Oberseminar, Institut für Mathematische Stochastik
Universität Münster

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A decomposition of the Brownian excursion

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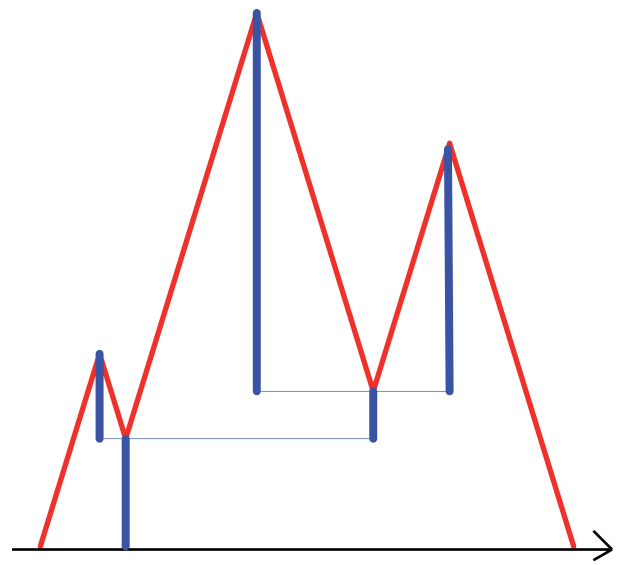
(mit Stephan Gufler (Technion Haifa) und Götz Kersting (GU FfM))

“**Walks and trees** are abstractly identical objects ... ”

(Ted Harris (1952))



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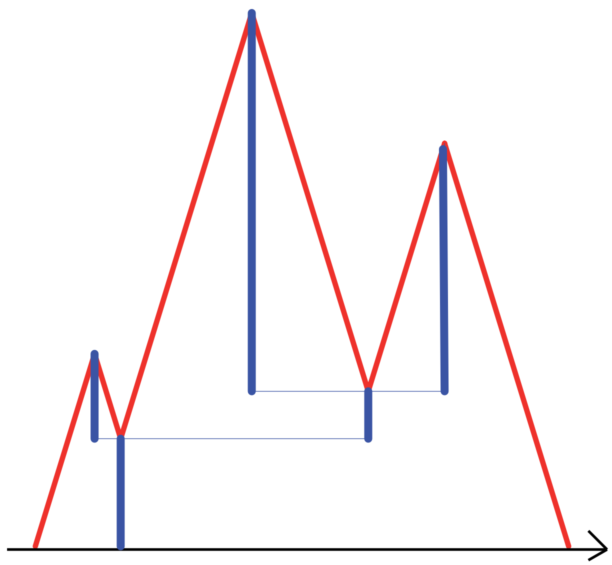
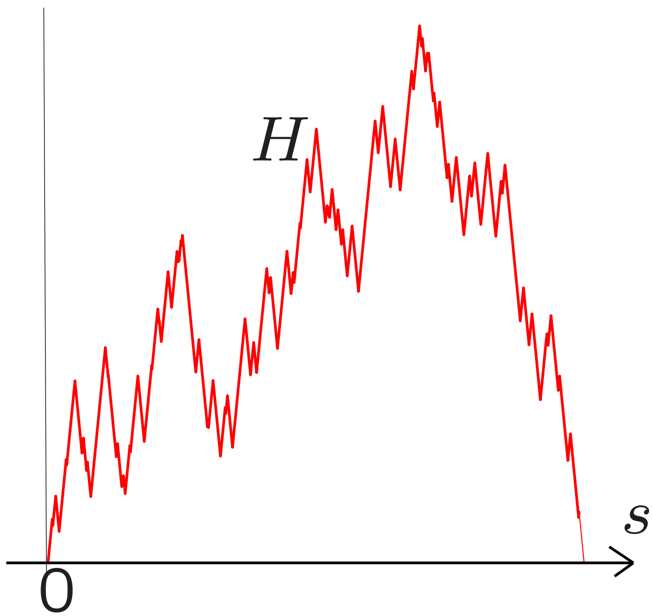


As known from work of Neveu, Pitman, Aldous, Le Gall, ...

Harris' paradigm holds also in a Brownian rescaling.

There the walk becomes an Itô excursion.



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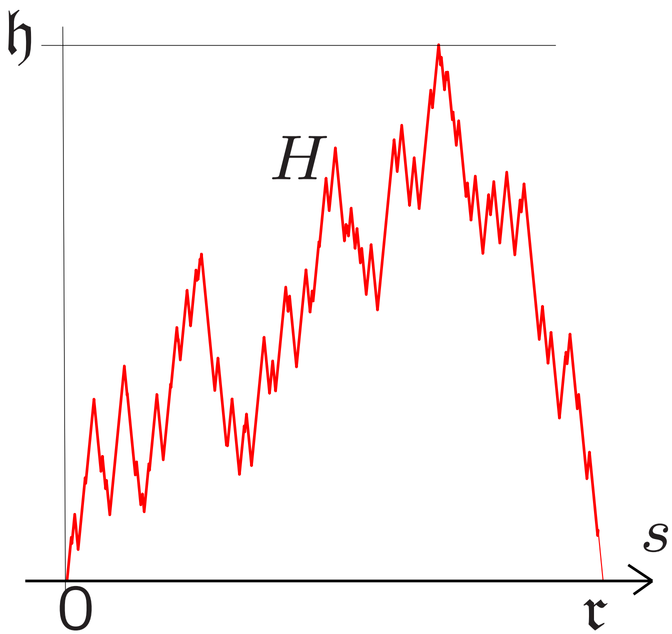


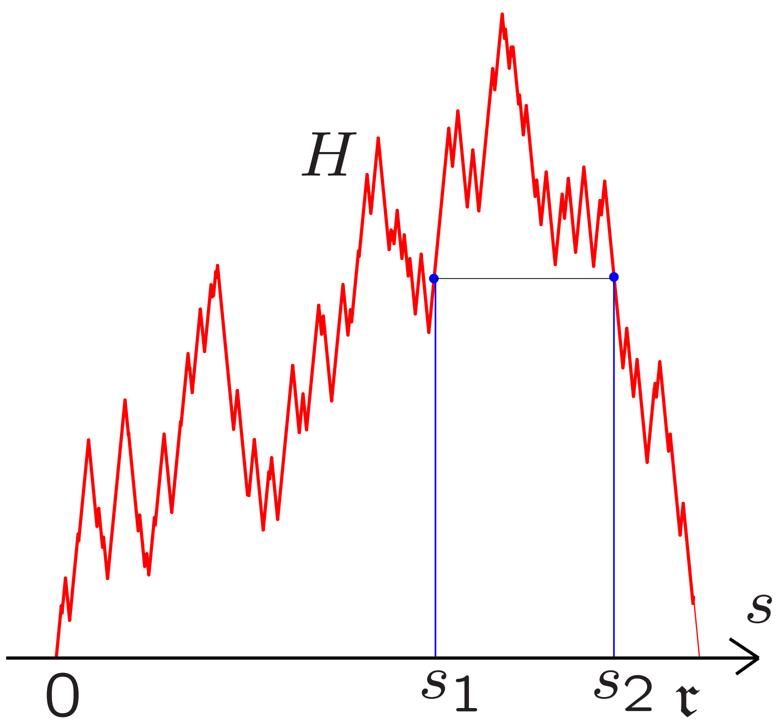
$\mathfrak{r}$: return time ("length") of H

$\mathfrak{h}$: (maximal) height of H

The *rooted, ordered* $\mathbb{R}$-tree $(T^H, d, \prec)$:

For $0 \le s_1 \le s_2 \le \mathfrak{r}$:

$s_1 \sim s_2 :\Longleftrightarrow H(s_1) = H(s_2) = \min\{H(s) : s \in [s_1, s_2]\}$



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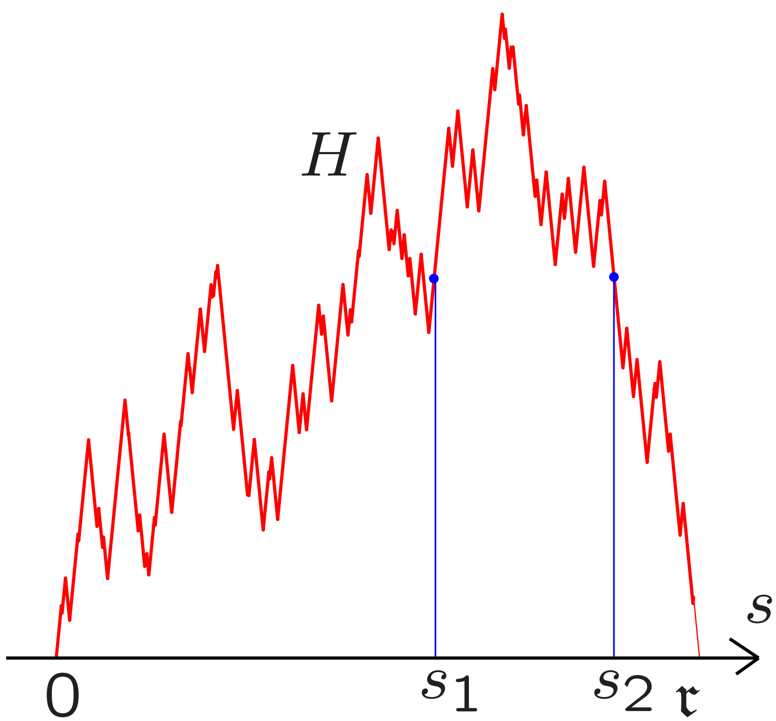

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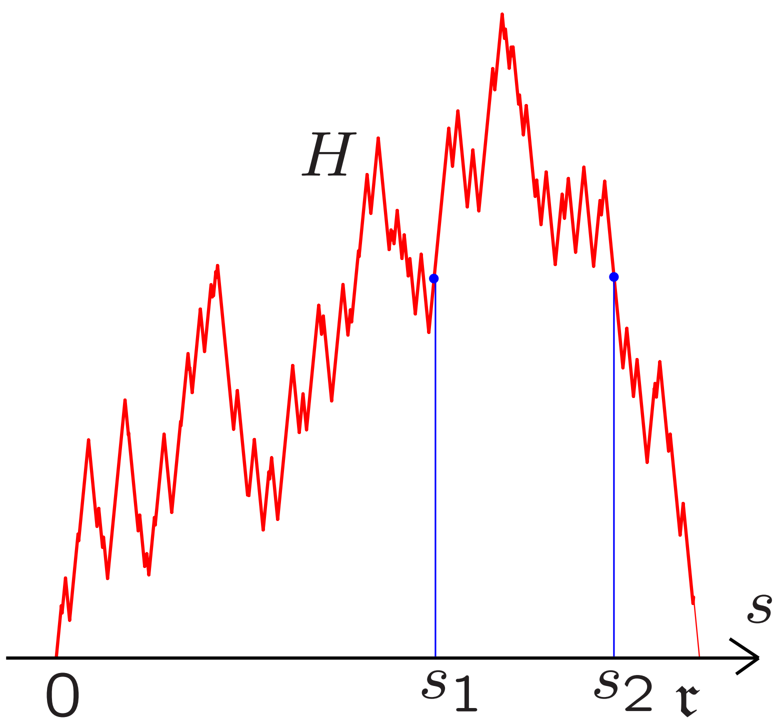

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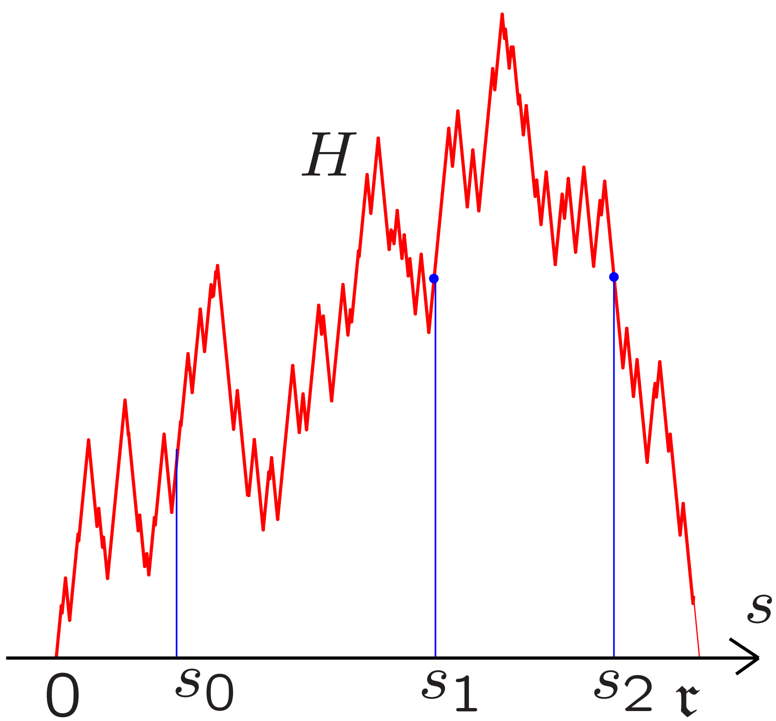

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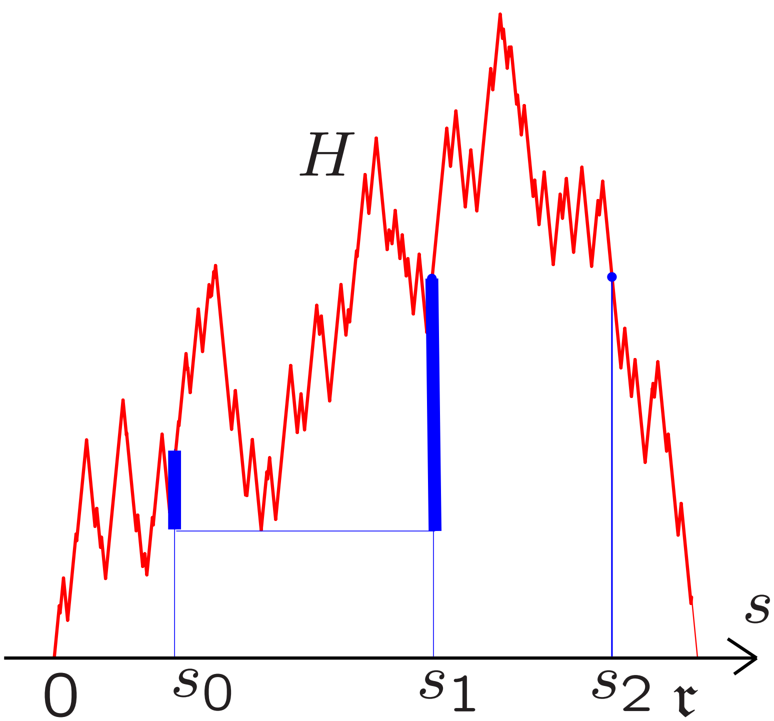

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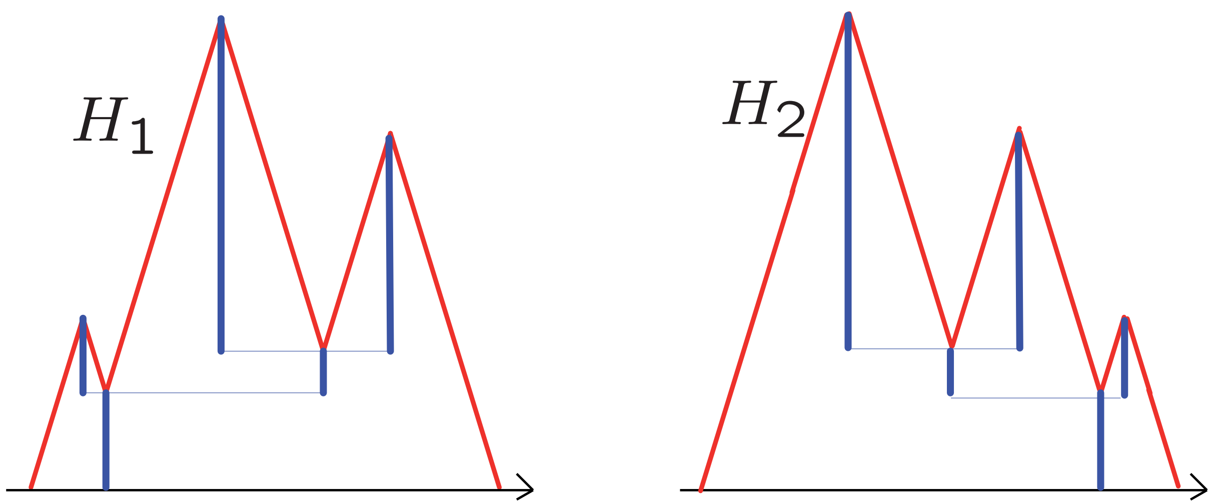
$$d(\langle s_0 \rangle, \langle s_1 \rangle) := H(s_0) + H(s_1) - 2\min\{H(s) : s \in [s_0, s_1]\}$$

The *isomorphy class* of $(T^H, d, \prec)$ will be denoted by $\mathbb{T}^H_\prec$.

The *root-preserving isometry class* of (T^H, d) will be denoted by $\mathbb{T}^H$.

Example:

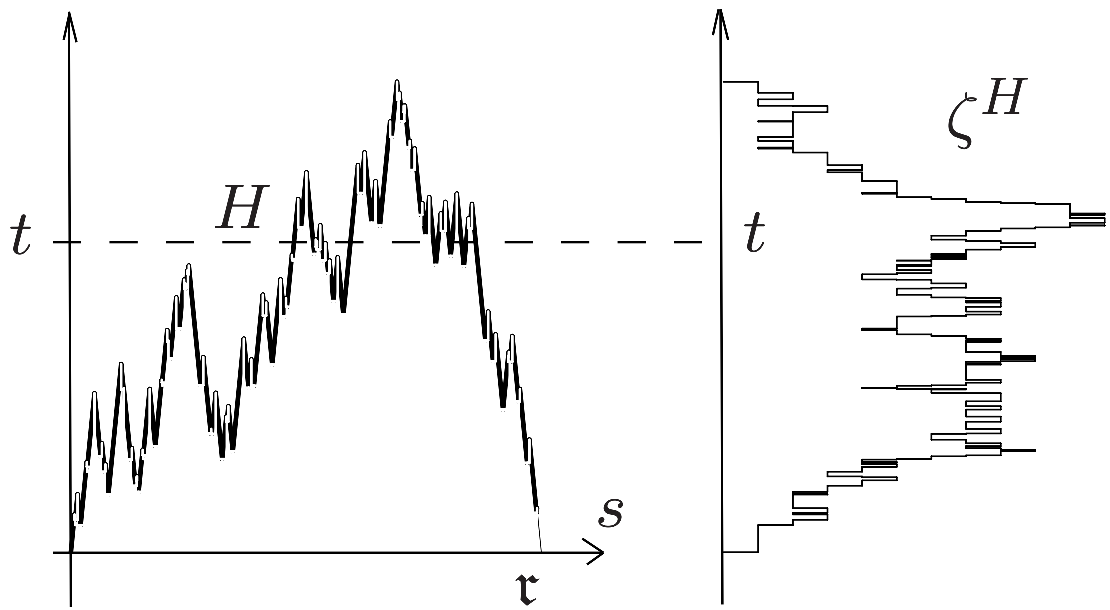


$$\mathbb{T}^{H_1}_\prec \neq \mathbb{T}^{H_2}_\prec$$

but

$$\mathbb{T}^{H_1} = \mathbb{T}^{H_2}.$$

"Counting" the number of subexcursions above height t:



$L^H(t, s)$ ... the local time accumulated by H at height t up to time s

$$\zeta_t^H := L^H(t, \mathfrak{r})$$

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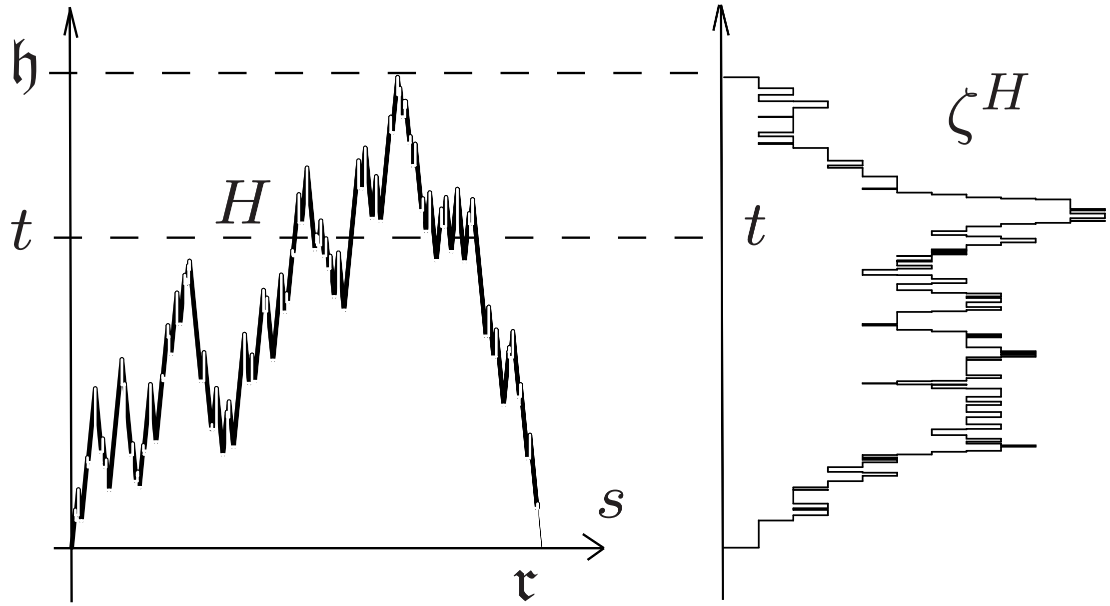

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$$\zeta_t^H := L^H(t,\mathfrak{r})$$

$$\zeta^H := \left(\zeta_t^H\right)_{0\le t\le \mathfrak{h}}$$ … the local time profile of H

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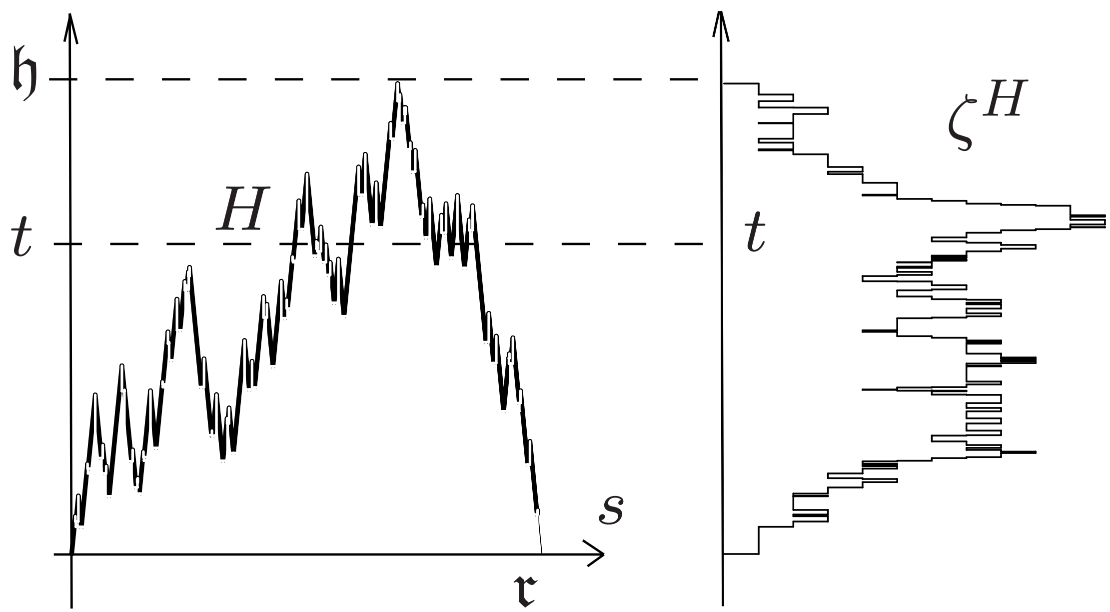
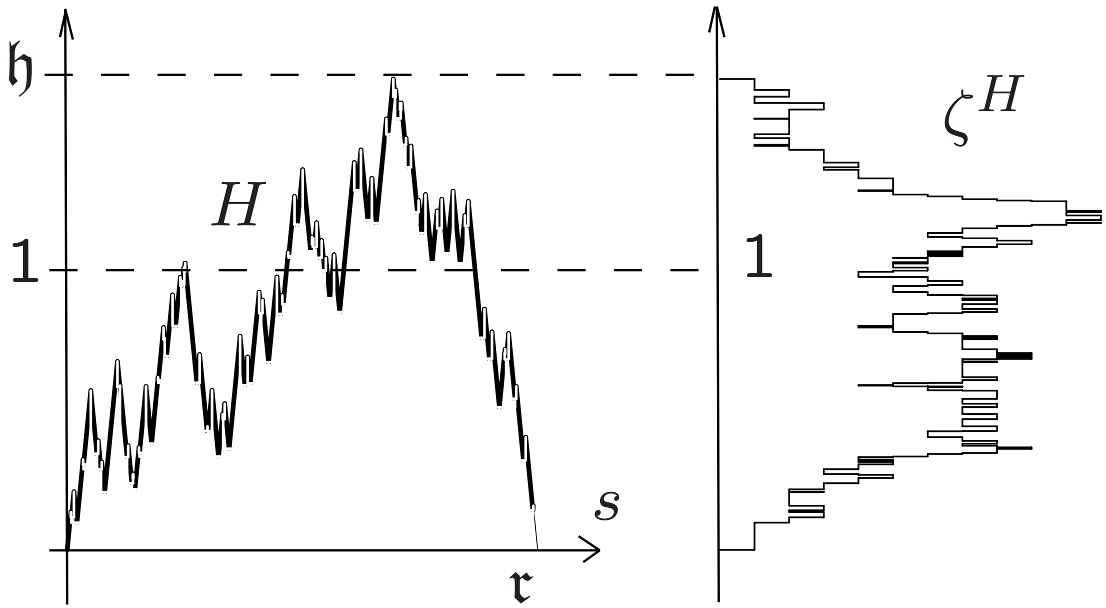


By the second Ray-Knight theorem, $H \mapsto \zeta^H$ transports the Itô excursion measure into the excursion measure of Feller’s branching diffusion $d\zeta_t = \sqrt{4\zeta_t}\, dW_t$.

We will condition on $\{\mathfrak{h} > 1\}$.



This turns the Itô excursion measure into a probability measure, under which H and ζ^H then are (path-valued) random variables.

How to go back

from ζ^H to H?

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Quote from D. Aldous (1998), *Brownian excursion conditioned on its local time*:

“Given a local time profile ζ, can we define a process whose law is, in some sense, the conditional law of H given $L(\cdot, \mathfrak{r}) = \zeta$?”

We will see that H is made up of

three independent ingredients $\zeta^H, \Lambda^H, \gamma^H$,

with

the pair $\left(\zeta^H, \Lambda^H\right)$ coding for $\mathbb{T}^H$,

and γ^H being responsible for the left-right order $\prec$.

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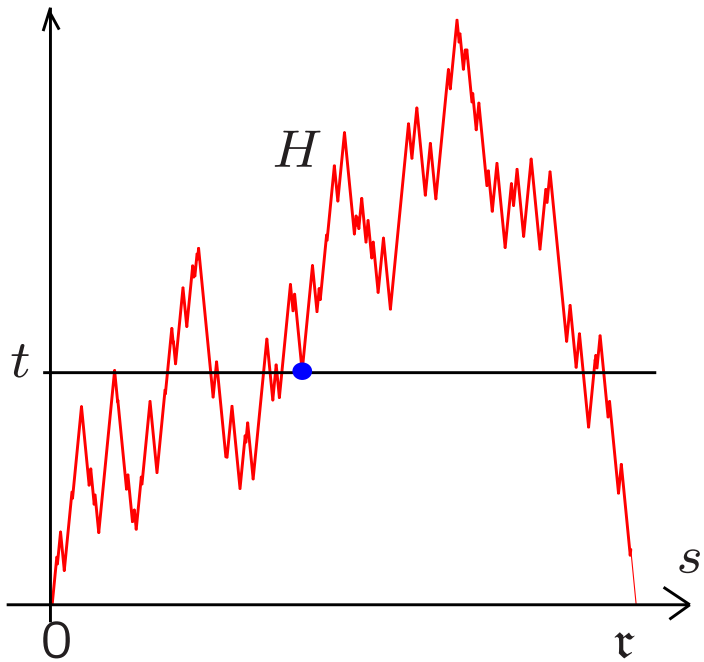
Let us now turn to the second ingredient, Λ^H.

This will be a point measure on $\mathbb{R} \times \{(i,j) : 1 \leq i < j \in \mathbb{N}\}$

whose points $(\tau, (i,j))$ are in 1-1 correspondence

with the local minima of H on $(0, \mathfrak{r})$.

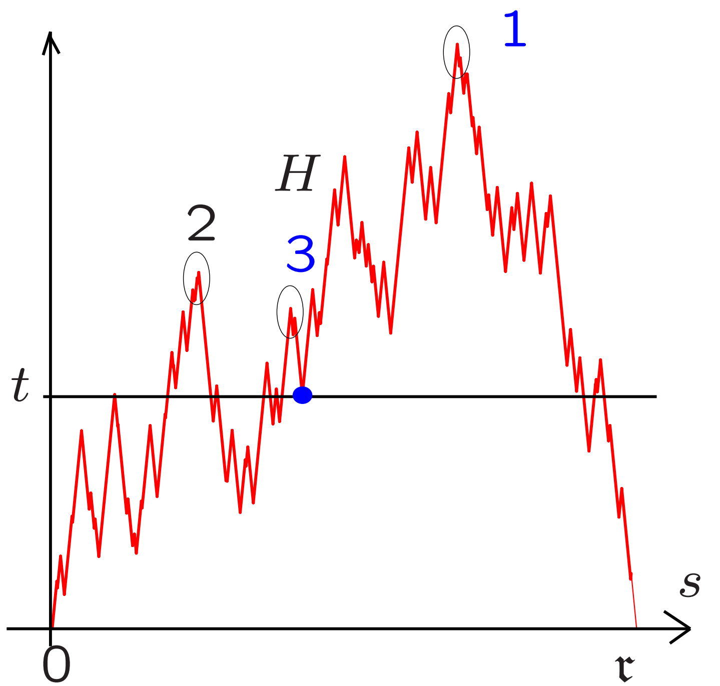
Let t be the height of a local minimum of H.

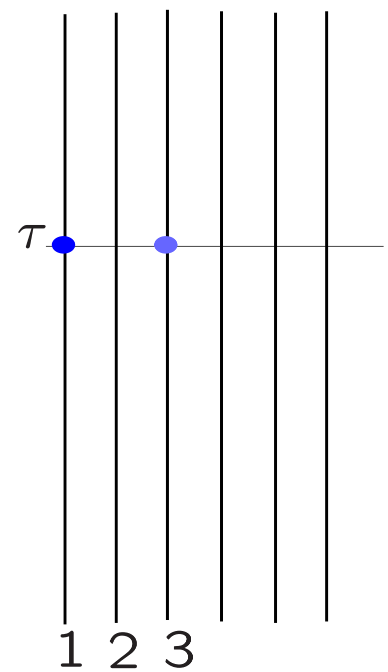


a local minimum of H at time t

Let t be the height of a local minimum of H.

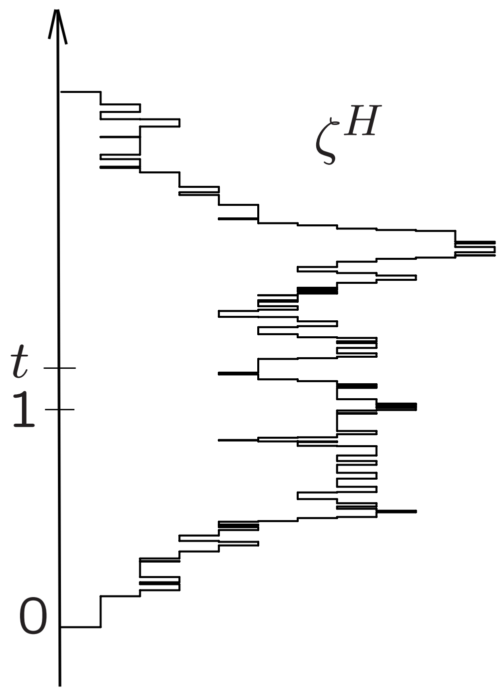
$i < j$ are the height ranks of the two subexcursions in H above t that are attached to this local minimum among all subexcursions in H above t.

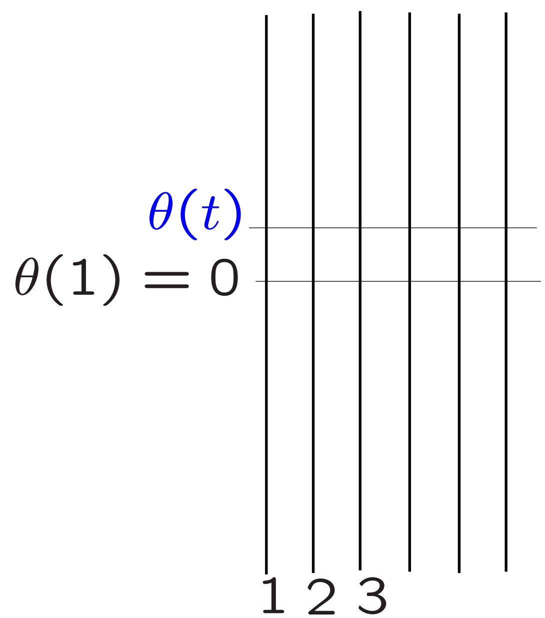




a local minimum of H at time t $\quad\leftrightarrow\quad$ a point $(\tau,(1,3))$ of Λ

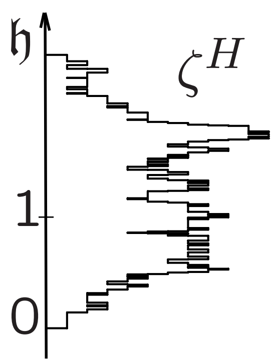
$$\tau := \theta(t) := \int_1^t \frac{4}{\zeta_u^H} du.$$

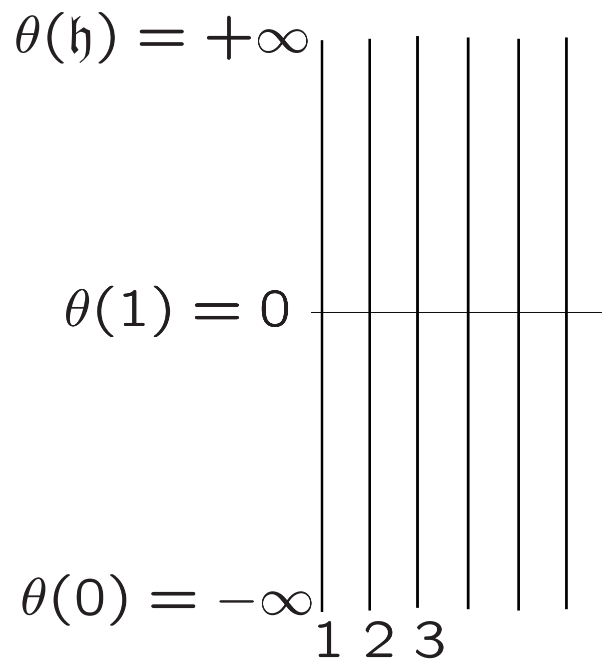




Almost surely,

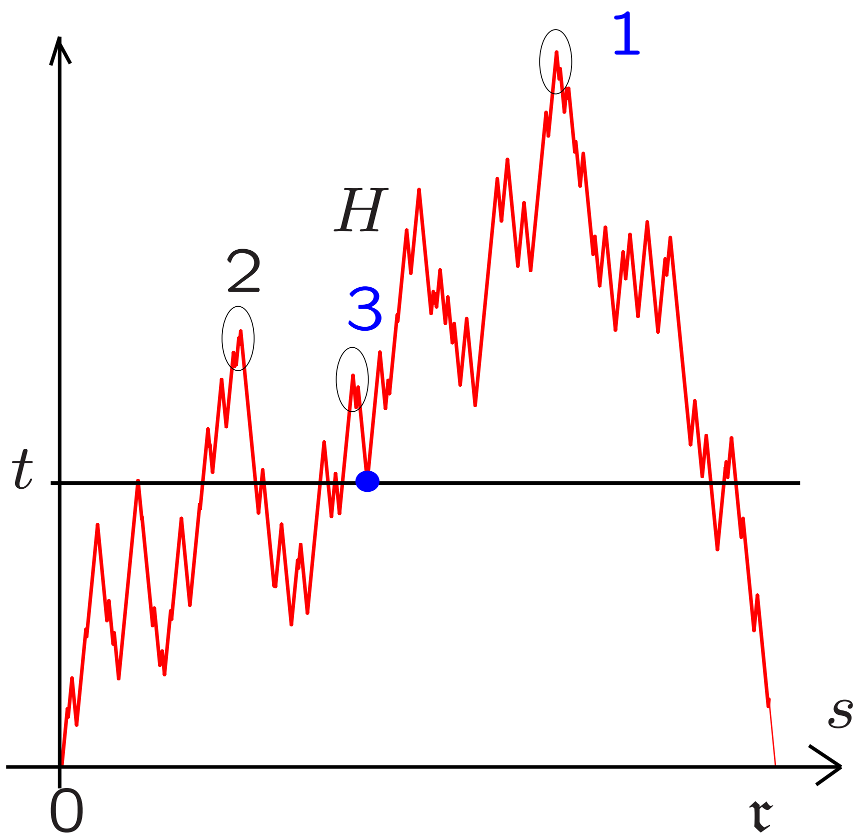
$t \mapsto \theta(t) := \int_1^t \frac{4}{\zeta_u^H}\, du$ maps $[0, \mathfrak{h}]$ bijectively to $[-\infty, +\infty]$.

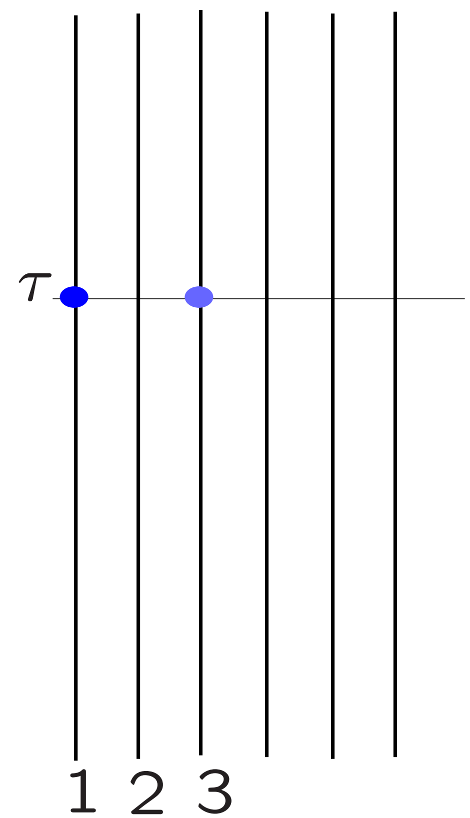




Λ^H is a random point measure on

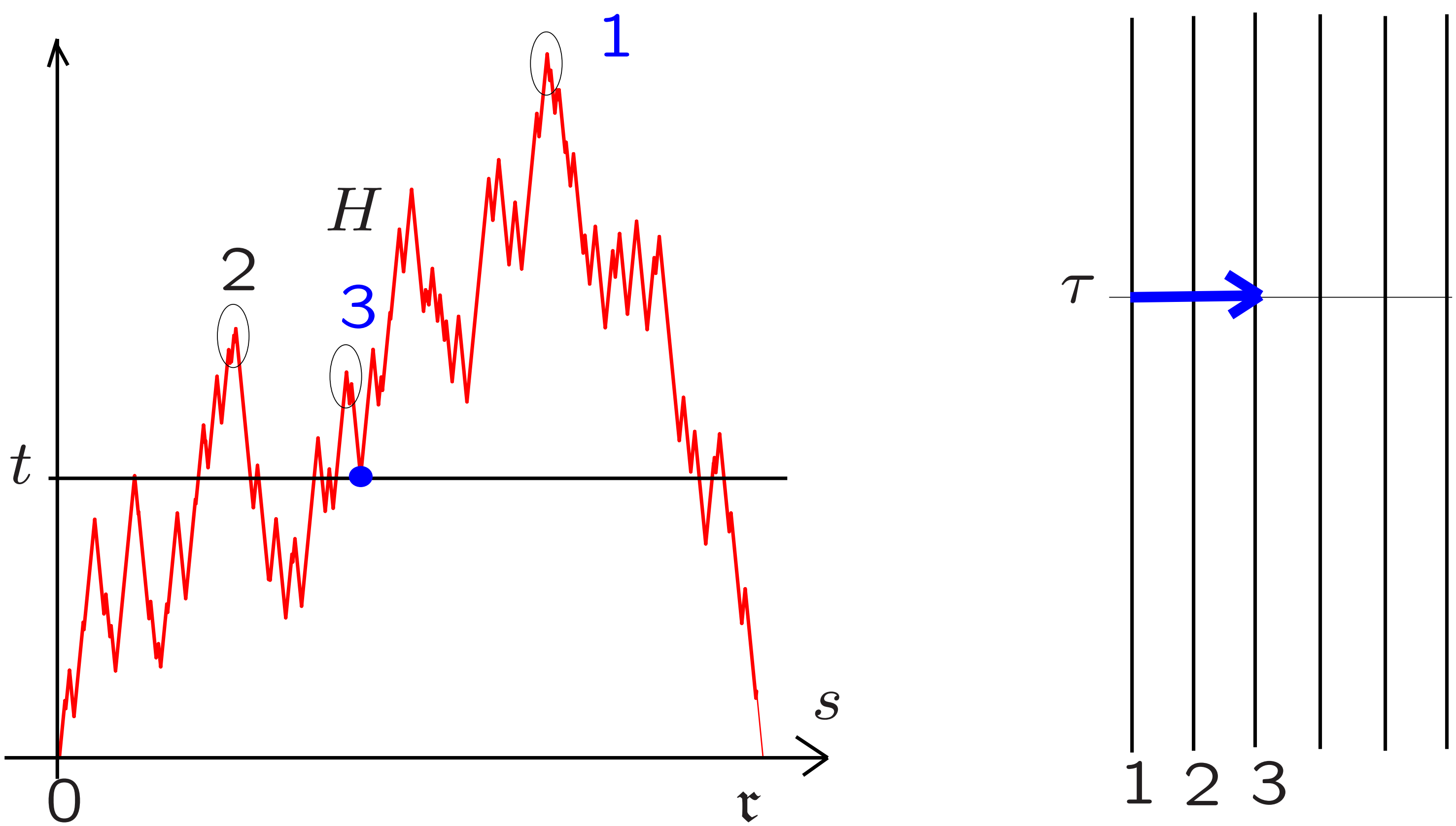
$$\mathbb{R} \times \{(i,j) : 1 \leq i < j \in \mathbb{N}\}$$





Visualize a point $(\tau, (i, j))$

by an arrow from i to j at time τ.



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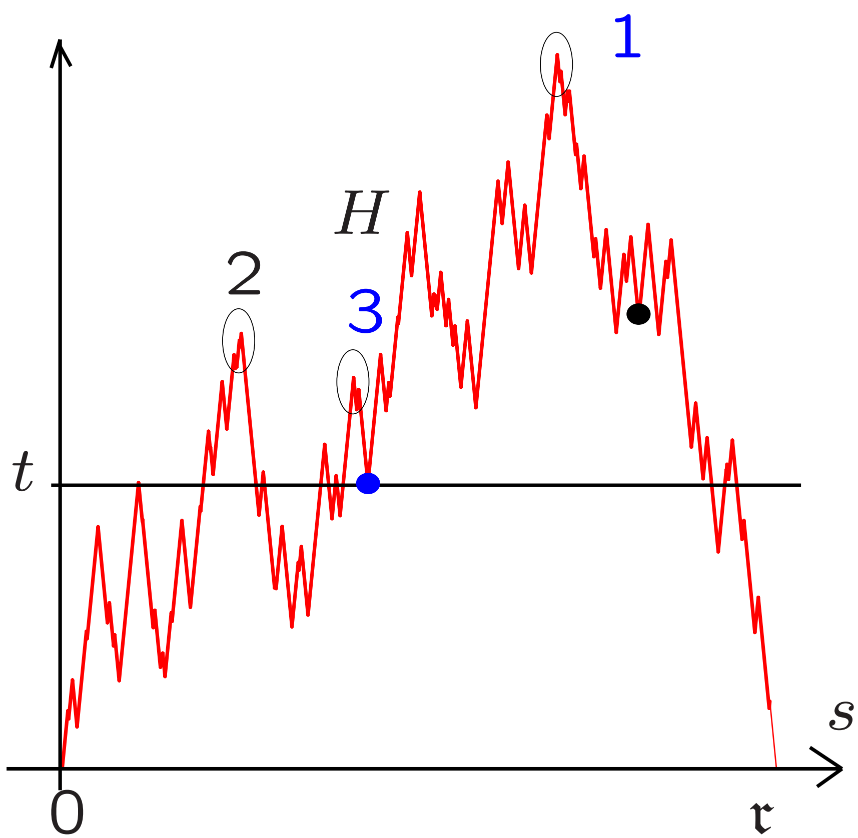
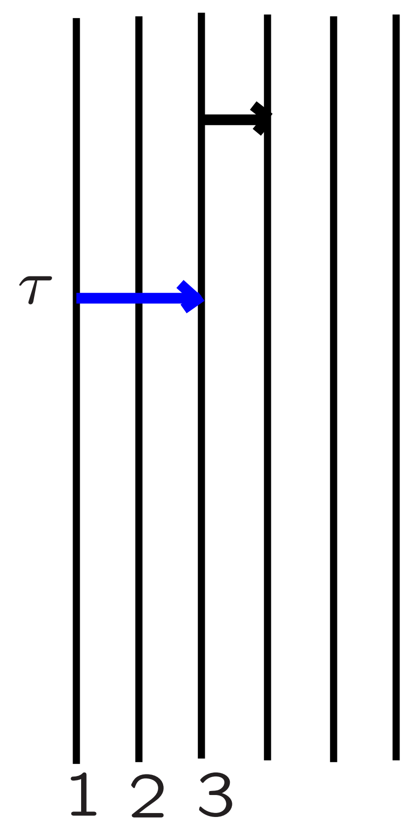
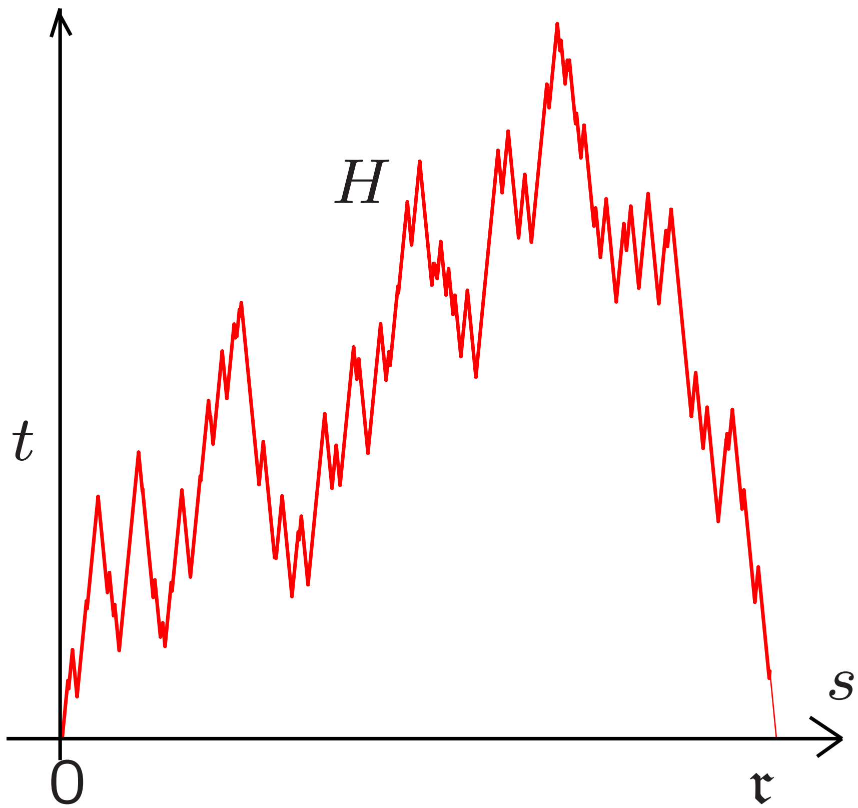
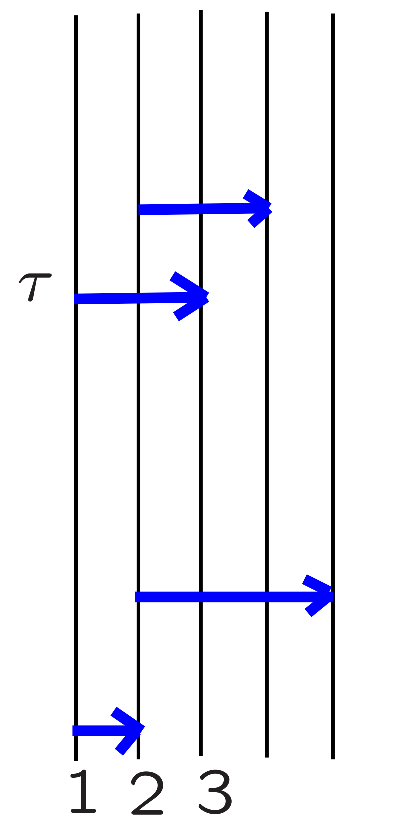




Then Λ^H becomes a random configuration of horizontal arrows on $\mathbb{R} \times \mathbb{N}$.





Theorem 1

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and properties of permutation invariance.

A precursor of this result is
J. & N. Berestycki (2009), *Kingmans coalescent and Brownian motion*.
Among others, they cite Le Gall (1989, 1993),
Aldous (1991,93,98), Warren and Yor (1998).
Gufler (2017) relates the Brownian excursion to the full lookdown picture
(between times $-\infty$ and $+\infty$) of Donnelly and Kurtz (1999).

The third ingredient $\gamma^H = \big(\gamma^H(a)\big)_{a \in \operatorname{supp} \Lambda^H}$

is a colouring of each of the points $a \in \Lambda^H$

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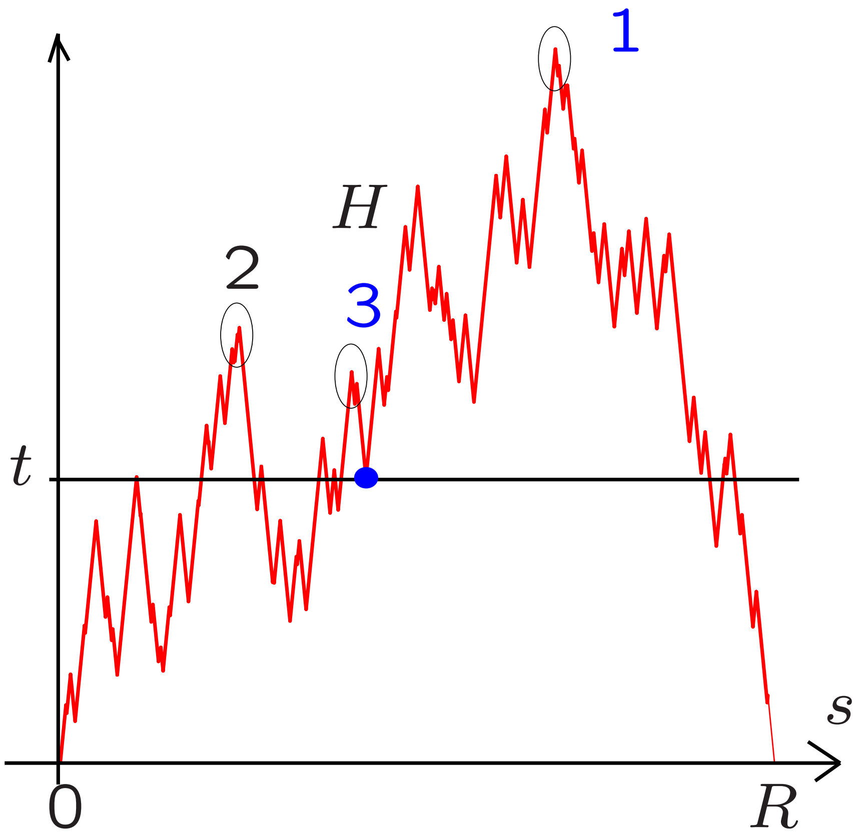
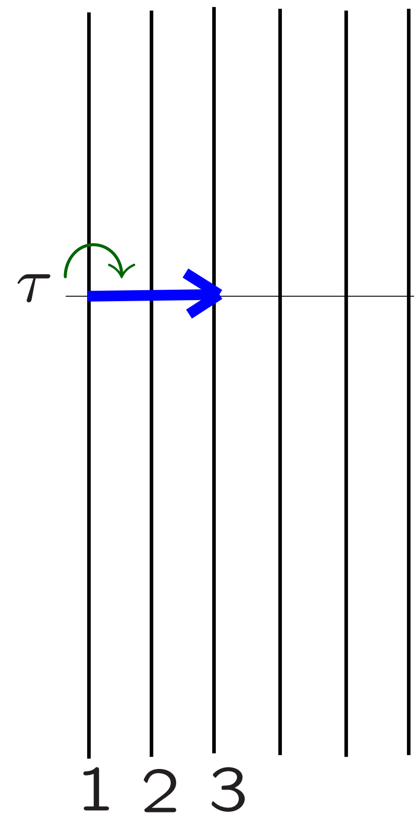
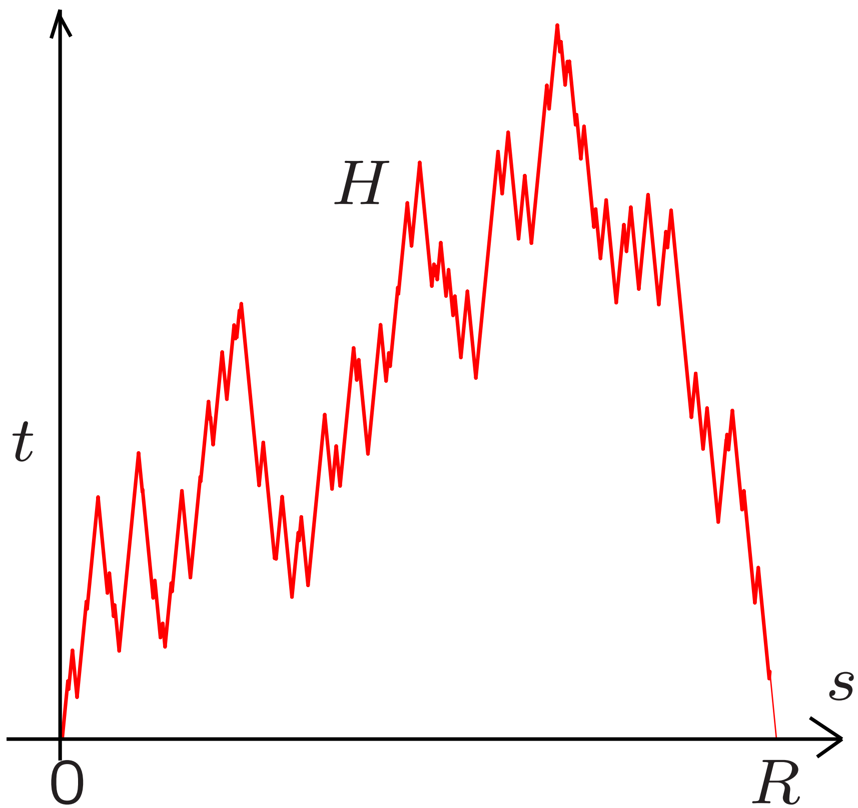


1
H
2
3
t
s
0
R



τ
1 2 3



t
H
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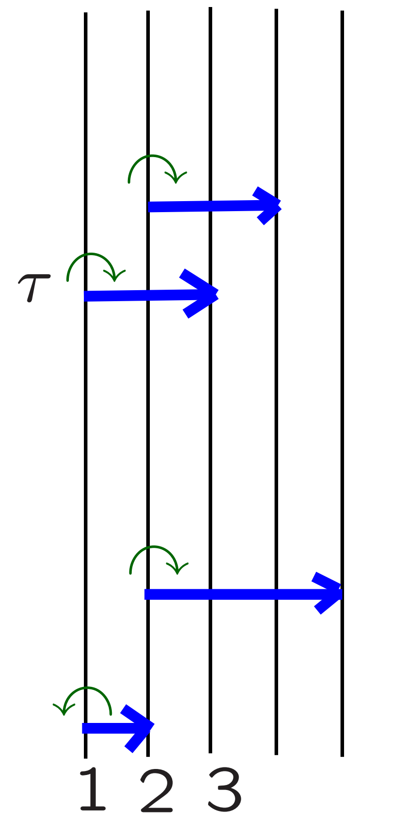
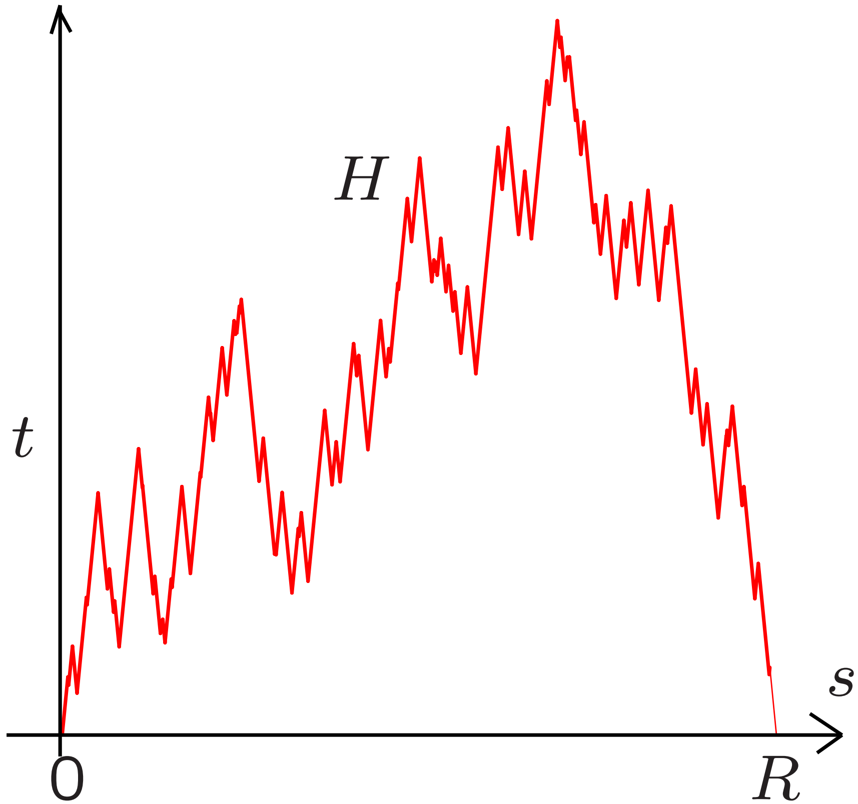
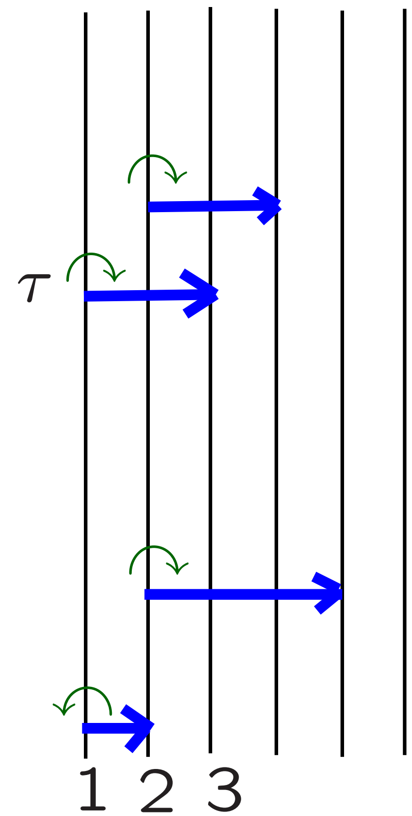


τ
1
2
3





Then $\gamma^H = \left(\gamma^H(a)\right)_{a \in \text{supp}\,\Lambda^H}$ is a fair coin tossing array.

How to reconstruct the (exploration) path H

from the triple (ζ, Λ, γ)?

First step: Obtaining from Λ

a complete metric space $(Z^{\Lambda}, \rho^{\Lambda})$,

the lookdown space.

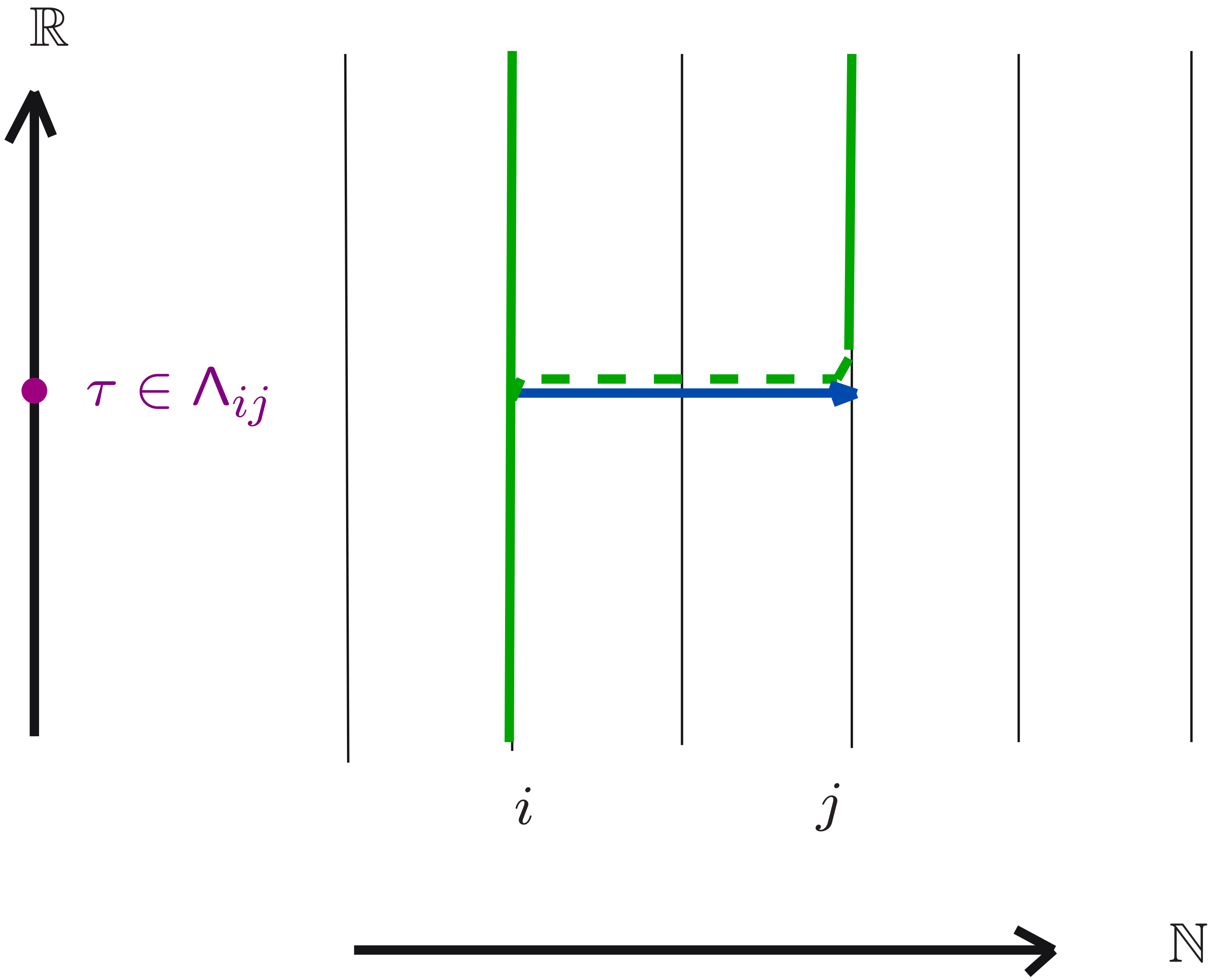
The lookdown space obtained from Λ:

Let Λ_{ij}, $1 \le i < j$,

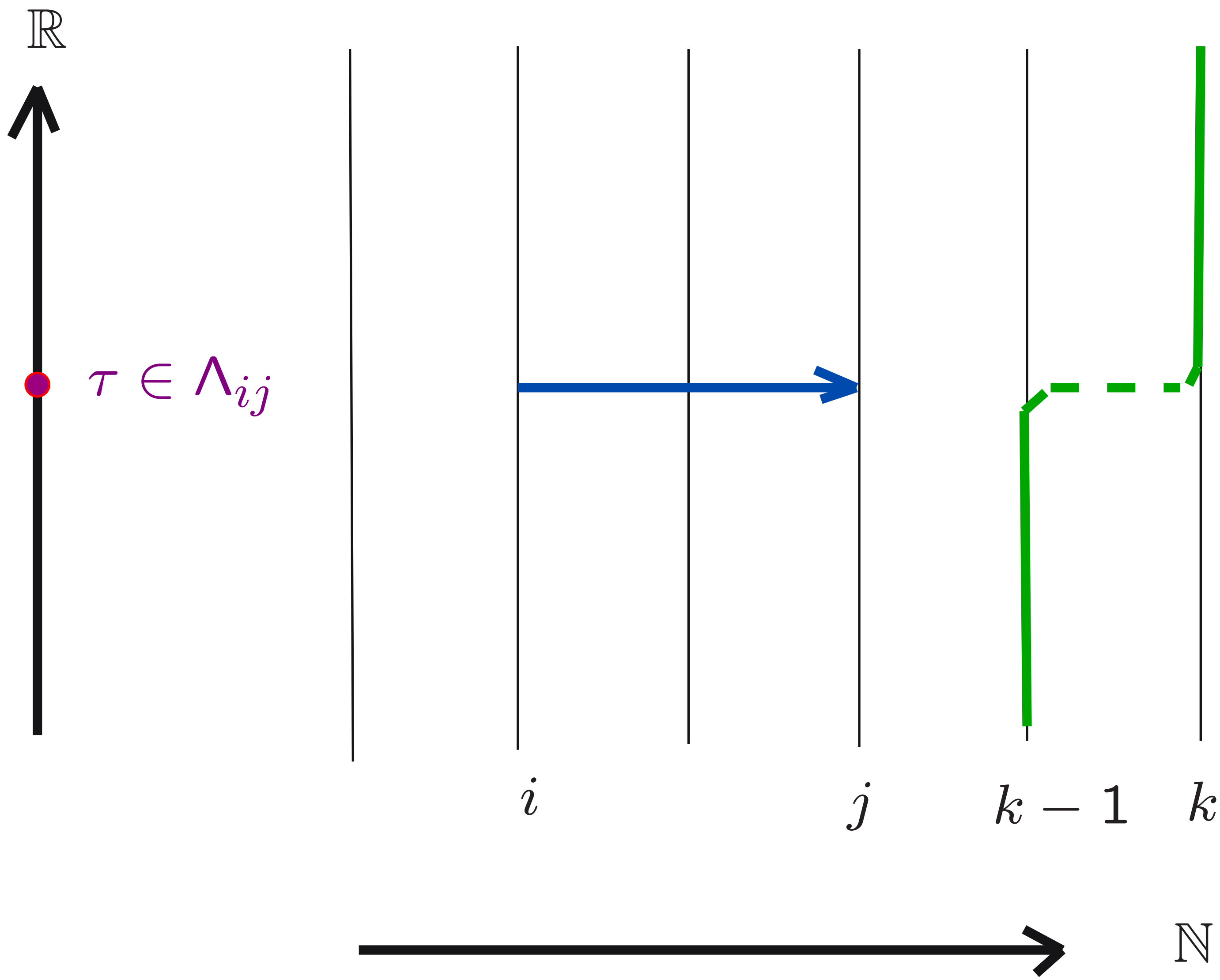
be independent rate 1 Poisson point processes.

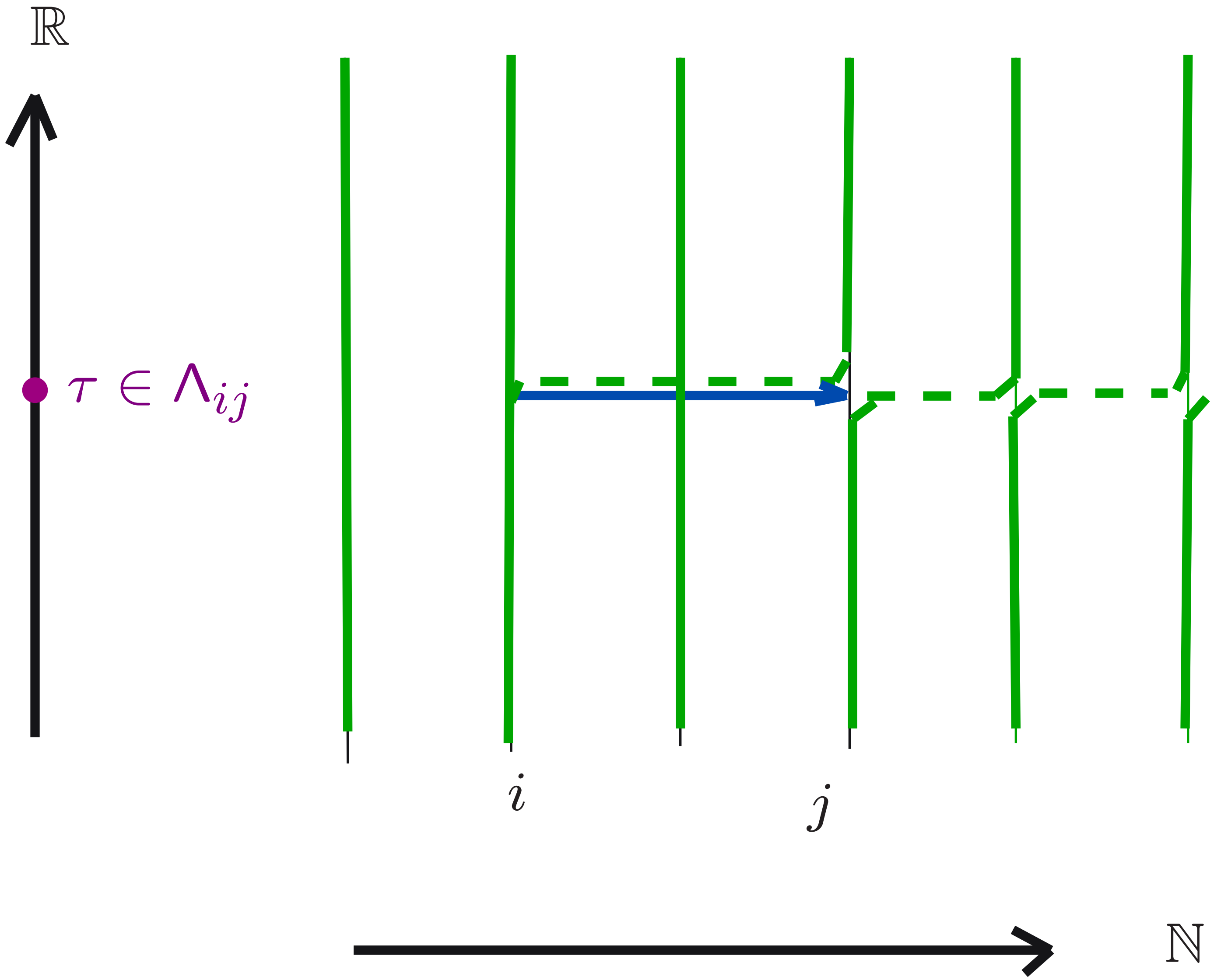
$\Lambda = (\Lambda_{ij})$ induces (random) geodesics on $\mathbb{N} \times \mathbb{R}$

via coalescent ancestral lineages



$\mathbb{R}$
$\tau \in \Lambda_{ij}$
i
j
$\mathbb{N}$


ℝ
τ ∈ Λ_ij
i
j
k − 1
k
ℕ



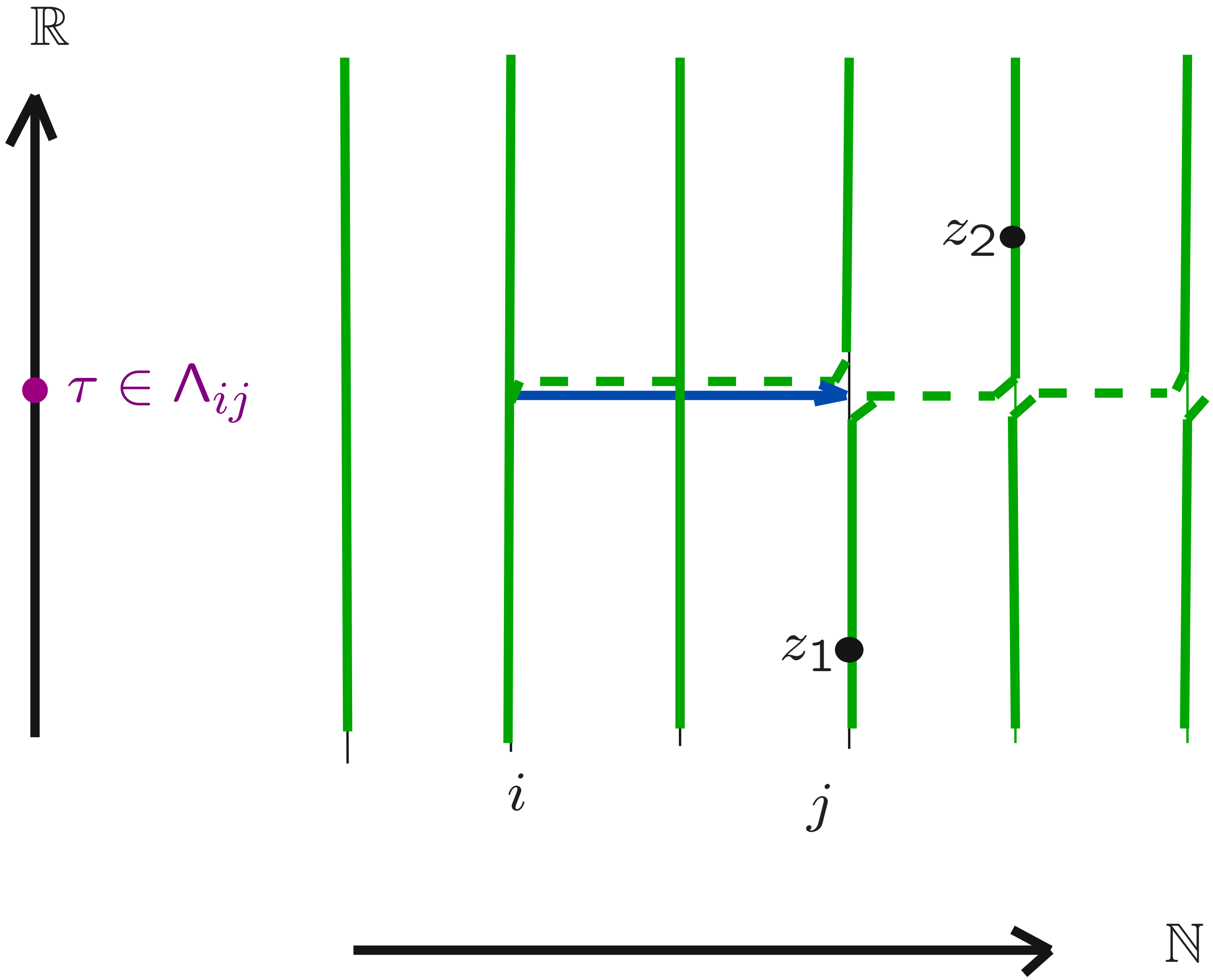
ℝ
$\tau \in \Lambda_{ij}$
i
j
ℕ

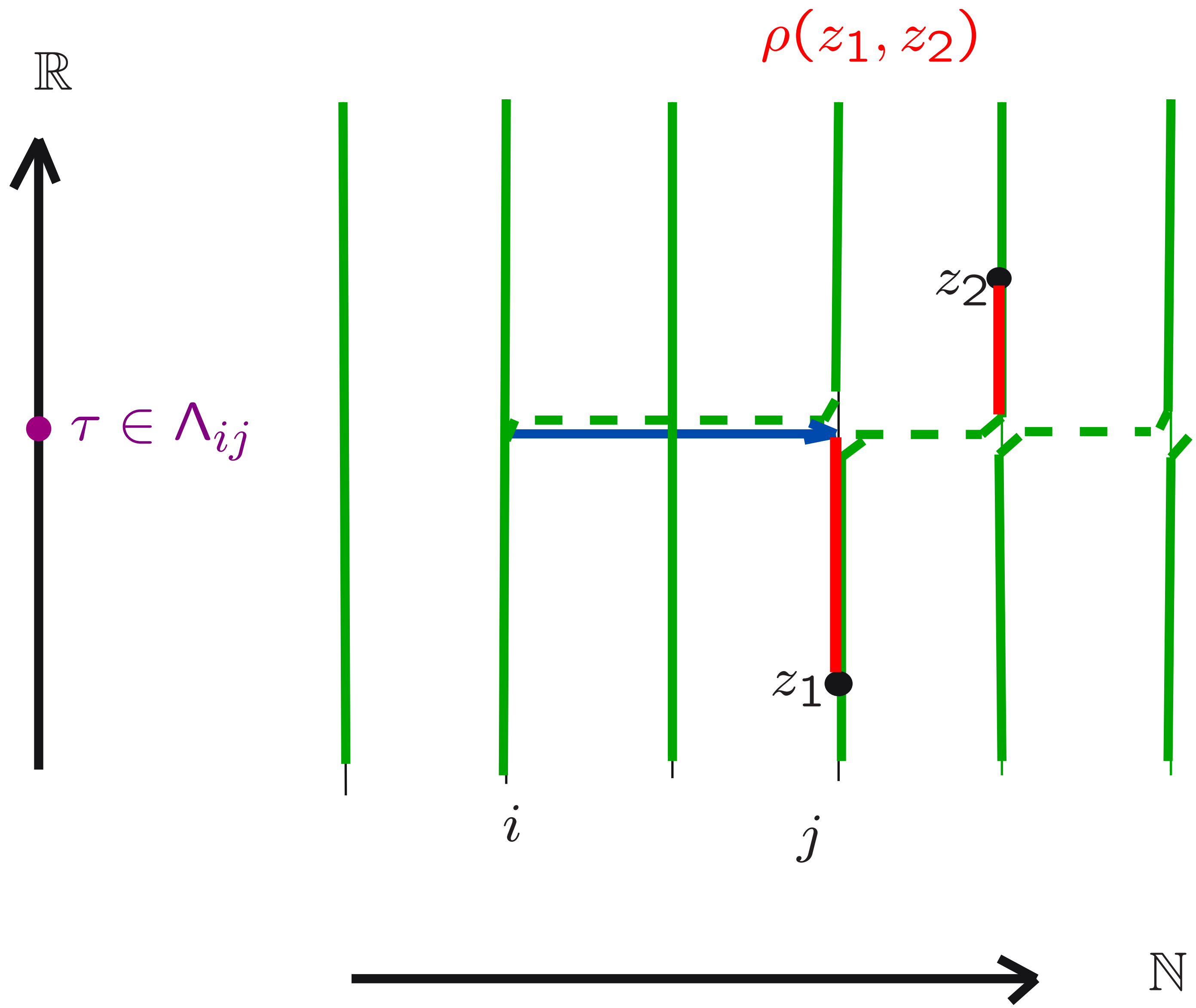
Let Λ_{ij}, $1 \leq i < j$,

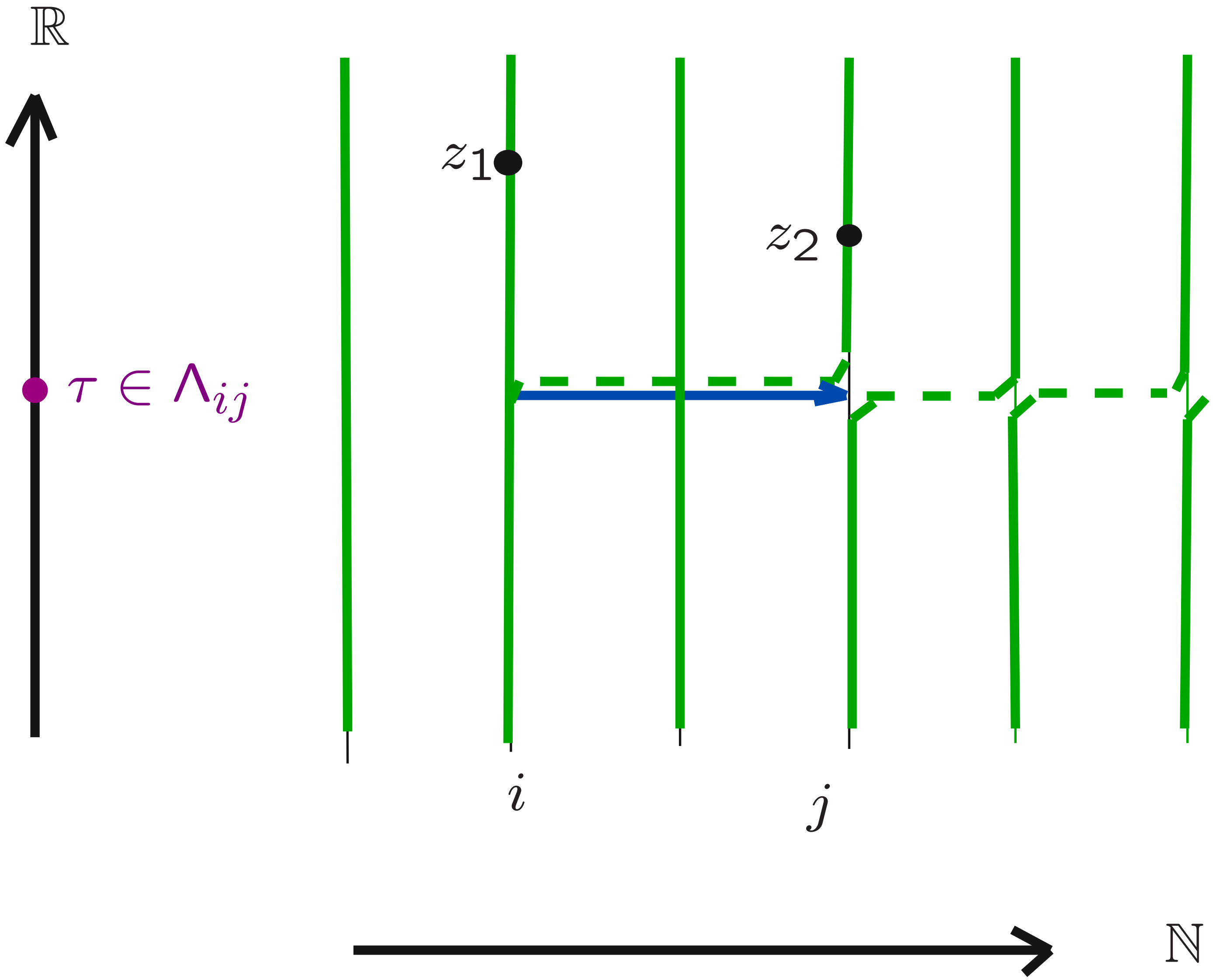
be independent rate 1 Poisson point processes.

$\Lambda = (\Lambda_{ij})$ induces a (random) semi-metric $\rho = \rho^{\Lambda}$ on $\mathbb{N} \times \mathbb{R}$

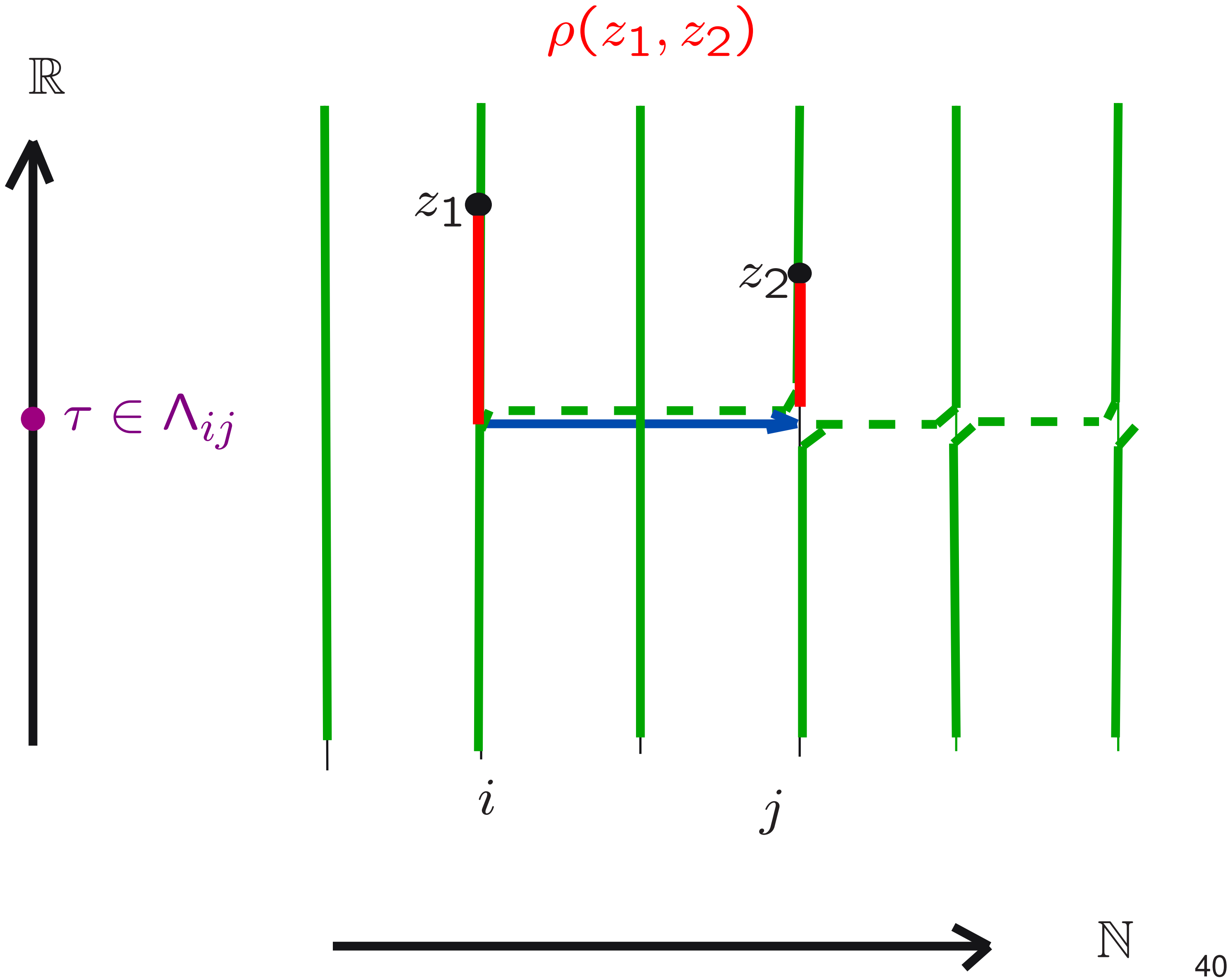
via vertical distances along the geodesics


ℝ
$\tau \in \Lambda_{ij}$
z_2
z_1
i
j
ℕ


$\rho(z_1, z_2)$
$\mathbb{R}$
$\tau \in \Lambda_{ij}$
z_2
z_1
i
j
$\mathbb{N}$



$\mathbb{R}$
z_1
z_2
$\tau \in \Lambda_{ij}$
i
j
$\mathbb{N}$



$\rho(z_1, z_2)$
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z_1
z_2
$\tau \in \Lambda_{ij}$
i
j
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and called the (random) lookdown space.

For $z = (\tau, i)$ we define $\tau(z) := \tau$

as the *height* of z

and extend this by continuity to Z.

(Z, ρ) is a (random) non-compact $\mathbb{R}$-tree,

and can be compactified to $\bar{Z} := Z \cup \{z_{\text{root}}, z_{\text{top}}\}$,

where we say that

$$z_n \to z_{\text{root}} \quad \text{if } \tau(z_n) \to -\infty =: \tau(z_{\text{root}}),$$

$$z_n \to z_{\text{top}} \quad \text{if } \tau(z_n) \to +\infty =: \tau(z_{\text{top}}).$$

Our program in this part of the talk
is to reconstruct H from (ζ, Λ, γ)

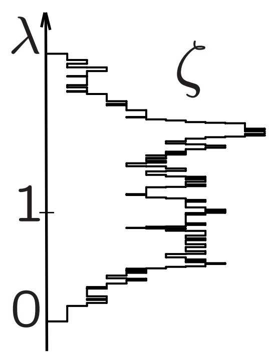
So far, we only worked in the Λ-world:

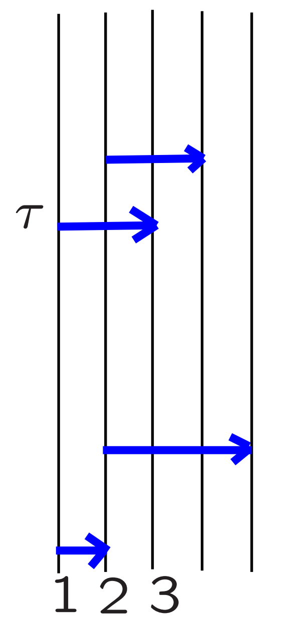
using Λ, we metrized and completed the set $\mathbb{N} \times \mathbb{R}$,
thus obtaining the semi-metric $\rho = \rho^{\Lambda}$.

Now we bring in the local time profile ζ,
in order to revert the “height change” $t \to \theta(t)$
by its inverse $t(\tau) := \theta^{-1}(\tau)$.

For given ζ and Λ, we define the semi-metric ρ_ζ on $\mathbb{N} \times \mathbb{R}$ by stretching ρ locally with the factor $\frac{1}{4}\zeta_{t(\tau)}$:

$$\rho_\zeta((i,\tau),(j,\tau+d\tau)) := \frac{1}{4}\zeta_{t(\tau))}\,\rho((i,\tau),(j,\tau+d\tau))$$

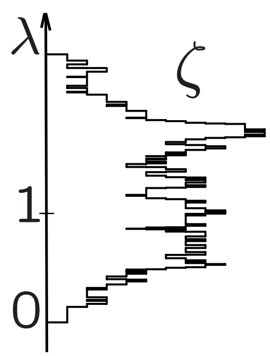


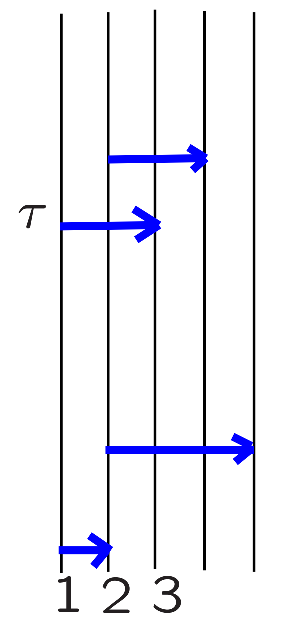


For given ζ and Λ, we define the semi-metric ρ_ζ on $\mathbb{N} \times \mathbb{R}$

by stretching ρ locally with the factor $\frac{1}{4}\zeta_{t(\tau)}$:

and extend this to a metric ρ_ζ on $\bar{Z}$.





Proposition 1:

For $\Lambda := \Lambda^H$ and $\zeta := \zeta^H$,

(T^H, d) and $\left(\bar{Z}, \rho_\zeta\right)$ are a.s. root-preserving isometric.

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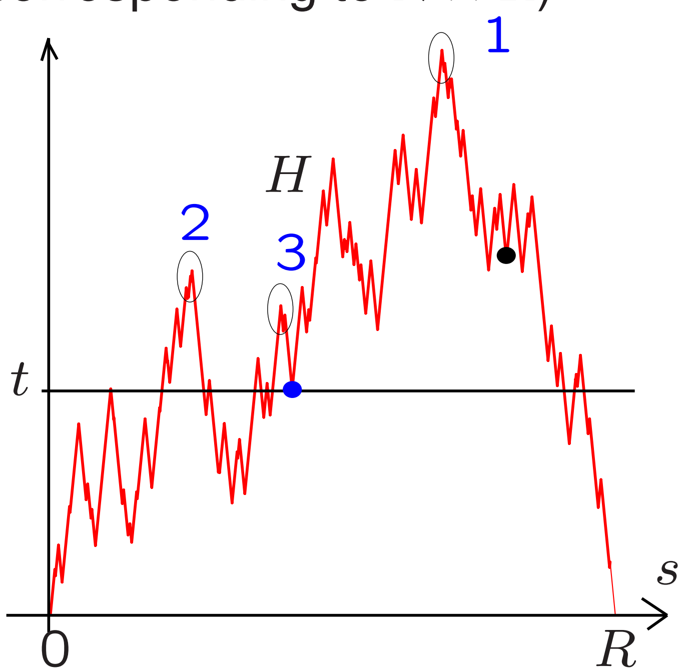
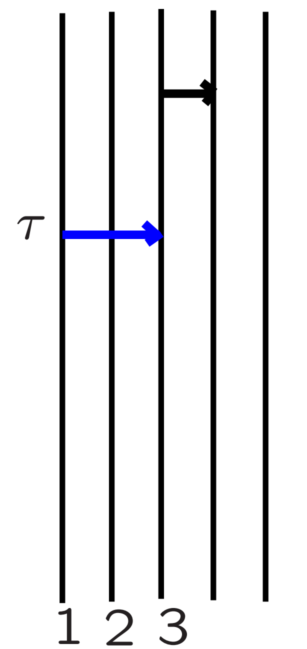


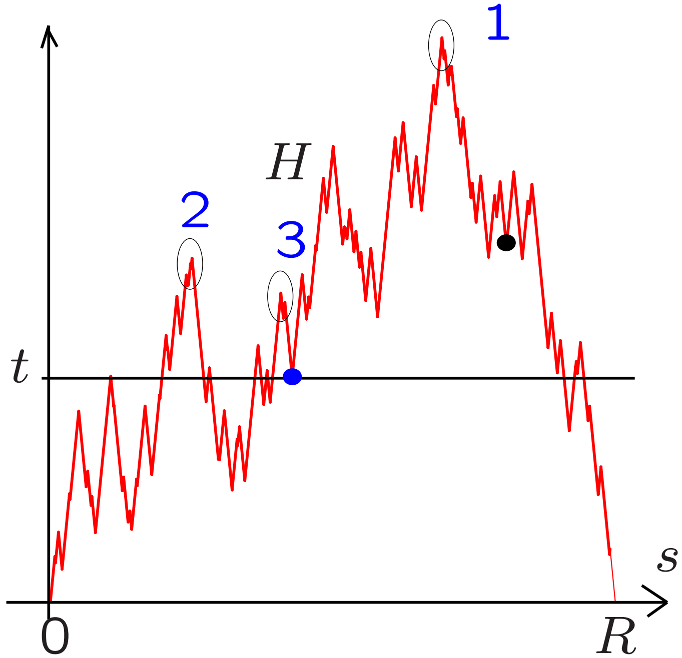


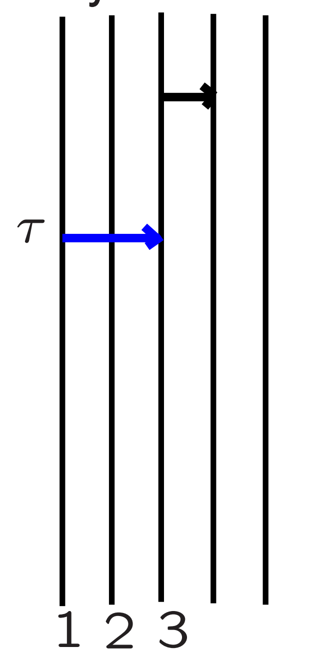
Proposition:

For $\Lambda := \Lambda^H$ and $\zeta := \zeta^H$,

(T^H, d) and $\left(\bar{Z}, \rho_\zeta\right)$ are a.s. root-preserving isometric.

Idea of proof: First show the isometry for the "skeleton points" (corresponding to $\mathbb{N} \times \mathbb{R}$), then proceed by continuity.





With the standing aim to reconstruct H from (ζ, Λ, γ),

we now proceed further to define

(the height process of) an exploration of Z^Λ.

To this purpose we endow Z^Λ with a

measure $\mu_\tau(dz)\, d\tau$

(which will help us to specify

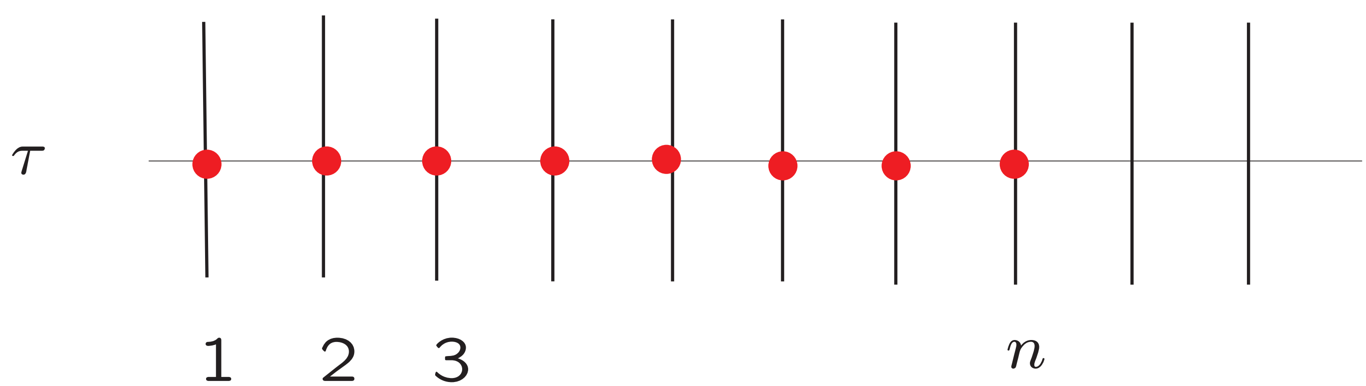
how much mass we have explored by which time).

Theorem 2 (S. Gufler, EJP, 2018)

For a.a. Λ the lookdown space $(Z^\Lambda, \rho^\Lambda)$ carries a family $(\mu_\tau)_{\tau\in\mathbb{R}}$ of probability measures such that for all $\tau \in \mathbb{R}$,

$$\mu_\tau = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{(i,\tau)},$$

in the weak topology on the probab. measures on $(Z^\Lambda, \rho^\Lambda)$.



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Elegant way of proof: By Theorem 1, embed the lookdown space into a Brownian excursion H and prove the assertion for $(Z^{\Lambda^H}, \rho^{\Lambda^H})$. The latter is achieved via the *uniform downcrossing representation for local times* due to Chacon, Le Jan, Perkins and Taylor (1981).

To prepare for an exploration process of Z^Λ:

Endowing Z^Λ with a total order $\prec$,
using the $\{\curvearrowleft, \curvearrowright\}$-valued array γ

Let $(Z, \rho) = (Z^\Lambda, \rho^\Lambda)$ be a lookdown space, and γ be a $\{\curvearrowleft, \curvearrowright\}$-valued array, indexed by the points of Λ.

Using γ we define a total order $\prec$ on Z as follows:

For $y, z \in Z$ connected by a single line of descent with z descending from y, we put $y \prec z$.

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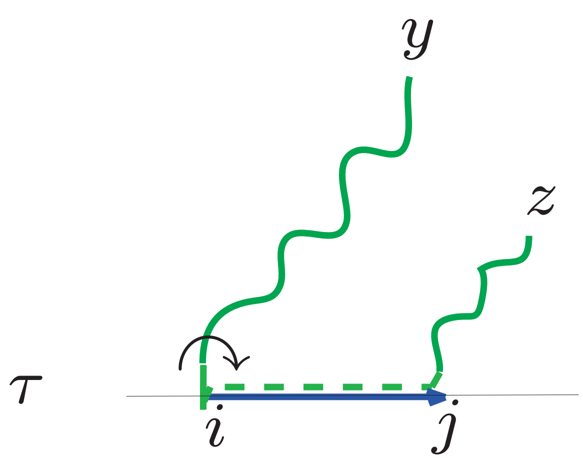
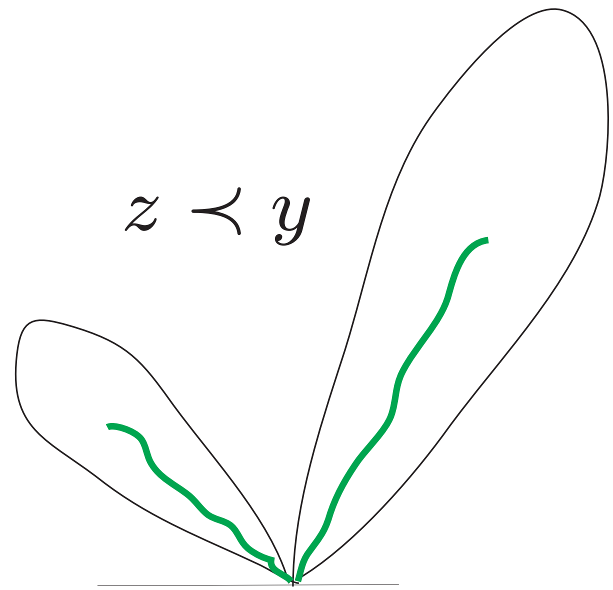
For $y, z \in Z$ not connected by a single line of descent, their most recent common ancestor is of the form (τ, i) for some $a = (\tau, (i, j)) \in \operatorname{supp} \Lambda$.

Assume that z descends from (τ, j).

We then put $z \prec y$ if $\gamma(a) = \curvearrowright$





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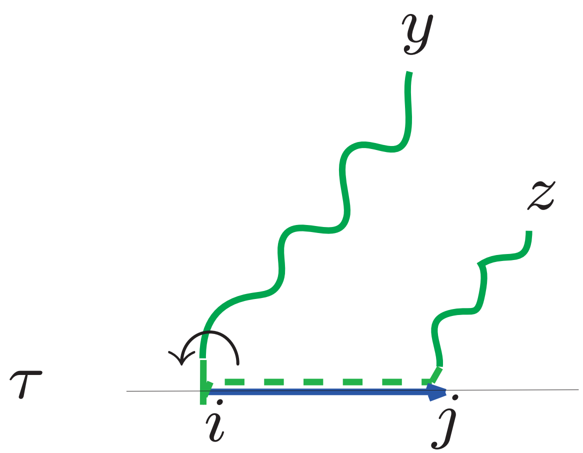

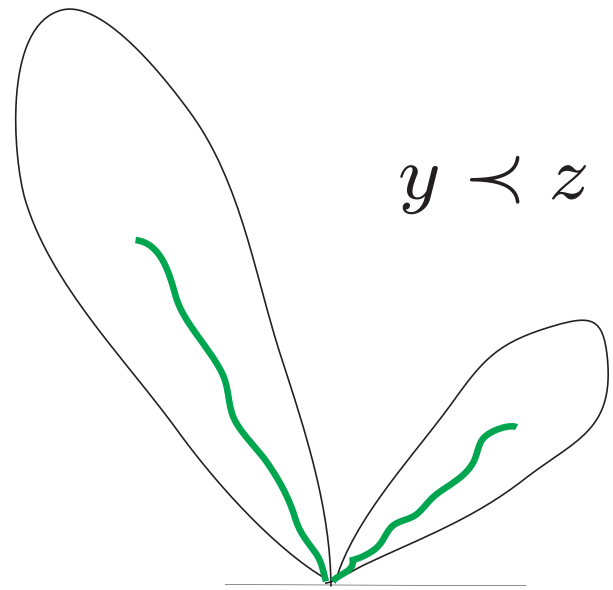

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A (standardized) exploration of $Z := Z^{\Lambda}$ using (the order $\prec$ induced by) γ:

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We think of an exploration starting at time $-\infty$ in z_{root}, arriving at time 0 in z_0, and ending at time $+\infty$ in z_{top}, with $\mathfrak{s}(z) :=$ the *time of the first exploration of* z.

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$$\hat{\mathfrak{s}}(z_{\text{root}}) := -\infty, \quad \hat{\mathfrak{s}}(z_0) := 0, \quad \hat{\mathfrak{s}}(z_{\text{top}}) := +\infty.$$

For $z_{\text{root}} \prec z \prec z' \prec z_{\text{top}}$, we decree that the time difference between the first explorations of z and z' is

$$\hat{\mathfrak{s}}(z') - \hat{\mathfrak{s}}(z) := \int_{-\infty}^{\infty} \mu_\tau(\{y : z \prec y \prec z'\})\, d\tau.$$

Altogether, for $z \in \bar{Z}_{\text{left}}$ this leads to

$$\hat{\mathfrak{s}}(z) := \int_{-\infty}^{\infty} \mu_\tau\{y : z_0 \prec y \prec z\} d\tau - \int_{-\infty}^{\infty} \mu_\tau\{y : z \prec y \prec z_0\} d\tau.$$

$\hat{\mathfrak{s}} : \bar{Z}_{\text{left}} \to [-\infty, +\infty]$ is strictly increasing (w. r. to $\prec$ and $<$) and its image is dense in $[\infty, +\infty]$.

For $s \in \widehat{\mathfrak{s}}(\bar{Z}_{\text{left}}) \subset [-\infty, +\infty]$, define

$\widehat{\mathfrak{z}}(s) := \widehat{\mathfrak{s}}^{-1}(s)$,

the individual whose time of first exploration is s.

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We now define

$\widehat{H}_s := \tau(\hat{\mathfrak{z}}(s)), \quad s \in [-\infty, +\infty]$,

the *height process of the exploration* $\hat{\mathfrak{z}}$ of $\bar{Z}_{\mathsf{left}}$

(the standardized exploration of $\bar{Z}_{\mathsf{left}}$ using γ)

This relates to a detective story:

Jonathan Warren and Marc Yor (1998),

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Quote from that paper's preamble:

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The latter correspond to to the local time profile ζ.

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This is precisely what we have achieved
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and what Warren and Yor had achieved
by completely different techniques...

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This suggests $dA_s := \frac{4}{\zeta^2_{H_s}} ds$ as an appropriate time change

between the two exploration processes (of Z and T^H).

More precisely, for

$$s_1 := \inf\{s > 0 : H_s = 1\}, \quad s_{\text{top}} := \text{argmax} H$$

we put $A_s := \int_{s_1}^{s} \frac{4}{\zeta_{H_u}^2} du$, $\quad 0 \leq s \leq s_{\text{top}}$.

In addition, we have our familiar "height change"

$$d\theta(t) = \tfrac{4}{\zeta_t} dt, \quad \theta(1) = 0.$$

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$$d\theta(t) = \frac{4}{\zeta_t} dt, \quad \theta(1) = 0.$$

Theorem 3:

For a Brownian excursion H,

with $\zeta := \zeta^H, \Lambda := \Lambda^H, \gamma := \gamma^H$ we have

$$H_s = \theta^{-1}(\widehat{H}_{A_s}), \quad 0 \le s \le s_{\mathrm{top}}.$$

In Warren&Yor's situation,

H ... Brownian motion started in 0 and reflected above 0,

T_1 ... time when H first reaches height 1,

$\zeta_t := L^H(t, T_1), \;\; t \geq 0.$

They put $\quad \theta(t) := \int_0^t \frac{1}{\zeta_u} du, \quad A_s := \int_0^s \frac{1}{\zeta_{H_u}^2} du$

and define the **Brownian burglar** $\widehat{H} = (\widehat{H}_s)_{0 \leq s < \infty}$ by

$$\theta(H_s) = \widehat{H}_{A_s}, \quad 0 \leq s \leq T_1.$$

Their main result is that $\widehat{H}$ is independent of ζ.

A (standardized) exploration of $Z := Z^{\Lambda}$ using γ and ζ

If we adjust the exploration speed of $\bar{Z}$ right away to the local time profile ζ, then we can reconstuct H directly from (ζ, Λ, γ), without the detour via the burglar $\widehat{H}$:

Define the ζ-profiled *time of the first exploration of* $z \in \bar{Z}$ by

$$\mathfrak{s}(z) := \int_{-\infty}^{\infty} \mu_\tau(\{y : y \prec z\}) \frac{\zeta^2_{t(\tau)}}{4} d\tau \,, \quad z \in Z,$$

$\mathfrak{s}(z_{\mathrm{root}}) := 0, \quad \mathfrak{s}(z_{\mathrm{top}}) := \lim_{z \to z_{\mathrm{top}}} \mathfrak{s}(z).$

For $s \in \mathfrak{s}(\bar{Z}) \subset [0, \mathfrak{r}]$, let $\mathfrak{z}(s)$ be

the individual whose ζ-profiled time of first exploration is s,

and extend $\mathfrak{z}$ by continuity to $[0, \mathfrak{r}]$.

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For a Brownian excursion H,

with $\Lambda := \Lambda^H$, $\zeta := \zeta^H$, $\gamma := \gamma^H$ we have

$\theta(H_s) = \tau(\mathfrak{z}(s)), \quad 0 \leq s \leq \mathfrak{r}$.

Corollary: The mapping $z \mapsto \langle \mathfrak{s}(z) \rangle$ is a root-, order- and measure-preserving isometry from $(\bar{Z}^{\wedge}, \rho_\zeta, \prec)$ to $(T^H, d, \prec)$.

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The correspondence between the sampling measures $\mu_\tau(dz)$ and the local time measures $L(t, ds)$ is then given by $\mu_\tau(\{y : y \prec z\}) = L(t(\tau), \mathfrak{s}(z))/\zeta_{t(\tau)}$.

Let us come back to Aldous' question:

"Given a local time profile ζ, can we define a process H^{ζ}

whose law is, in some sense,

the conditional law of H given $L(\cdot, \mathfrak{r}) = \zeta$?"

Our construction accomplishes this because

ζ is independent of (Λ, γ):

we can change the local time profile (almost) ad libitum!

An example:

Genealogies of continuum populations under (neutral) competition:

Recall:

A Brownian excursion H conditioned to height > 1 corresponds to an independent triple (ζ, Λ, γ)

where

ζ is a Feller branching diffusion excursion conditioned to survive time 1,
Λ is the Poisson process of points $(\tau, (i, j))$ in the lookdown space,
γ is a fair coin-tossing that colours the points of Λ by $\curvearrowleft$ or $\curvearrowright$.

Thus, a Girsanov reweighting of the law of ζ does not affect Λ nor γ.

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This gives a direct way to obtain a genealogy of (say)

Feller's logistic branching diffusion

$$\zeta_t = (b\zeta_t - c\zeta_t^2)dt + 2\sqrt{\zeta_t}dW_t:$$

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Feller's logistic branching diffusion

$$\zeta_t = (b\zeta_t - c\zeta_t^2)dt + 2\sqrt{\zeta_t}dW_t:$$

The only change in the underlying genealogy

is through the time change induced by ζ.

This allows for interesting comparisons

with the genealogy that is obtained

when exposing H to a *local time drift*

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Does this reweighting affect Λ?

We conjecture this, but do not yet have a proof.