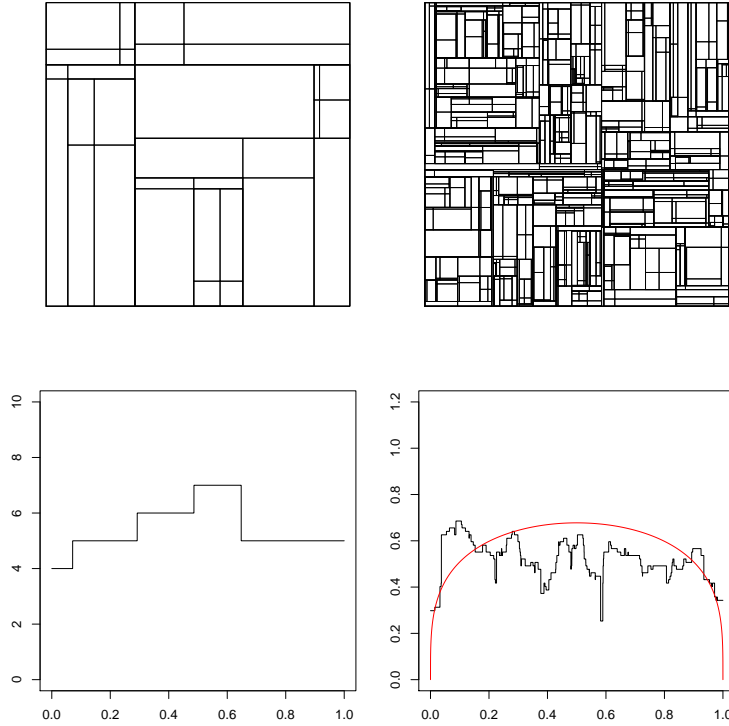


Figure 2: Left column: A realisation of the decomposition of the unit square for a quad tree of size  $n = 10$  and its process  $(C_n(t))_{t \in [0,1]}$ . Right column: A realisation of the decomposition of the unit square for a quad tree of size  $n = 300$  and the process  $(X_n(t))_{t \in [0,1]}$ . The smooth curve indicates the function  $t \mapsto (t(1 - t))^{\beta/2}$ .

In the Computer Science setting this is the measure for the complexity of a partial match query in a random (point) quad tree where the first component is specified as  $t$ , the second component is



arbitrary and  $n$  data are inserted in the uniform model, see [6]. We have  $C_0(t) = 0$  and  $C_1(t) = 1$  for all  $t \in [0, 1]$ . We consider the process  $(C_n(t))_{t \in [0, 1]}$  as a process in  $(\mathcal{D}[0, 1], d_{sk})$ .

For a recursive decomposition of this process we denote the numbers of points among the first  $n$  points which fall into each of the four rectangles generated by the first point  $(U_1, V_1) =: (U, V)$  by  $I^{(n)} = (I_1^{(n)}, I_2^{(n)}, I_3^{(n)}, I_4^{(n)})$ . Hence, conditionally on  $(U, V)$ , the vector  $I^{(n)}$  has the multinomial distribution  $M(n-1; UV, U(1-V), (1-U)V, (1-U)(1-V))$ , where a numbering of the four quadrants is used. Moreover, conditionally on  $(U, V)$  and  $I^{(n)}$  we have that each point set within a rectangle is a set of independent and identically distributed points each with the uniform distribution on the particular rectangle and that the four point sets are also independent. Hence for processes  $(C_j^{(r)}(t))_{t \in [0, 1]}$  which are independent and independent of  $(U, V, I^{(n)})$ , and  $(C_j^{(r)}(t))_{t \in [0, 1]}$  distributed as  $(C_j(t))_{t \in [0, 1]}$  for  $r = 1, \dots, 4$  and  $j \in \mathbb{N}_0$  we obtain the recurrence

$$(C_n(t))_{t \in [0, 1]} \stackrel{d}{=} \left( 1 + \mathbf{1}_{\{t < U\}} \left[ C_{I_1^{(n)}}^{(1)} \left( \frac{t}{U} \right) + C_{I_2^{(n)}}^{(2)} \left( \frac{t}{U} \right) \right] + \mathbf{1}_{\{t \geq U\}} \left[ C_{I_3^{(n)}}^{(3)} \left( \frac{t-U}{1-U} \right) + C_{I_4^{(n)}}^{(4)} \left( \frac{t-U}{1-U} \right) \right] \right)_{t \in [0, 1]}. \quad (65)$$

The arguments  $t/U$  and  $(1-t)/(1-U)$  adjust that a vertical line  $x_1 = t$  within the whole square  $[0, 1]^2$ , after scaling, corresponds to the line  $x_1 = t/U$  in the left rectangles (if  $t < U$ ) and to the line  $x_1 = (1-t)/(1-U)$  in the right rectangles (if  $t \geq U$ ). Note that equation (65) has exactly the form (1), where the indicators and rescalings in time in (65) give the random linear maps  $A_r(n)$  for  $r = 1, \dots, 4$ , and we have  $b(n) = 1$ .

The first asymptotic analysis of this process was done by Flajolet et al. [15], where the one-dimensional averaged complexity  $C_n(\xi)$  was considered with  $\xi$  uniformly distributed on  $[0, 1]$  and independent of the sequence  $(U_i, V_i)_{i \in \mathbb{N}}$ . In [15] it is shown that, as  $n \rightarrow \infty$ ,

$$\mathbb{E}[C_n(\xi)] \sim \kappa n^\beta \quad \text{with} \quad \kappa = \frac{\Gamma(2\beta + 2)}{2(\Gamma(\beta + 1))^3}, \quad \beta = \frac{\sqrt{17} - 3}{2},$$

where  $\Gamma$  denotes the gamma function; see also Chern and Hwang [8] for more refined analysis of this expectation. Recently, Curien and Joseph [9] showed

$$\mathbb{E}[C_n(t)] \sim \chi(t(1-t))^{\beta/2} n^\beta, \quad \text{with} \quad \chi = \frac{\kappa}{B\left(\frac{\beta}{2} + 1, \frac{\beta}{2} + 1\right)}, \quad (66)$$

where  $B(\cdot, \cdot)$  denotes the beta function (Euler integral). The analysis beyond expectations, in particular of variances and limit laws either for the process  $(C_n(t))_{t \in [0, 1]}$  itself or its marginals or the averaged complexity  $C_n(\xi)$  or the worst case complexity  $\sup_{t \in [0, 1]} C_n(t)$  remained open.

We now discuss how our general framework from section 3 can be applied to a proper normalization of  $(C_n(t))_{t \in [0, 1]}$  and highlight the use and verification of conditions **C1**–**C5**, which can be shown to hold with the choice  $s = 2$ . The details are worked out in [6]. The resulting functional limit law allows to settle the open questions raised in the previous paragraph.

Let us first use the normalization  $X_0(t) := 0$  and

$$X_n(t) := \frac{C_n(t)}{\chi n^\beta}, \quad n \geq 1, t \in [0, 1], \quad (67)$$

and write  $X_n := (X_n(t))_{t \in [0, 1]}$ . See Figure 2 for a simulation of  $X_n$ . For  $X_n$  we obtain the

recurrence

$$X_n \stackrel{d}{=} \left( \frac{1}{\chi n^\beta} + \mathbf{1}_{\{t < U\}} \left[ \left( \frac{I_1^{(n)}}{n} \right)^\beta X_{I_1^{(n)}}^{(1)} \left( \frac{t}{U} \right) + \left( \frac{I_2^{(n)}}{n} \right)^\beta X_{I_2^{(n)}}^{(2)} \left( \frac{t}{U} \right) \right] \right. \\ \left. + \mathbf{1}_{\{t \geq U\}} \left[ \left( \frac{I_3^{(n)}}{n} \right)^\beta X_{I_3^{(n)}}^{(3)} \left( \frac{t-U}{1-U} \right) + \left( \frac{I_4^{(n)}}{n} \right)^\beta X_{I_4^{(n)}}^{(4)} \left( \frac{t-U}{1-U} \right) \right] \right)_{t \in [0,1]}$$

with assumptions on independence and identical distributions as in (65). This suggests that a limit process  $X = (X(t))_{t \in [0,1]}$  satisfies

$$X \stackrel{d}{=} \left( \mathbf{1}_{\{t < U\}} \left[ (UV)^\beta X^{(1)} \left( \frac{t}{U} \right) + (U(1-V))^\beta X^{(2)} \left( \frac{t}{U} \right) \right] \right. \\ \left. + \mathbf{1}_{\{t \geq U\}} \left[ ((1-U)V)^\beta X^{(3)} \left( \frac{t-U}{1-U} \right) + ((1-U)(1-V))^\beta X^{(4)} \left( \frac{t-U}{1-U} \right) \right] \right)_{t \in [0,1]}, \quad (68)$$

where  $U$  and  $V$  are independent  $[0, 1]$ -uniform random variables and  $(X^{(r)}(t))_{t \in [0,1]}$ , for  $r = 1, \dots, 4$ , are independent copies of the process  $X$ , also independent of  $(U, V)$ . Note that (68) is a fixed-point equation of type (4).

This heuristic derivation of equation (68) can be turned into a rigorous approach as follows. First, note that the operators  $A_1^{(n)}$  and  $A_1$  on  $\mathcal{D}[0, 1]$  are given as follows: For  $f \in \mathcal{D}[0, 1]$  the random functions  $A_1^{(n)}(f)$  and  $A_1(f)$  are

$$t \mapsto \mathbf{1}_{\{t < U\}} \left( \frac{I_1^{(n)}}{n} \right)^\beta f \left( \frac{t}{U} \right), \quad \text{and} \quad t \mapsto \mathbf{1}_{\{t < U\}} (UV)^\beta f \left( \frac{t}{U} \right), \quad (69)$$

and direct integration shows that condition **C2** is satisfied for the choice  $s = 2$ .

For condition **C3** first an appropriate process  $X = (X(t))_{t \in [0,1]}$  which solves (68) has to be constructed. Since we do not know the completeness of  $\zeta_2$  on an appropriate subspace of  $\mathcal{M}_2(\mathcal{D}[0, 1])$  and also are not able to guess  $X$  as a well-known process (as in the example in section 4.1.1) such a process  $X$  has to be constructed individually. In view of (66) the normalization (67), the choice  $s = 2$  and Lemma 24 we additionally need to have  $\mathbf{E}[\|X\|_\infty^2] < \infty$  and  $\mathbf{E}[X(t)] = (t(1-t))^{\beta/2}$  for  $t \in [0, 1]$ . In [6] a sequence of random continuous functions is constructed from a discrete recurrence approximating (68) which converges uniformly. The construction uses concentration inequalities and tail bounds for the saturation level of random quad trees. Its limit  $X$  is the stochastic process as needed. Moreover, it can also be shown that it has continuous paths almost surely.

Our normalization does not imply that  $\mathcal{L}(X_n) \in \mathcal{M}_2(\mathcal{L}(X))$  for all  $n \geq 1$ , since the normalization in (67) does violate condition (56). Thus, the processes  $X_n$  cannot be compared with  $X$  using the  $\zeta_2$  distance. To overcome this technical issue one can instead consider the normalization

$$\frac{C_n(t) - \mathbf{E}[C_n(t)]}{\chi n^\beta}, \quad t \in [0, 1], \quad n \geq 1. \quad (70)$$

and the shifted limit  $(X(t) - (t(1-t))^{\beta/2})_{t \in [0,1]}$ . Then condition **C3** is satisfied. This also shows the necessity to allow the perturbation  $h_n$  in Corollary 10 and condition **C5** in our general setup. The centering of the sequence  $X_n$  and the solution  $X$  only affects the additive term  $b^{(n)}$  and the toll term  $b$ . In particular, condition **C2** remains valid in the centered setting and we have

$\|A_1^{(n)} - A_1\|_s \rightarrow 0$  for any  $s > 0$ . Similarly  $\|A_r^{(n)} - A_r\|_s \rightarrow 0$  for  $r = 2, 3, 4$ . Convergence of the additive term  $b^{(n)}$  is equivalent to uniformity of the expansion in (66). This is shown in [6]. It is also easily seen that (39) holds, hence condition **C1** is true.

For condition **C4** an appropriated rate of convergence of the coefficients in (38) is needed. Note that such a rate can only be derived if a rate in the asymptotic expansion of the means in (66) is available. Hence, as a technical step in [6] a polynomial additive error term of the order  $O(n^{\beta-\varepsilon})$  for some  $\varepsilon > 0$  is shown to hold valid uniformly in  $t \in [0, 1]$ . This implies that the convergence rates  $\gamma(n)$  in (38) satisfy  $\gamma(n) = O(n^{-\varepsilon})$  as  $n \rightarrow \infty$ . Hence, for the sequence  $(R(n))_{n \geq 1}$  in condition **C4** we can choose  $R(n) = n^{-\varepsilon'}$  with  $0 < \varepsilon' \leq \varepsilon$  sufficiently small such that we obtain  $L^* < 1$  in **C4**.

Finally note that the jumps of your piecewise constant processes  $X_n$  occur at the random times  $U_1, \dots, U_n$  so that interval lengths between consecutive jumps may become arbitrarily small. Condition **C5** allows to cover such instances of processes if the probability for close jumps can be controlled. In our example it is easy to see that the smallest interval between jumps is of length at least  $n^{-3}$  with probability of order  $O(1/n)$ . Hence condition **C5** is satisfied with the choice  $r_n = n^{-3}$  there. Moreover, the sequences  $(r_n)_{n \in \mathbb{N}}$  and  $(R(n))_{n \in \mathbb{N}}$  are chosen such that condition (54) is fulfilled. Hence, our main result Theorem 23 applies and we first obtain distributional convergence of the centered normalized sequence in (70) which also implies

$$X_n \xrightarrow{d} X$$

in  $(\mathcal{D}[0, 1], d_{sk})$ . Here, we may also apply Theorem 12 to infer convergence of moments of  $\|X_n\|$  towards the moments of  $\|X\|$ .

The use of some other search trees to support partial match queries leads to distributional recurrences related to (65), e.g., the 2-d-trees. For the application of our framework in this case, see [6].

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